

B.1 - B.3: Basic Algebra - Part 1

Properties of Exponents.

Suppose $a > 0$.

1. $a^0 =$

2. $a^1 =$

3. $a^{-1} =$

4. $a^m \cdot a^n =$

5. $\frac{a^m}{a^n} =$

6. $(a^m)^n =$

Example 0.1. Simplify:

• $(a^2b)^3$

• $(a^3b^2)^3(a^4b)^2$

• $(a^4b^{-2})^{-1}(a^{-1}b^2)^{-3}$

• $\left(\frac{a^2}{b^{-3}}\right)^{-2} \left(a^{-2} \cdot \frac{b^{-5}}{c^3}\right)^3$

The n^{th} root

We say that $a = \sqrt[n]{b}$ if and only if

Also, a can be written as

Example 0.2. Simplify the following:

- $\sqrt[3]{8}$

- $\sqrt[3]{16}$

- $\sqrt[3]{\frac{1}{64}}$

- $\sqrt{\frac{1}{81}}$

- $\sqrt[5]{64}$

- $\sqrt[3]{a^4b^2c^9}$

Example 0.3. Rationalize (remove square roots from) the denominator in the following:

- $\frac{1}{\sqrt{2}}$

- $\frac{x}{\sqrt[3]{y}}$

- $\frac{2}{\sqrt{x}}$

- $\frac{x-2}{\sqrt{x}-\sqrt{2}}$

Rational Exponents

The expression $a^{m/n}$ can be rewritten as

The expression $a^{-m/n}$ can be rewritten as

Example 0.4. Remove all fractional exponents. Make all exponents positive.

• $a^{2/5}$

• $a^{-2/5}$

• $\sqrt[4]{w\sqrt{x}}$

• $\sqrt[5]{\frac{a^2\sqrt[3]{b}}{c^{4/3}}}$

Example 0.5. Simplify having no root symbols or fractions (but fractions in the exponent are allowed!)

• $\sqrt{\sqrt{x}}$

• $\sqrt{w^2\sqrt[3]{x}}$

• $\sqrt{\frac{a^2b^3}{\sqrt[3]{c}}}$

• $\sqrt{\frac{\sqrt[3]{x^4}\sqrt{y^3}}{\sqrt[5]{z^4}}}$

B.4 - B.5: Basic Algebra - Part 2

Factoring Techniques.

1. Finding a common factor:

- $2x^3 - 12x^2 + 6x =$

- $18a^4b^2 - 30a^3b^3 =$

2. A difference of squares:

$$x^2 - a^2 =$$

$$4x^2 - \frac{1}{9} =$$

3. Difference/Sum of Cubes:

- $x^3 - a^3 =$

- $8x^3 - 27 =$

- $x^3 + a^3 =$

- $x^3y^3 + \frac{1}{8} =$

4. Grouping:

- $x^3 - x^2 + x - 1$

- $2x^3 - 4x^2 - x + 2$

5. Trial and Error:

- $x^2 - 2x + 3$

- $2x^2 - 9x + 10$

A little better method:

If you want to factor $ax^2 + bx + c$,

1.

2.

3.

Example 0.6. Factor the following quadratics:

- $x^2 - 2x - 3 =$

- $6x^2 - 5x - 6 =$

- $2x^2 - 9x + 10 =$

- $6x^2 - 35x + 36 =$

Example 0.7. Factor the following expressions:

1. $a^3b - ab^3$

2. $6x^5(x+1)^3 + 3x^6(x+1)^2$

3. $x^2\sqrt{x^2+4} - (x^2+4)^{3/2}$

4. $\frac{x^2}{(x^2+1)^{2/3}} + \sqrt[3]{x^2+1}$

Simplifying Fractions

Example 0.8. Simplify the following expressions:

- $\frac{x^3 - 9x}{x^2 + 6x + 9}$

- $\frac{x^2 + 2x - 3}{x^2 + 4x + 4} \cdot \frac{x^2 - 4}{x^2 + 4x - 5}$

- $\frac{\frac{1}{a} - 2}{\frac{1}{a^2} - 4}$

- $\frac{\frac{1}{ab} + \frac{2}{ab^2}}{\frac{3}{a^3b} - \frac{4}{ab}}$

1.1 - 1.4: Coordinates and graphs - Part 1

Preliminaries.

A $\sqrt{2}$ is a number which has a decimal expansion that never ends and never repeats.

Sets of real numbers:

- Natural numbers -

- Integers -

- Rational numbers -

- Irrational numbers -

Intervals

An _____ consists of all numbers x such that

A _____ consists of all numbers x such that

Example 0.9. Use the number line to describe the following intervals:

• $[1, 2]$

• $(0, 3)$

• $[1, 2)$

• $(1, 2]$

• $(-\infty, 2]$

• $(3, \infty)$

• $(-\infty, -.001)$

• $(-\infty, \infty)$

Absolute Value

The _____ of a real number x is the

Example 0.10. Simplify the following:

• $|-14|$

• $|10|$

• $|4 - \pi|$

• $|2 - e|$

• $|x - 3|, x \leq 3$

• $|x - 4|, x \geq 4$

The _____ between a and b on the number line is

Example 0.11. Express each of the following sets using absolute value:

- The distance between x and y is 2.
- The distance between y and x is 2.
- The distance between x and 4 is a .
- The distance between x and z is strictly less than 2.
- The distance between a and b is c or greater.

Solving Equations

Example 0.12. Solve the following equations for x :

- $12x - 4 = 68$

- $2[3 - (2x - 1)] = 4.$

- $\frac{x}{2} + \frac{2x}{3} = 2.$

- $a = \frac{x}{3 + x}$

The Zero-Product Property

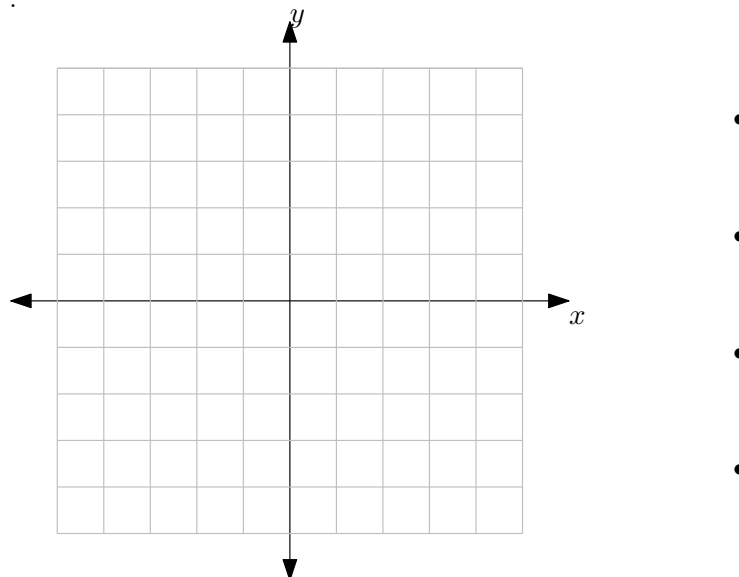
If $pq = 0$, then

Example 0.13. Solve the following equations using factoring:

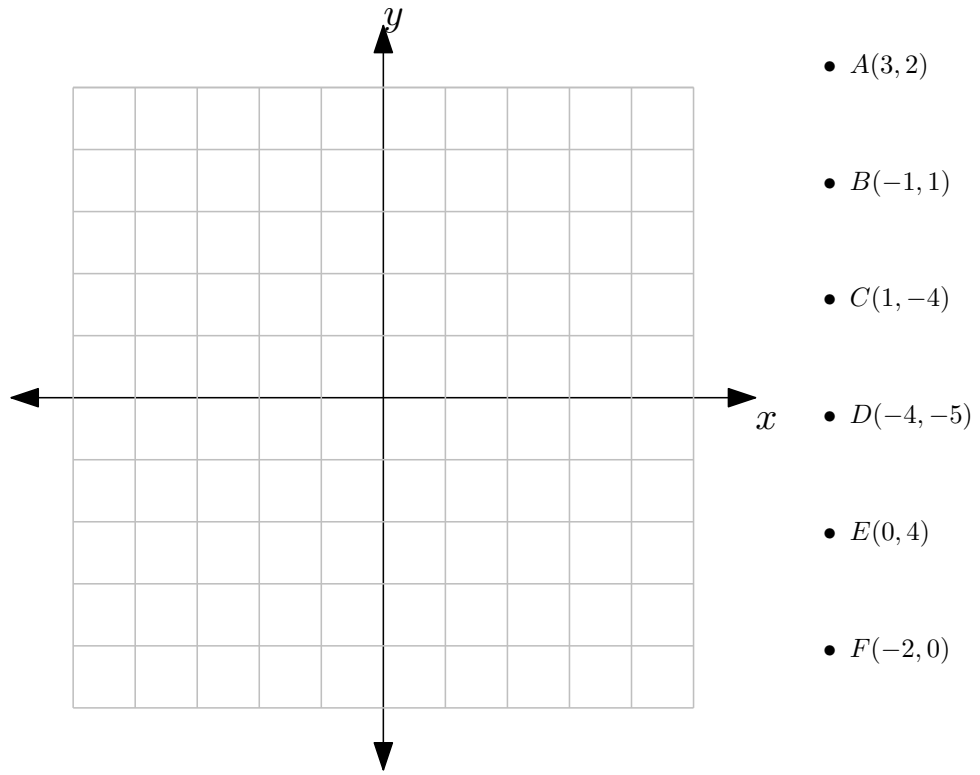
- $x^2 - 2x - 3 = 0.$

- $x^3 - 9x = 0.$

Rectangular Coordinates

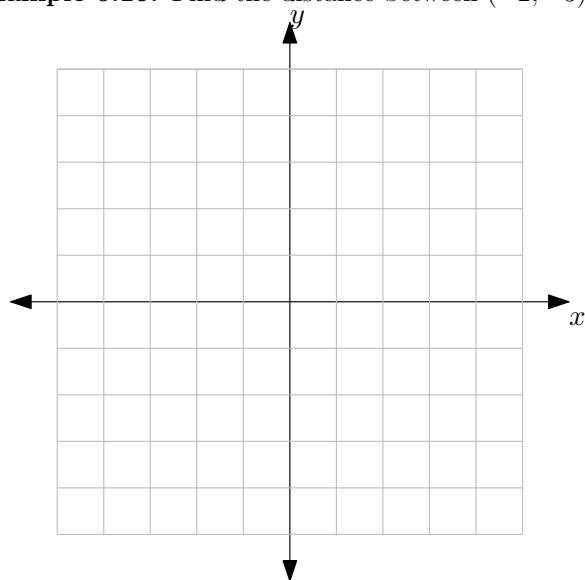


Example 0.14. Plot the following points:

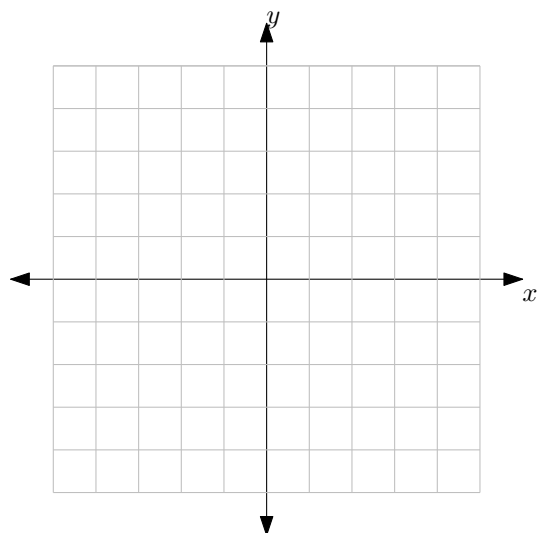


The Distance Formula

Example 0.15. Find the distance between $(-2, -3)$ and $(4, 5)$.



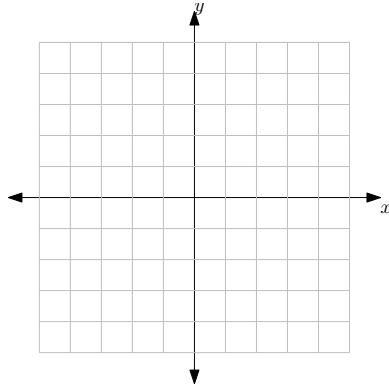
Example 0.16. Find the distance between (x_1, y_1) and (x_2, y_2) .



The **distance** between (x_1, y_1) and (x_2, y_2) is

The Midpoint Formula

Example 0.17. Find the midpoint between $(-3, 2)$ and $(3, 4)$.



The **midpoint** between (x_1, y_1) and (x_2, y_2) is

Example 0.18. Find the midpoint between the following points:

• $(-1, 1)$ and $(2, 3)$

• $(1, -3)$ and $(5, -3)$

• $(a, 0)$ and $(0, b)$

• (a, b) and (b, a)

1.5 - Coordinates and Graphs - Part 2

The **graph** of an equation in two variables is the set of all points with coordinates satisfying the equation.

Example 0.19. Determine whether the following points are on the graph of the given equation:

- $(1, \frac{5}{4}), x + 8y = 11.$

- $(\sqrt{2}, -\sqrt{3}), x^2 - y^2 = 5$

Intercepts

. Suppose we have an equation and its graph.

If a point of the graph is on the x -axis, then

its x -coordinate is called an x -intercept.

Therefore, the x -coordinate of the point is a .

If a point of the graph is on the y -axis, then

its y -coordinate is called a y -intercept.

Therefore, the y -coordinate of the point is b .

Example 0.20. Find the intercepts for the following equations:

- $4x - 6y = 12$

- $\sqrt{2-x} + 1 = y$

- $y^2 = x^3 - x$

Example 0.21. Find the intercepts for the following equations:

- $2x + 3y = 7$

- $x^2 + 6y = y^2$

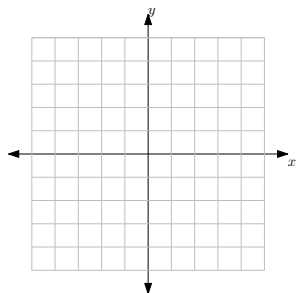
- $xy = x^2 + 1$

1.6 - Lines

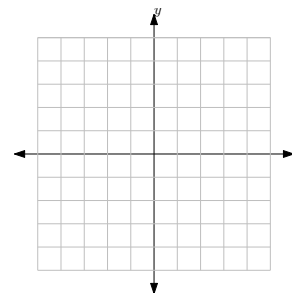
The **slope** of a line going through (x_1, y_1) and (x_2, y_2) is

Example 0.22. Find the slope of the line going through the following points:

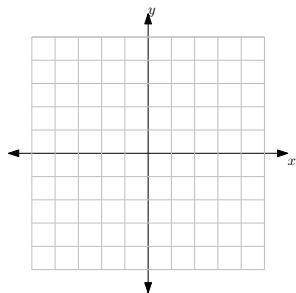
- $(1, 1)$ and $(3, 5)$



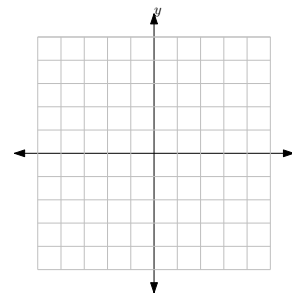
- $(3, 4)$ and $(0, 4)$



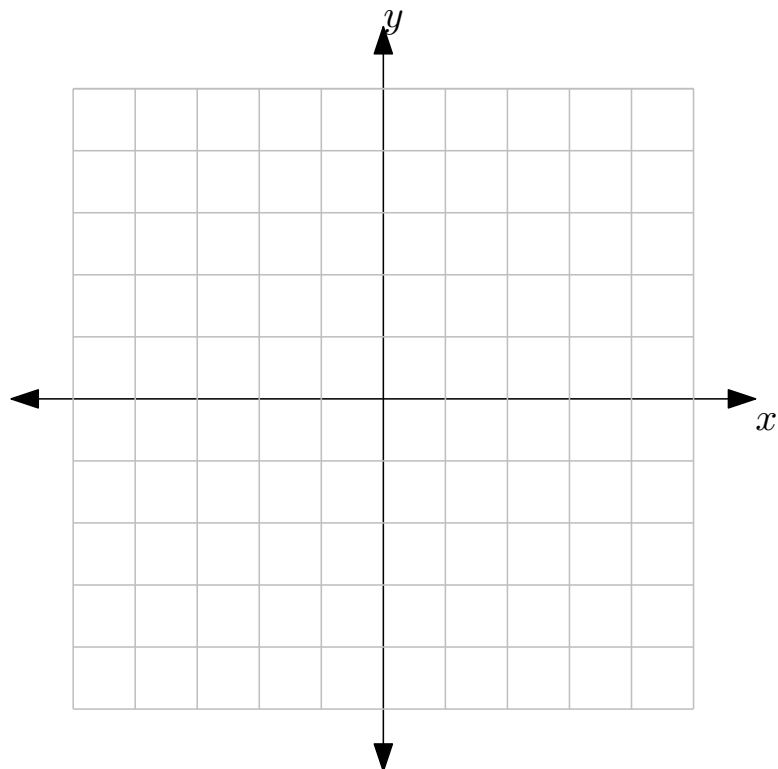
- $(-2, 3)$ and $(-1, 1)$



- $(-1, -2)$ and $(3, -2)$



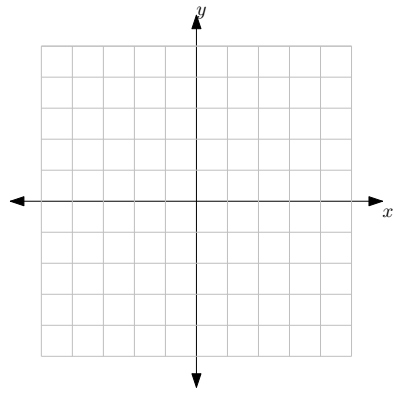
Comparing Slopes



The slope of a _____ line is _____ .

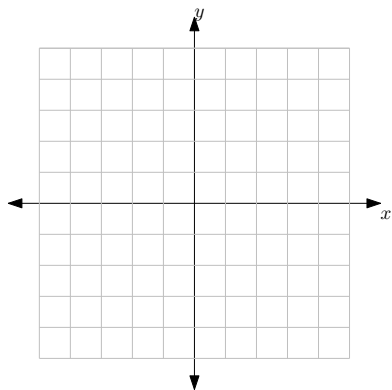
The slope of a _____ line is _____ .

Example 0.23. Suppose a line has slope _____ and goes through $(2, -1)$ and (x, y) .
What is the slope of the line?



Point-Slope Form

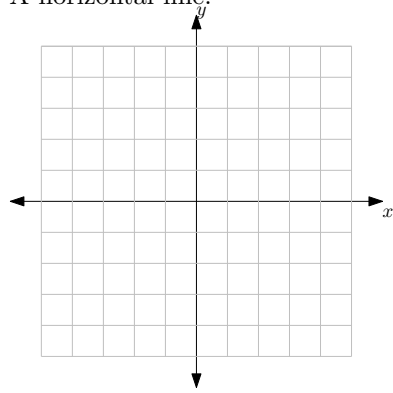
Example 0.24. A line has slope 4 and goes through the point $(2, -1)$. What if (x, y) is on the line?



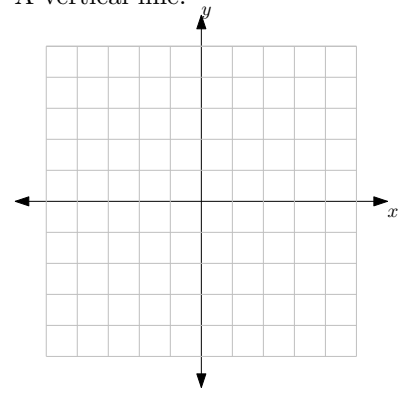
The **point-slope formula** for a line with slope m , going through (x_1, y_1) is

The equations for

A horizontal line:

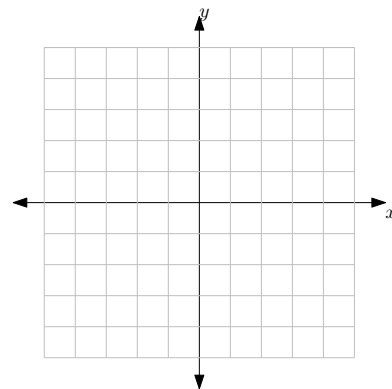


A vertical line:

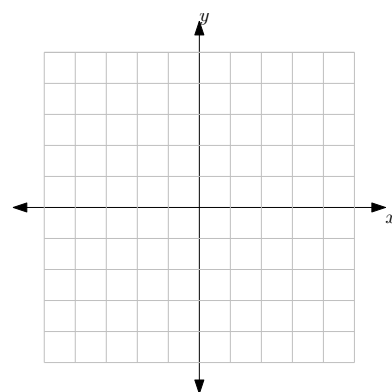


Example 0.25. Find the equation for the following lines:

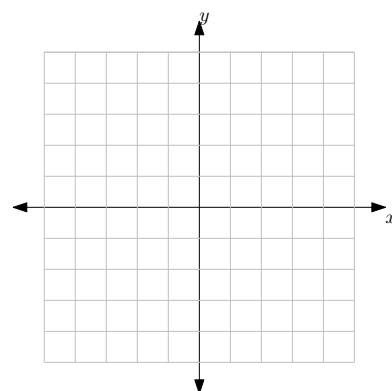
- a vertical line through $(2, -1)$



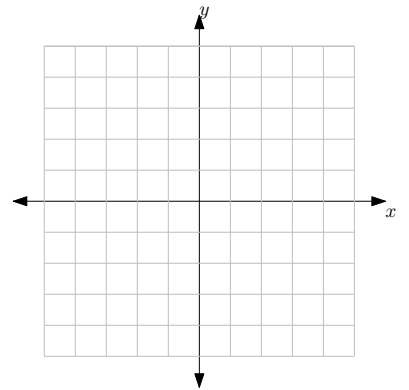
- a horizontal line through $(3, \frac{1}{2})$



- $m = \frac{1}{2}$, goes through $(1, 3)$

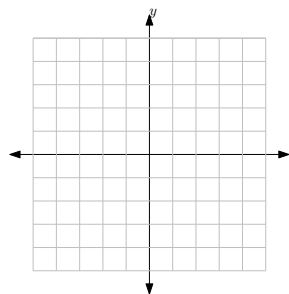


- Goes through $(1, 2)$ and $(-3, 4)$

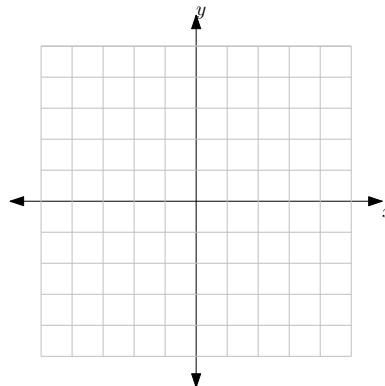


Slope-Intercept

Example 0.26. Find the equation of the line of slope $-\frac{1}{3}$ going through $(0, 4)$.



If a line has slope m and a y -intercept b , then the **slope-intercept formula** for the line is



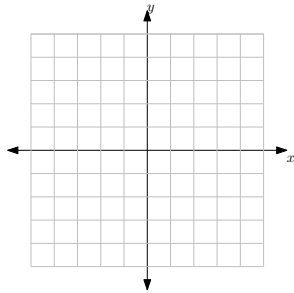
Example 0.27. Find the equation for the line going through $(1, 3)$ with slope $\frac{1}{2}$ in slope-intercept form.

The equation of any line can be expressed as

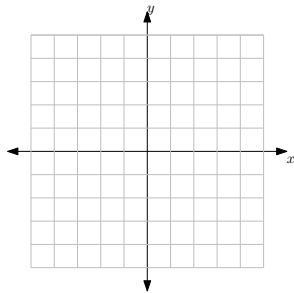
provided that

Example 0.28. Graph the following lines:

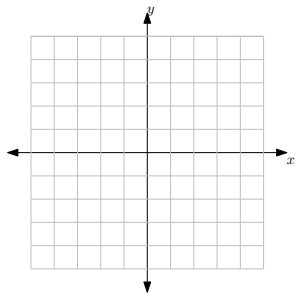
- $3x + 4y = 6$



- $y = \frac{1}{2}x + 2$



- $y - 2 = 4(x + 2)$



Parallel and Perpendicular Lines

Two lines have slopes m_1 and m_2 .

1. The lines are parallel if and only if
2. The lines are perpendicular if and only if

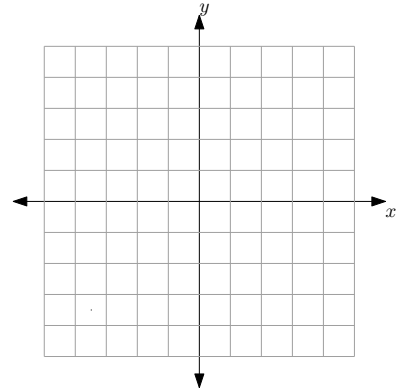
Example 0.29. Find the equation of the following lines:

- A line parallel to $3x - 4y = 12$ going through $(1, 2)$
- A line perpendicular to $x + y = 6$ going through the midpoint between $(3, 4)$ and $(-1, 2)$.

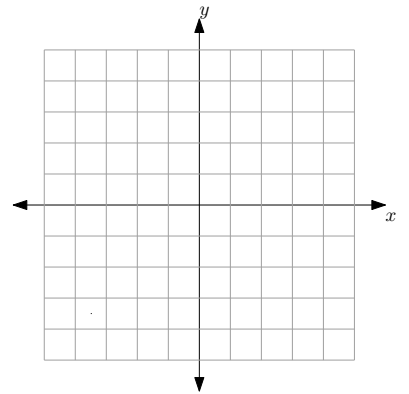
1.7 - Circles and Symmetry

Types of Symmetry

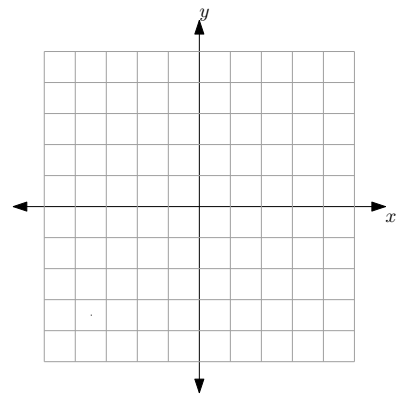
1.



2.

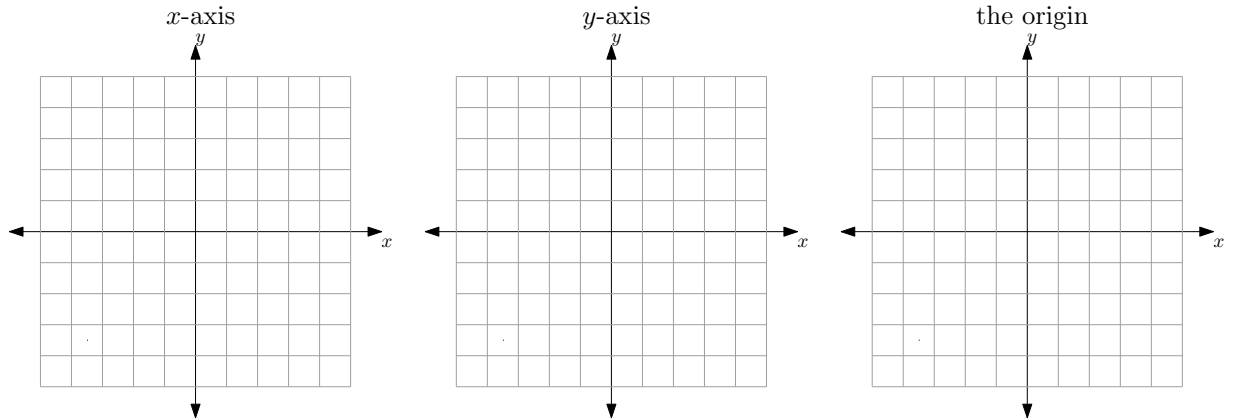
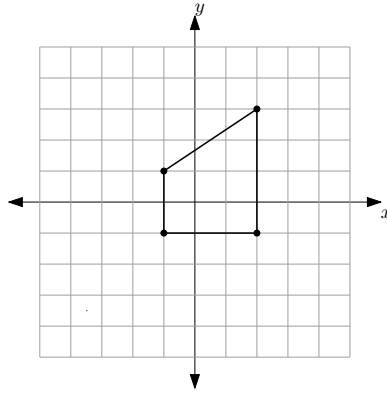


3.



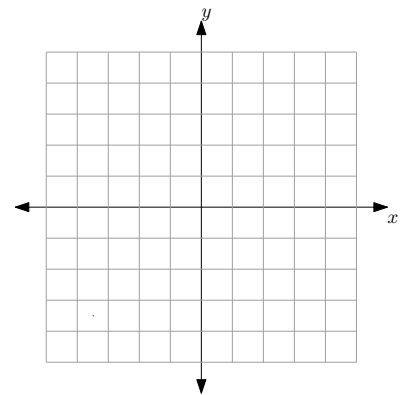
Reflections

Example 0.31. Find the reflections of the following graph about the:



Given a point (x, y) , its reflection

- over the x -axis is
- over the y -axis is
- through the origin is



Testing for Symmetry - x axis

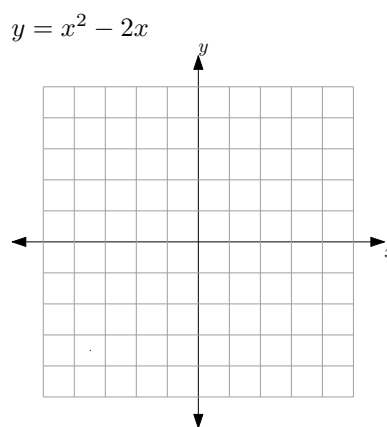
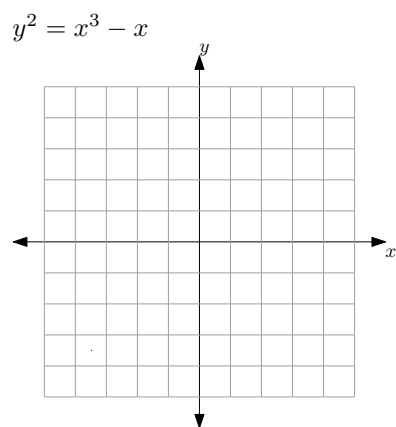
If the graph of an equation goes through (a, b) and is symmetric to the x -axis, then the graph also goes through

To test for symmetry over the x -axis,

The procedure:

- 1.
- 2.

Example 0.32. Determine if either of the following equations are symmetric to the x -axis:



Testing for Symmetry - y axis

If the graph of an equation goes through (a, b) and is symmetric to the y -axis, then the graph also goes through

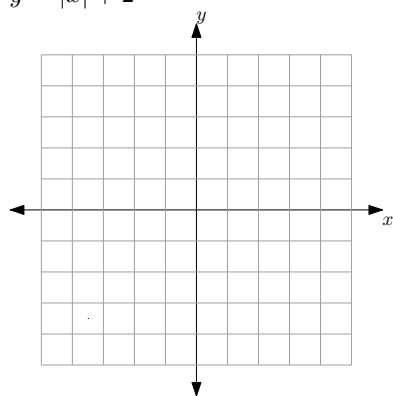
To test for symmetry over the y -axis,

The procedure:

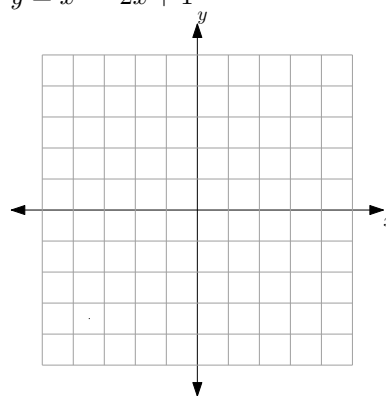
- 1.
- 2.

Example 0.33. Determine if either of the following equations are symmetric to the y -axis:

$$y = |x| + 2$$



$$y = x^2 - 2x + 1$$



Testing for Symmetry - Origin

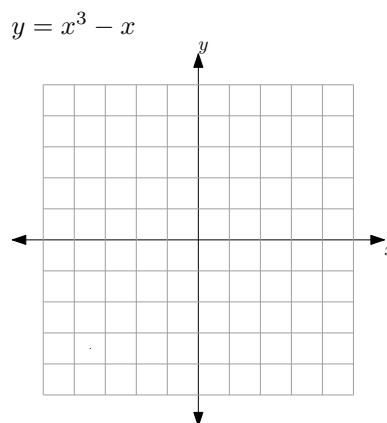
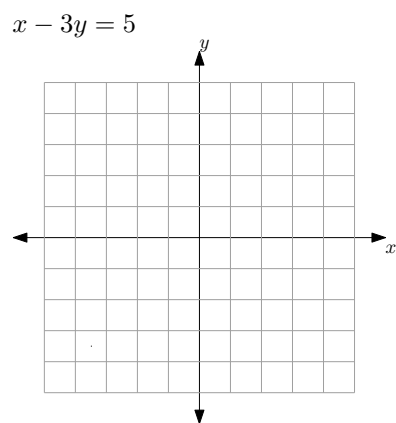
If the graph of an equation goes through (a, b) and is symmetric to the origin, then the graph also goes through

To test for symmetry to the origin,

The procedure:

- 1.
- 2.

Example 0.34. Determine if either of the following equations are symmetric to the x -axis:



Equations of circles

A **circle** with center (h, k) and radius r is the set of points (x, y) such that

Example 0.35. Find the equation for the circle with center $(1, 2)$ and radius 3.

The equation for a circle with center (h, k) and radius r is

Example 0.36. Find the equation for the circle with center $(-1, 3)$ and radius $\sqrt{2}$.

Completing the square

Expand the following squares:

- $(x - 1)^2 =$

- $(x - 3)^2 =$

- $(x + \frac{1}{2})^2 =$

- $(x + 3)^2 =$

For each of the following, what needs to be added to make a perfect square?

- $x^2 + 2x$

- $x^2 + 6x$

- $x^2 - x$

- $x^2 - \frac{1}{2}x$

To determine what is needed to complete the square,

Identifying Circles.

Example 0.37. Find the center and radius of the circles whose equation is given below:

- $x^2 + y^2 - 6x = 7$

- $x^2 + y^2 - 4x + 6y = -12$

- $4x^2 + 4y^2 - 4x + 8y = 11$

Example 0.38. Find the intercepts of the circle $x^2 + y^2 - 7x + 8y = 12$.

Example 0.39. Find the equation of a circle whose diameter is the segment \overline{AB} , where $A = (-3, 1)$ and $B = (5, -5)$.

2.1 - Quadratic Equations

Completing the square ... again.

To make $x^2 + bx$ a perfect square, for example $x^2 - 6x$,

1.

2.

A **quadratic equation** is an equation of the form

The solutions to a quadratic equation (and any polynomial equation for that matter) are called .

Example 0.40. Solve the following quadratic equation:

- $x^2 - 2x - 3 = 0$

- $x^2 - 2x - 4 = 0$

The Quadratic Formula

Example 0.41. Solve the equation $3x^2 + 6bx + 4 = 0$ (in terms of b)

The Quadratic Formula

Example 0.42. Solve the equation $ax^2 + bx + c = 0$ (in terms of a, b, c).

Thus, the quadratic formula says that the solutions to $ax^2 + bx + c = 0$ are

Example 0.43. Solve the following quadratic equations:

- $x^2 - 12x + 35 = 0$

- $x^2 - 12x + 36 = 0$

- $x^2 - 12x + 37 = 0$

Therefore, the number of solutions you get to a quadratic equation is either

The product and sum of roots.

Suppose the roots of $x^2 + bx + c = 0$ are r_1 and r_2 .

Then $x^2 + bx + c =$

So

Example 0.44. Find the roots of the following equations:

• $x^2 - 2x - 3 = 0$

• $x^2 + x - 1 = 0$

Product of roots:

Product of roots:

Sum of roots:

Sum of roots:

Example 0.45. Find the sum and product of the roots of the following equations:

• $x^2 + 4x - 7 = 0$

• $2x^2 + 6x - 135 = 0$

The Discriminant.

The **discriminant** of the quadratic equation $ax^2 + bx + c = 0$ is

The discriminant can be used to tell how many solutions a quadratic equation will have.

- If $b^2 - 4ac > 0$, then there are two real solution(s).
- If $b^2 - 4ac = 0$, then there are one real solution(s).
- If $b^2 - 4ac < 0$, then there are no real solution(s).

Example 0.46. Find the number of real solutions for $2x^2 - 3x - 1 = 0$.

Example 0.47. Find a value of k which makes $kx^2 + 4x - 7 = 0$ have no real solutions.

2.2 - Other Types of Equations

Absolute Value

Example 0.48. Solve the following equations.

- $|2x - 3| = 1$

- $|x + 4| = |x - 1|$

- $|x| + |x + 2| = 4$

n^{th} roots

Example 0.49. Solve the following equations.

- $x^2 = 4$

- $x^2 = -4$

- $x^3 = 8$

- $x^3 = -8$

- $x^4 = 16$

- $x^4 = -16$

When taking n^{th} roots,

- if n is _____, then

- if n is _____, then

Fractional Exponents.

Example 0.50. Solve the following equations.

- $x^{\frac{1}{3}} = 2$

- $x^{\frac{1}{3}} = -2$

- $x^{\frac{2}{3}} = 4$

- $x^{\frac{2}{3}} = -4$

- $x^{\frac{3}{2}} = 8$

- $x^{\frac{3}{2}} = -8$

When solving problems with fractional exponents,

Using Factoring.

Example 0.51. Solve the following equations.

- $x^5 - 18x^4 + 80x^3 = 0$

- $6x^3 - 5x^2 - 6x = 0$

- $x^4 - 2x^2 - 3 = 0$

- $t^{4/3} - 4t^{2/3} - 5 = 0$

- $6t^{-2} - 20t^{-1} + 6 = 0$

- $x^4 + x^2 - 1 = 0.$

Radicals.

Example 0.52. Solve the following equations.

- $\sqrt{x-2} = 10$

- $\sqrt{x+3} - 1 = x$

- $\sqrt{x-5} - \sqrt{x+4} + 1 = 0$

- $\sqrt{2x+3} - 2\sqrt{x-2} = 1$

2.4 - Applications

Example 0.53. On the first three exams in a history class, you make a 85%, 94%, and a 73%. The final exam is counts as two exams. What would you need to score on the final in order to average a 90%?

Example 0.54. Jackie can paint a house in 4 hours. Cletus can paint the same house in 3 hours. How long would it take for them to complete the house together?

Example 0.55. We have a limitless supply of pure acid and of a 15% acid solution. If we must obtain 20 liters of a 40% acid by mixing pure acid and the 15% acid solution, how much of each should be mixed?

Example 0.56. A right triangle has a hypotenuse of 13 inches. If the perimeter of the triangle is 30 inches, what are the side lengths?

Example 0.57. A circular water puddle grew in radius by 3 inches. The resulting change in the area of the puddle 53π square inches. What was the original radius of the circular puddle?

Example 0.58. You have \$10,000 to invest and two choices of accounts in which to invest; one gives 6% simple interest and the other gives 10% simple interest. How much should you initially invest in order to make \$4,200 in interest after 5 years?

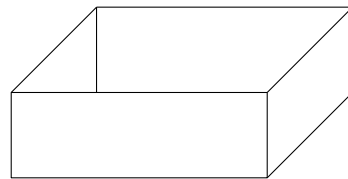
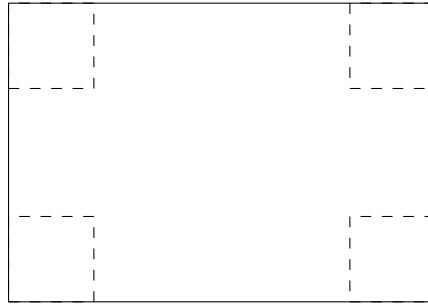
Example 0.59. If you invest \$2000 at 6% for a year and \$8,000 at 10% for a year, what is the effective interest rate?

Example 0.60. A bullet is shot directly up in the air with an initial velocity of 200 m/s. The equation relating the height of the bullet and time elapsed is

$$h = -10t^2 + 500t + 490$$

When does the bullet reach 1000 meters? When does it hit the ground?

Example 0.61. A rectangular piece of cardboard with area 192 square inches has a square with side 3 inches cut from each corner, and the resulting piece is folded into a box. If the volume of the box is 128 cubic inches, what was the original dimension of the box?



Example 0.62. Robin Banks left a bank at 52 mph. Officer Willie Catchup arrived at the bank 15 minutes behind Robin leaving and chases him at 60 mph. The state line is 100 miles away. Will Robin get caught (does Willie catch up)?

2.5 - Inequalities

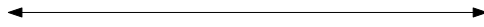
Properties of Inequalities

1. •
2. •
3. •
4. •

Example 0.63. Solve the following inequalities:

- $4x + 3 < -5$

- $2x - 1 \leq 7(x + 2)$



Example 0.64. Solve the following inequality:

$$\frac{1}{2} \leq \frac{2-x}{3} \leq 2$$



Suppose that $a > 0$. Then

- $|u| < a$ •
- $|u| \leq a$ •
- $|u| > a$ •
- $|u| \geq a$ •

Example 0.65. Solve the following inequalities:

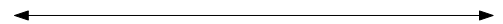
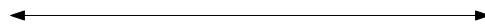
- $|x - 1| \leq 2$

- $|2x - 3| \geq 1$



- $|\frac{-1}{2}x + 3| < 2$

- $|\frac{-x+3}{-2}| > 2$



Example 0.66. Solve the following inequalities:

$$\bullet \left| \frac{3(x-2)}{4} + \frac{4(x-1)}{3} \right| \leq 2$$

$$\bullet |(x+h)^2 - x^2| < 3h^2$$



$$\bullet |-x + 3| < -2$$

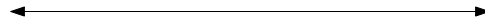
$$\bullet \left| -\frac{1}{2}x + 3 \right| > 0$$



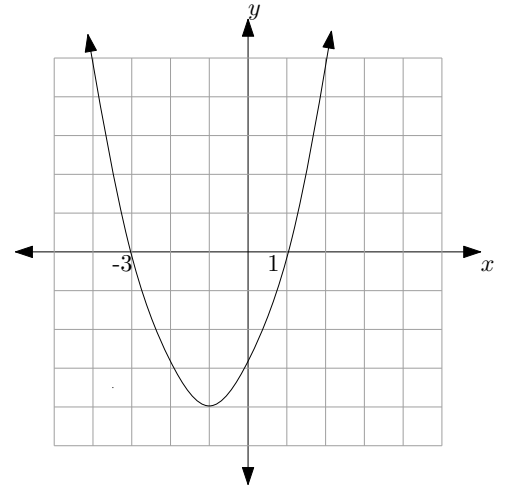
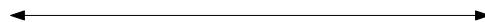
2.6 - More on Inequalities

Example 0.67. Use the graph of $y = x^2 - 2x - 3$ to determine when

(a) $x^2 - 2x - 3 < 0$



(b) $x^2 - 2x - 3 \geq 0$



Observation:

If you have an inequality that looks like any of

$$\frac{P}{Q} < 0$$

$$\frac{P}{Q} \leq 0$$

$$\frac{P}{Q} > 0$$

$$\frac{P}{Q} \geq 0$$

where P and Q are polynomials in x with no common factors, find the

When x is between the key values, the
the polynomial

of

Example 0.68. Solve the following inequality:

$$x^3 - 4x \geq 0$$

(a)

(b)



(c)

Interval	Test Number				

(d)

Example 0.69. Solve $x^3 - 4x < x^2 - 5x + 1$

(*)

(a)

(b)



(c)

Interval	Test Number

(d)

Example 0.70. Solve $\frac{x-1}{x+2} \leq 0$

(*)

(a)

(b)



(c)

Interval	Test Number

(d)

Example 0.71. Solve $\frac{2x+1}{x-1} - \frac{2}{x-3} \leq 1$

(*)

(a)

(b)



(c)

Interval	Test Number

(d)

3.1 - Definition of Functions

What is a function?

Representations of Functions

•

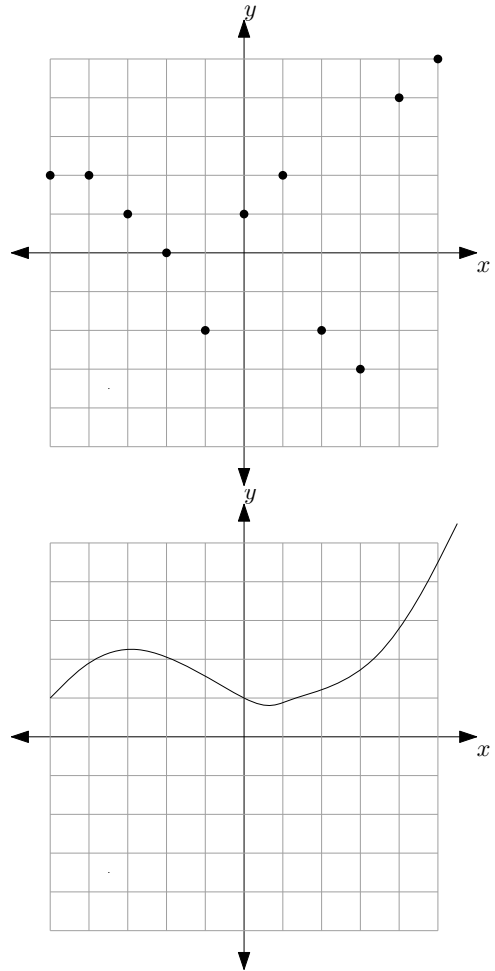
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Team	Results	Element	Atomic Weight
Washington State	Win	Hydrogen	1.008
UNLV	Win	Helium	4.003
Citidel	Win	Lithium	6.941
Iowa	Win	Beryllium	9.012
Michigan	Win	Boron	10.81
Illinois	Loss	Carbon	12.01
Penn State	Loss	Nitrogen	14.01

Representations of Functions (cont.)

•



The set of all possible inputs is called the domain .

The set of all possible outputs is called the range .

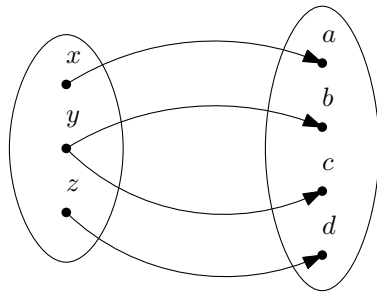
The Definition of a function.

Let A and B be sets. A **function** from A to B is a rule of correspondence

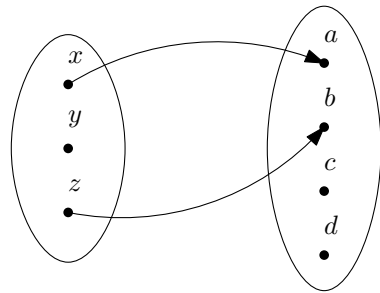
that

Example 0.72. Which of the following correspondences are functions?

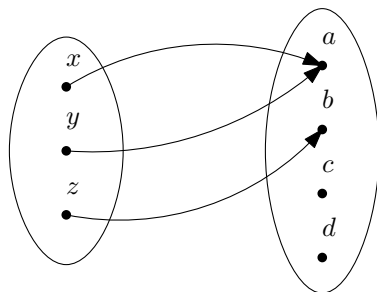
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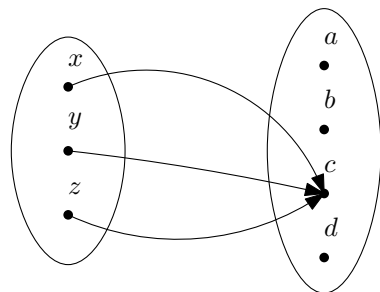
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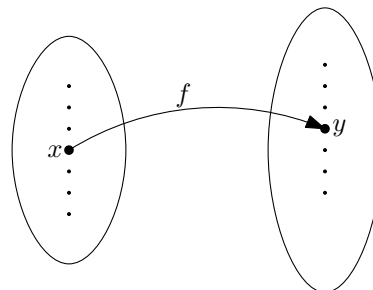


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Functions defined by equations

If f is a function and x is an input value, then



The input variable is called the .

The output variable is called the .

Domain and Range

Unless otherwise specified, the **domain** of a function f is the set of

The **range** of a function f is the set of

Example 0.73. Find the domain of the following functions:

- $f(x) = x^2 - x$

- $f(x) = \sqrt{1 - 4x}$

- $f(x) = \frac{1}{x - 1}$

- $f(x) = \frac{1}{x^2 + 4}$

Example 0.74. Find the domain of the following functions:

- $f(x) = \sqrt{\frac{1}{x^2 - 1}}$

- $f(x) = \sqrt[3]{\frac{x-2}{x+1}}$

Example 0.75. Let $f(x) = \frac{x^2}{x-4}$ Find

- $f(2)$

- $f(-1)$

- $f(4)$

- $f(a+2)$

Example 0.76. Let $f(x) = x^2 - x - 2$ Evaluate

- $f(x - 2)$

- $f(2x)$

- $f(x + h)$

- $f(x + h) - f(x)$

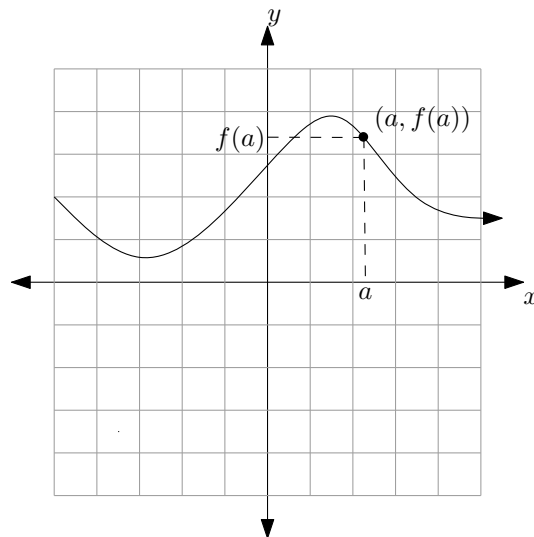
Example 0.77. Find the domain and range for $y = f(x) = \frac{x}{x + 1}$.

3.2 - 3.3: Graphs of Functions

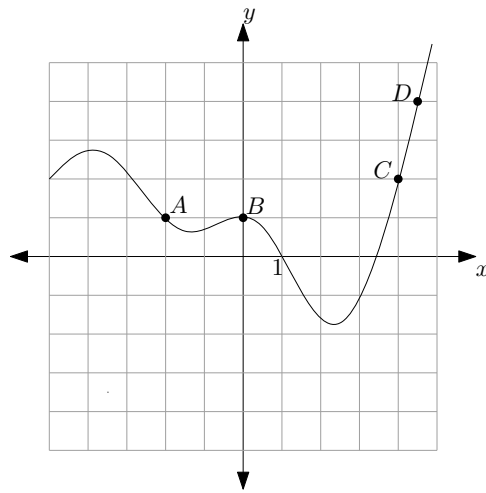
The **graph of a function** f in the x - y plane consists of those points (x, y) such that

•

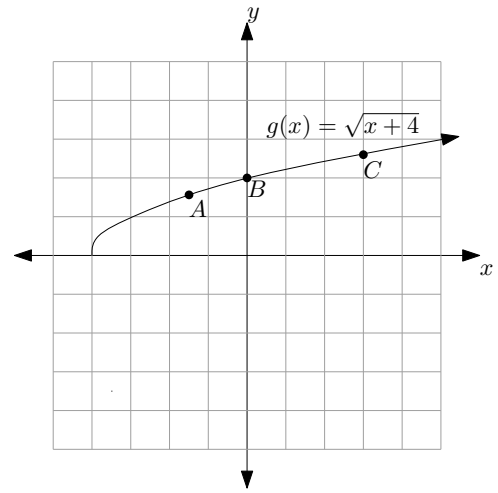
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Example 0.78. Evaluate the following:



- $f(0)$
- $f(-2)$
- $f(4)$
- $f(4.5)$



Find the coordinates for

- A
- B
- C

When is a graph a function?

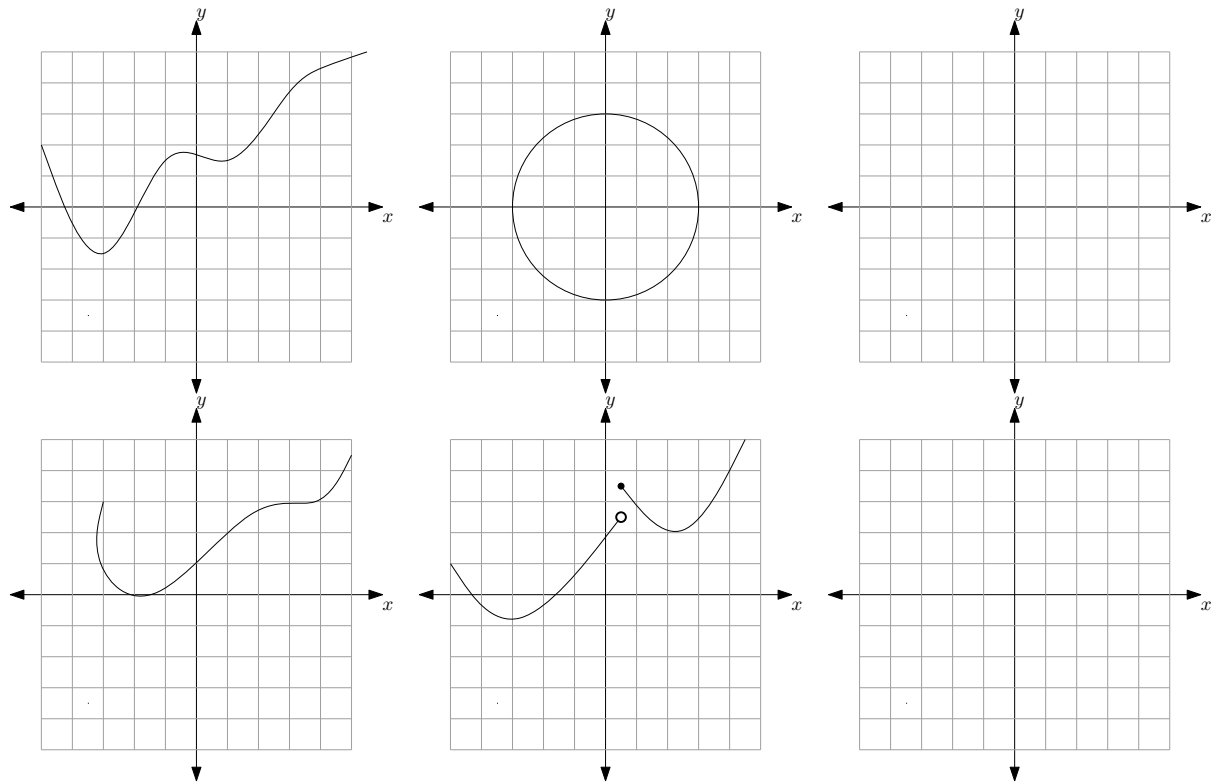
Recall a function from A to B is a rule of correspondence that

If a graph is to represent a function f , then for any a in the domain,

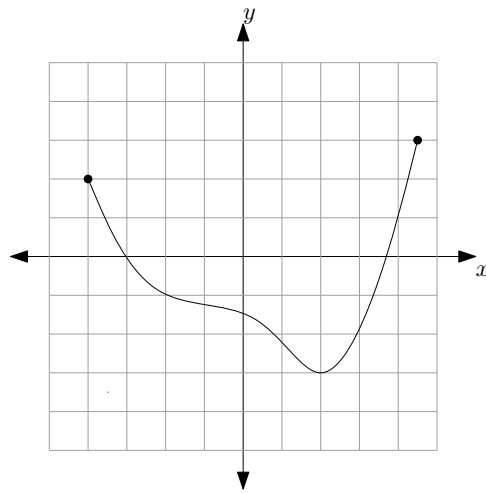
The Vertical Line Test.

A graph in the x - y plane represents a function $y = f(x)$ provided that

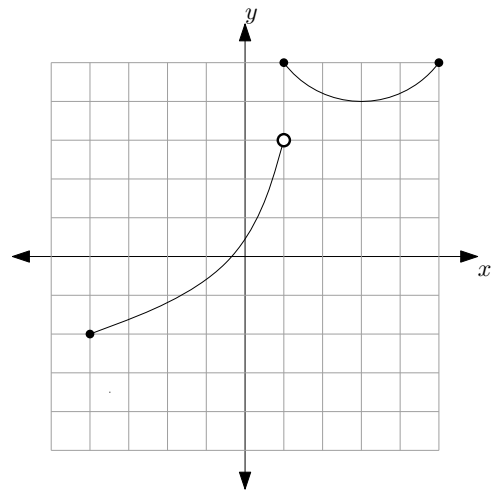
Example 0.79. Do these graphs represent a function?



Example 0.80. Find the domain and range of f in each graph:

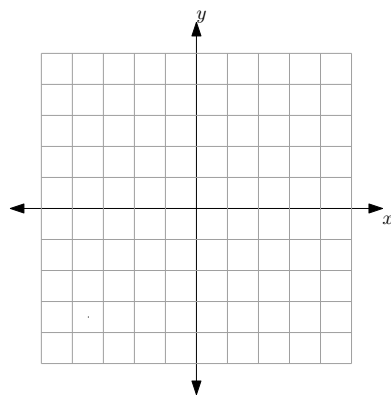


- Domain:
- Range:
- $f(-2) =$
- $f(4) =$

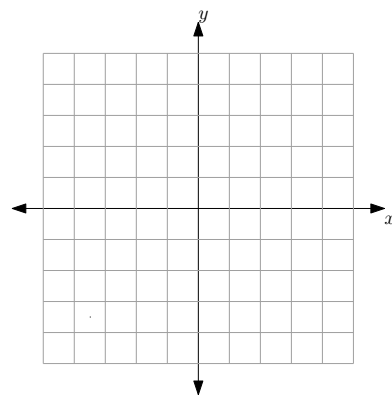


- Domain:
- Range:
- $f(1) =$
- $f(3) =$

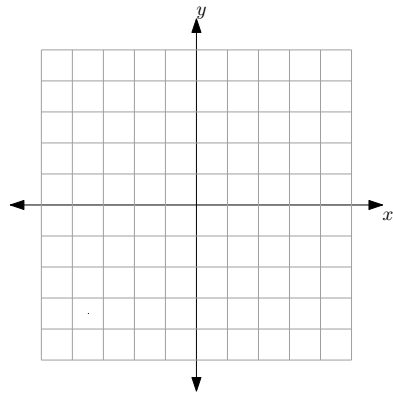
The Six Basic Graphs



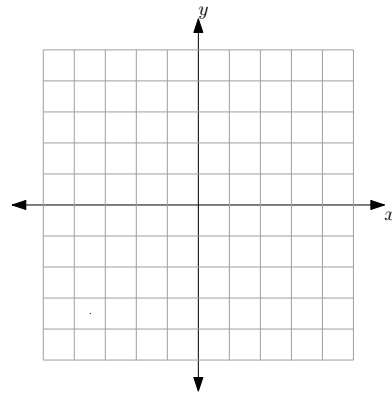
-
- $y = f(x) =$
- Domain:
- Range:



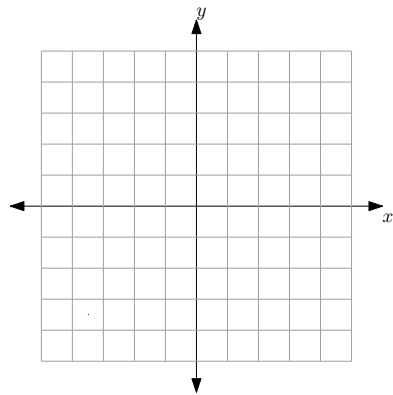
-
- $y = f(x) =$
- Domain:
- Range:



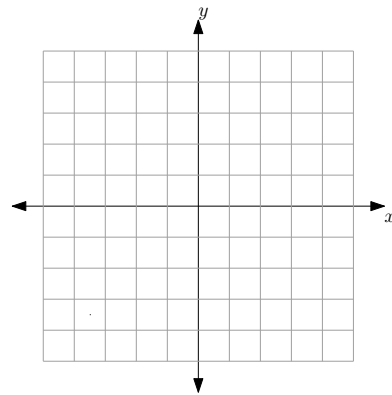
-
- $y = f(x) =$
- Domain:
- Range:



-
- $y = f(x) =$
- Domain:
- Range:



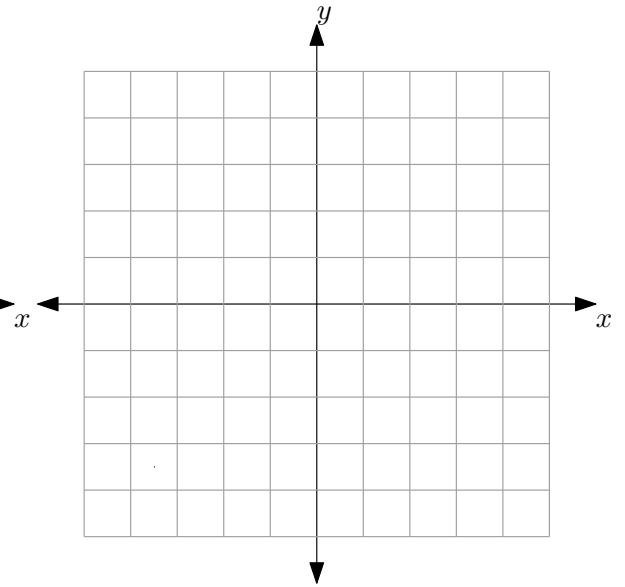
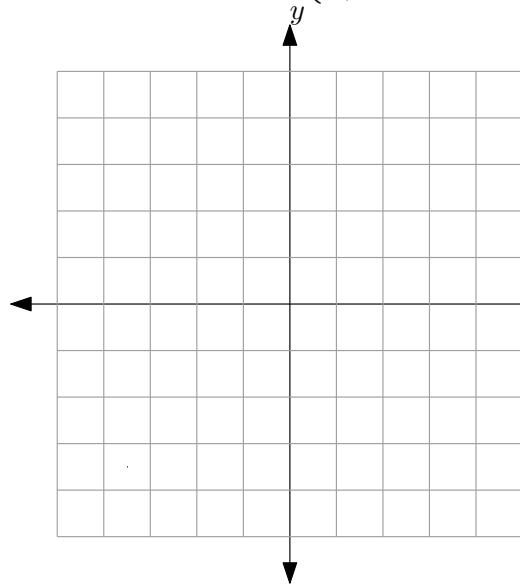
-
- $y = f(x) =$
- Domain:
- Range:



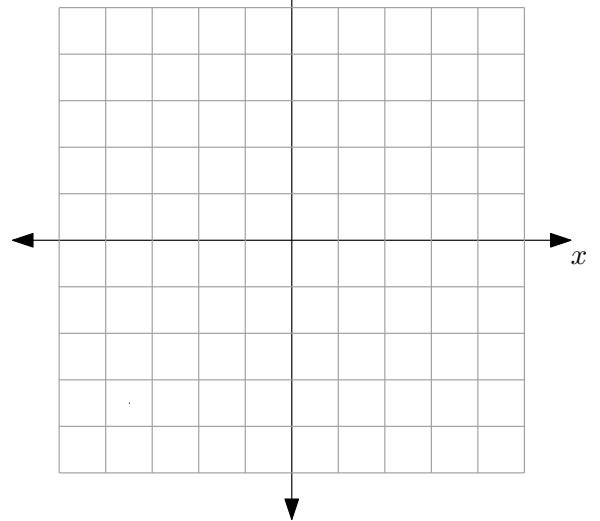
-
- $y = f(x) =$
- Domain:
- Range:

Piecewise Functions

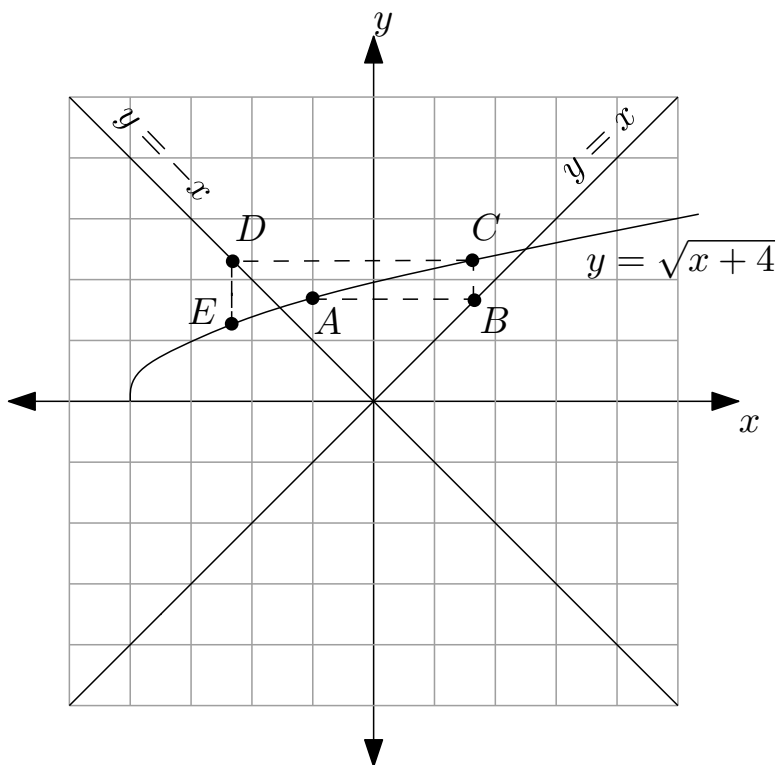
Example 0.81. Let $g(x) = \begin{cases} |x| & -2 < x \leq 2 \\ \sqrt{x} & 2 < x \leq 4 \end{cases}$



- $g(-1)$
- $g(2)$
- $g(-2)$
- $g(4)$
- $g(3)$



Example 0.82. Consider the following graph:



Find the coordinates for the the above 5 points.

Shapes of Graphs

Consider the graph of $y = f(x)$:

A function is **increasing** on an interval if for each a, b in the interval,

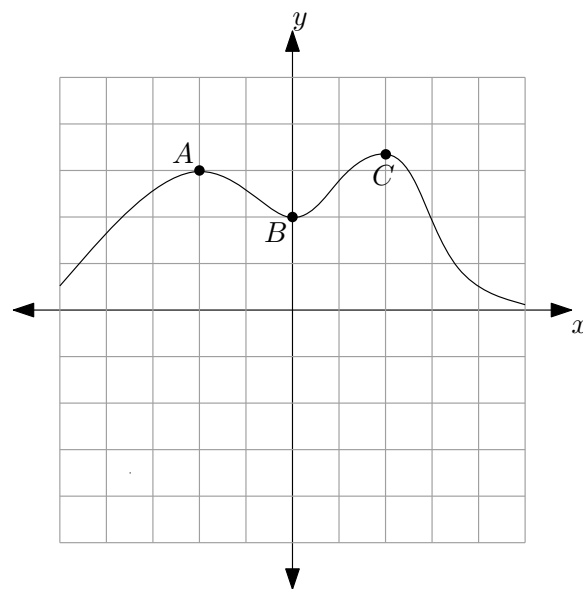
A function is **decreasing** on an interval if for each a, b in the interval,

A value M is the **maximum** provided that

-
- For every x in the domain,

A value m is the **minimum** provided that

-
- For every x in the domain,



A **turning point** is a place where a graph goes from

Using the graph above,

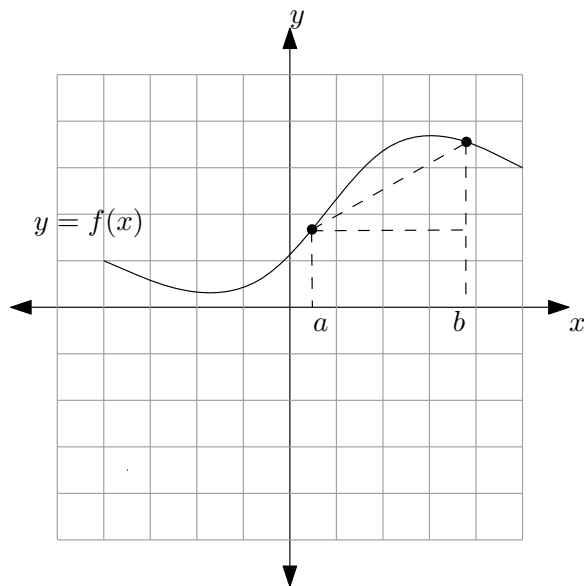
- what are the turning points?

- where is f increasing?

- where is f decreasing?

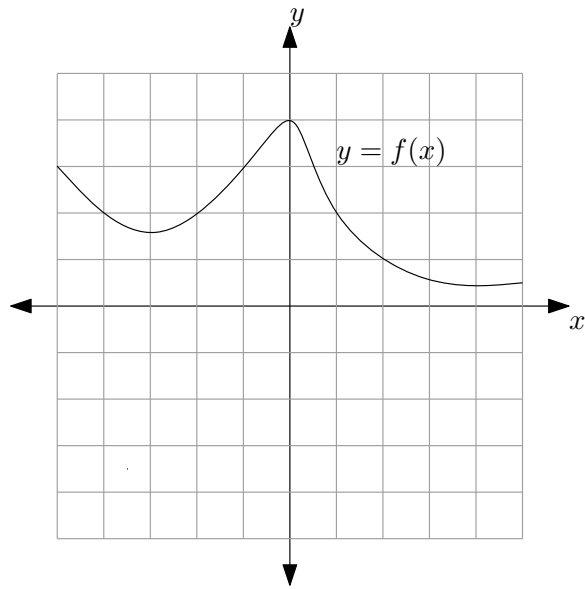
- what is the max, min?

The Average Rate of Change



The **average rate of change** of a function f on the interval $[a, b]$ is

Example 0.83. Find the average rates of change of $f(x)$ over the following intervals:



• $[0, 1]$

• $[0, 2]$

• $[-2, 0]$

• $[2, 5]$

Example 0.84. Find the average rate of change of the given function over the given interval:

- $f(x) = x^2 - x$, $[0, 2]$

- $f(x) = x^2 - x$, $[1, 3]$

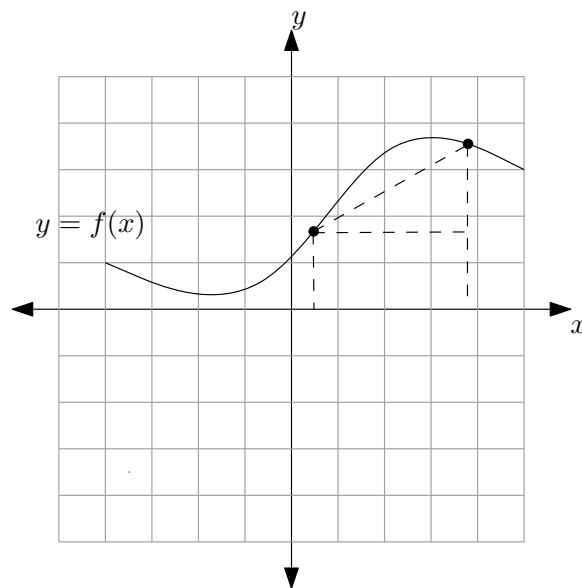
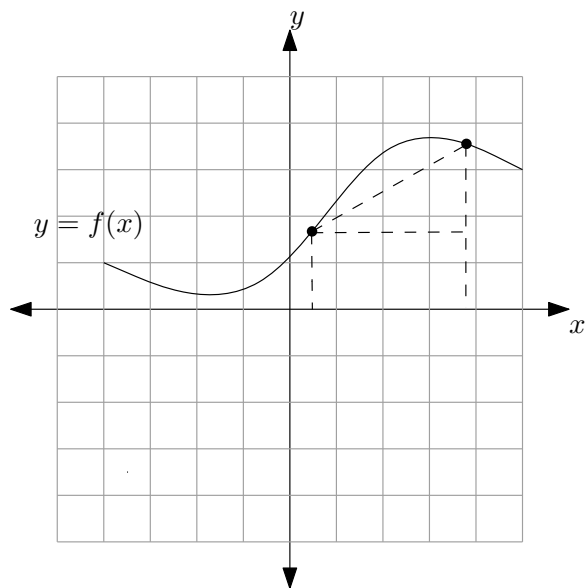
- $g(t) = |t - 1|$, $[0, 2]$

- $h(t) = t^2$, $[a, b]$

- $f(t) = \sqrt{t}$, $[4, a]$
Solve for $a = 5, 4.1, 4.01$.

- $h(z) = \frac{1}{z}$, $[1, 1 + x]$
Solve for $x = 1, 0.1, 0.01$.

The Difference Quotient



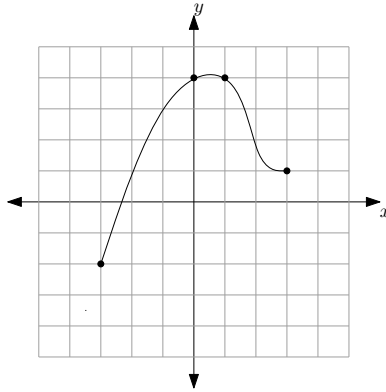
Example 0.85. Simplify each difference quotient for the following functions:

$$f(x) = \sqrt{x}$$

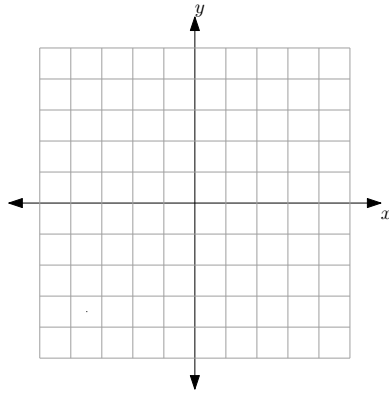
$$f(x) = \frac{1}{x}$$

3.4 - Techniques in Graphing

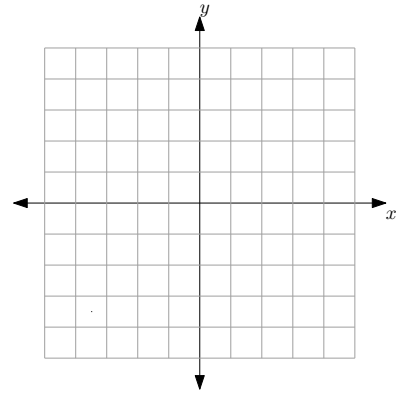
Use the following graph and perform the given operations:



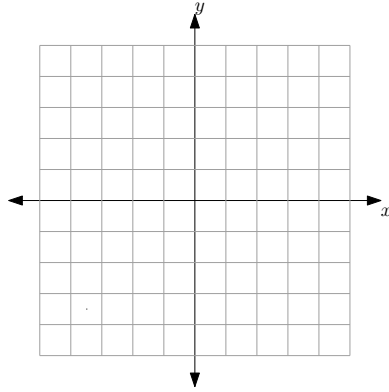
- right 3.



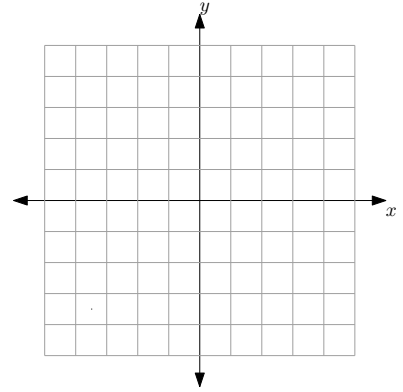
- reflect over y -axis



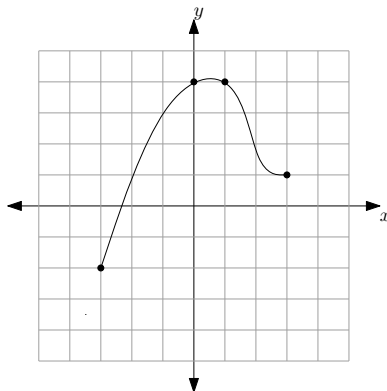
- 1 down



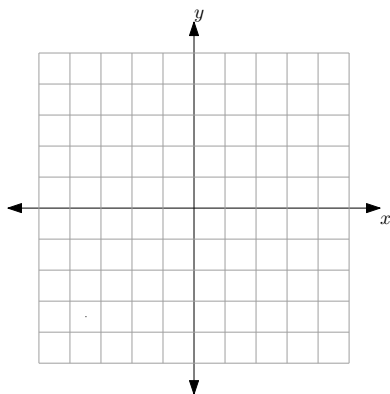
- reflect over x -axis



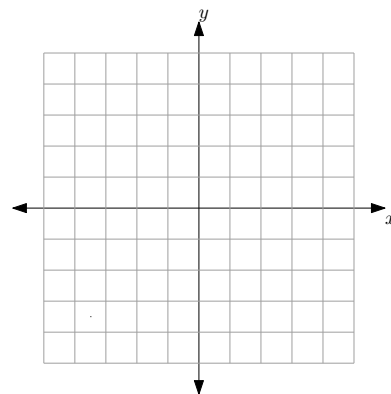
Use the following graph and preform the given operations:



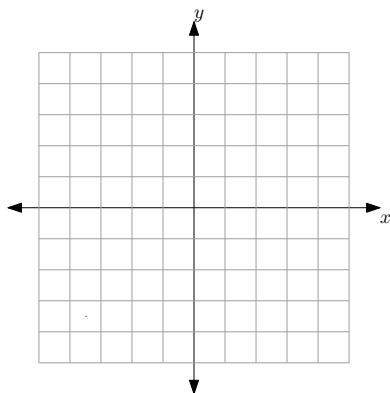
- reflect over x , left 3



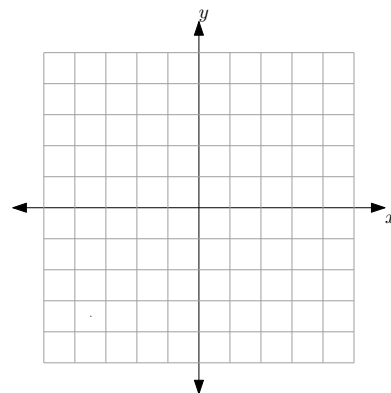
- left 3, reflect over x



- up 2, reflect over x

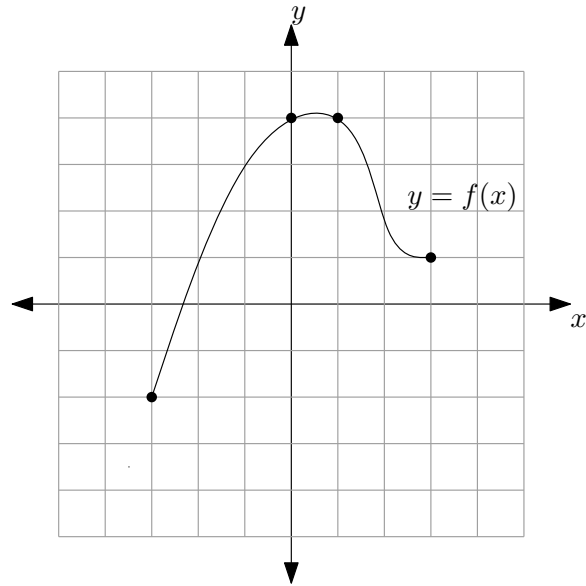


- reflect over x , up 2

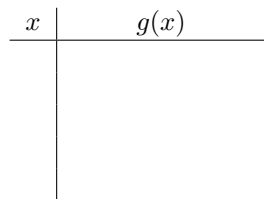
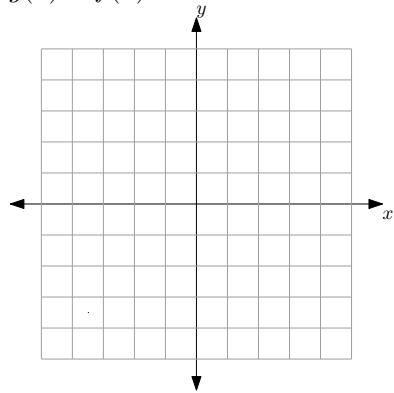


Observation: In some cases, the order of the movements

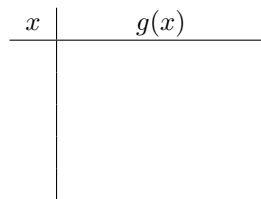
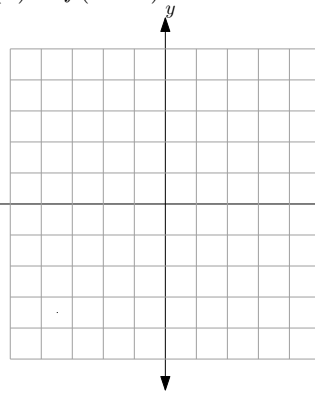
Translating and Reflecting functions



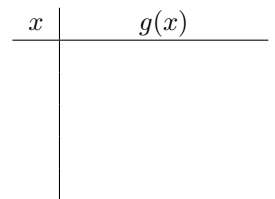
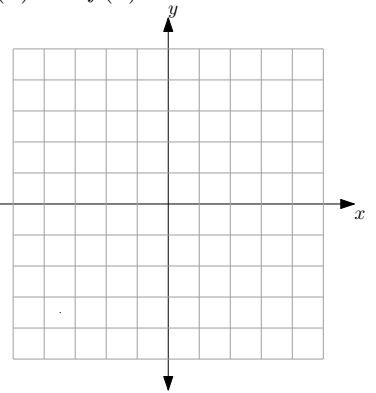
$$g(x) = f(x) + 1$$



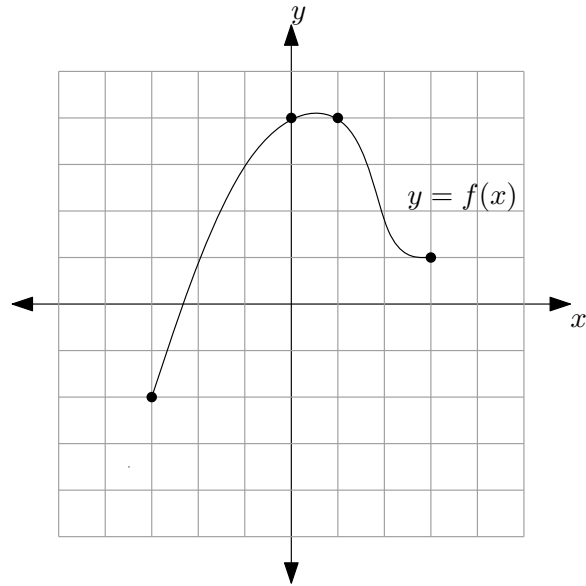
$$g(x) = f(x + 1)$$



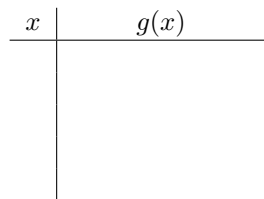
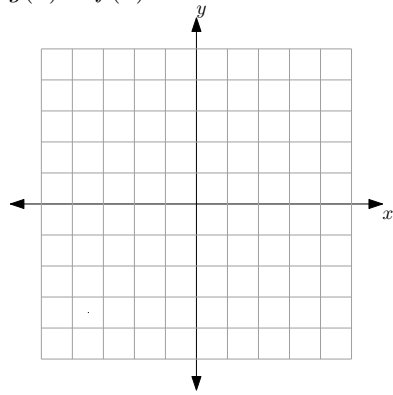
$$g(x) = -f(x)$$



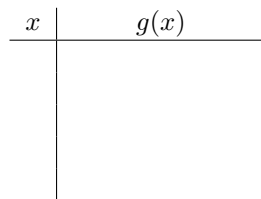
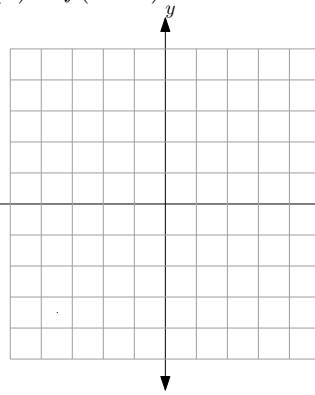
Translating and Reflecting functions



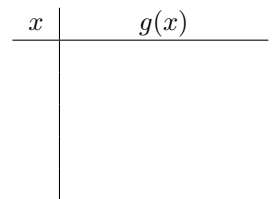
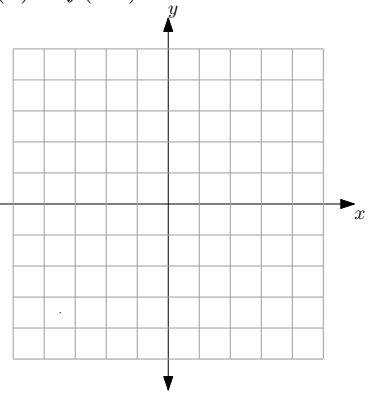
$$g(x) = f(x) - 2$$



$$g(x) = f(x - 2)$$



$$g(x) = f(-x)$$



Summary

Given the graph of a function $y = f(x)$ and c positive,

- $y = f(x) + c$

- $y = f(x) - c$

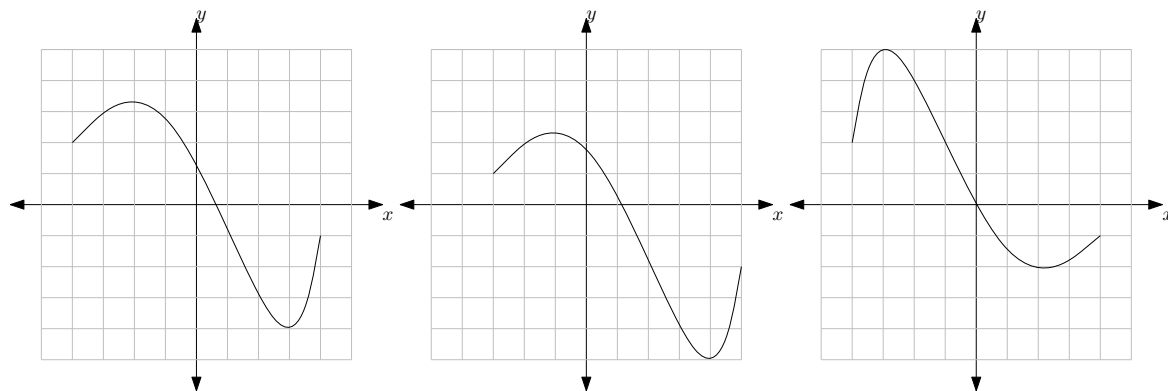
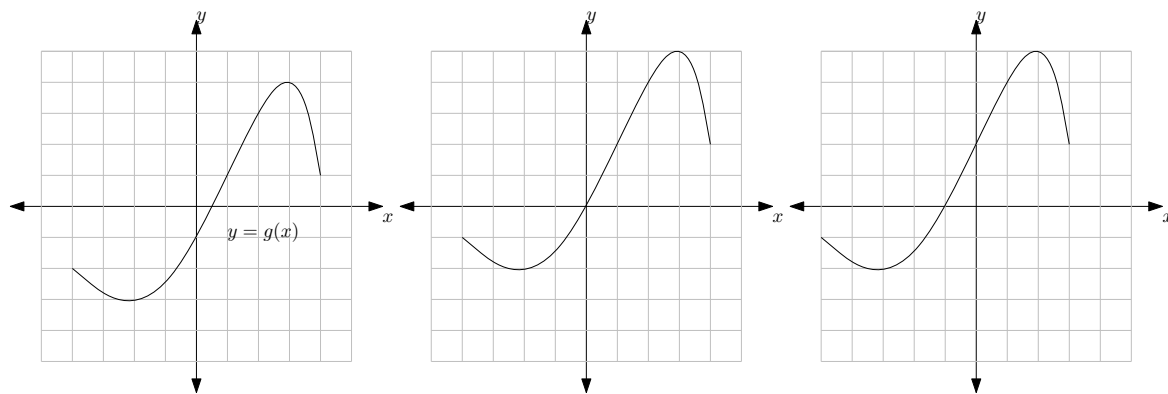
- $y = f(x - c)$

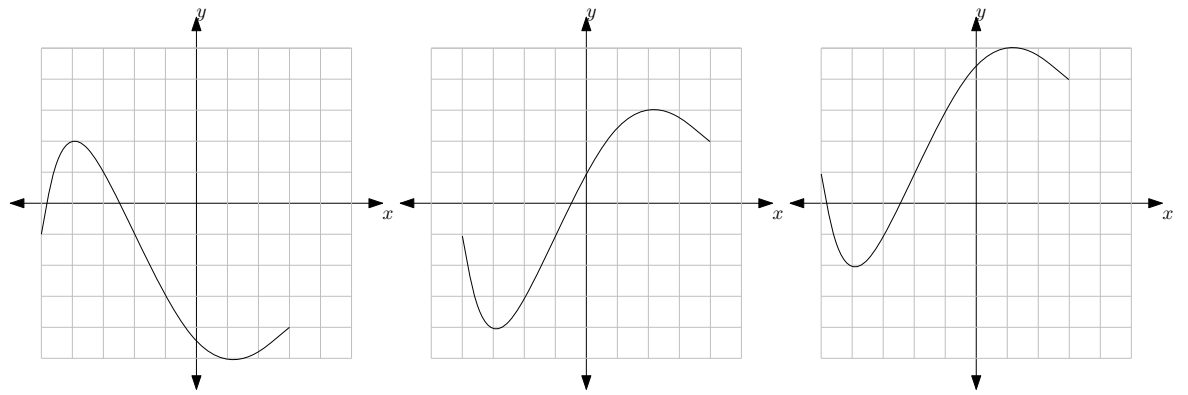
- $y = f(x + c)$

- $y = f(-x)$

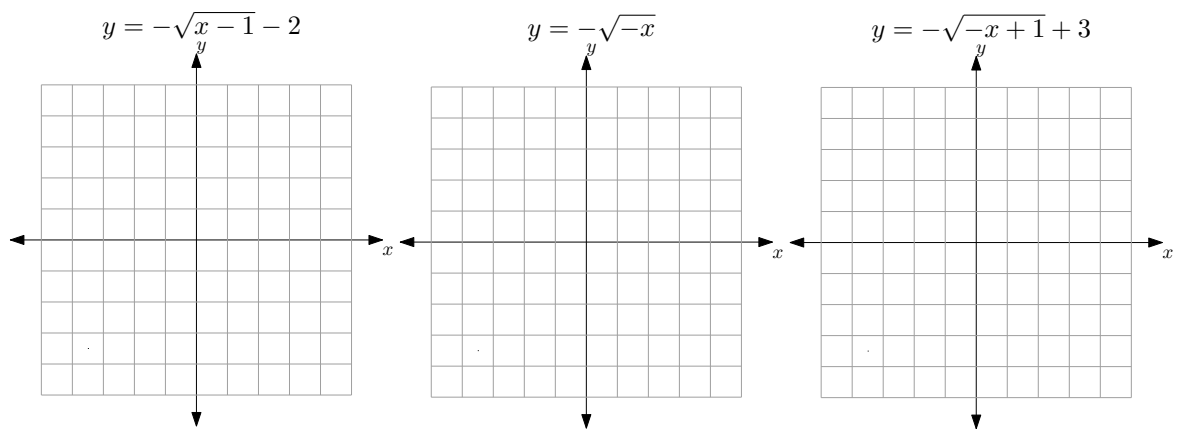
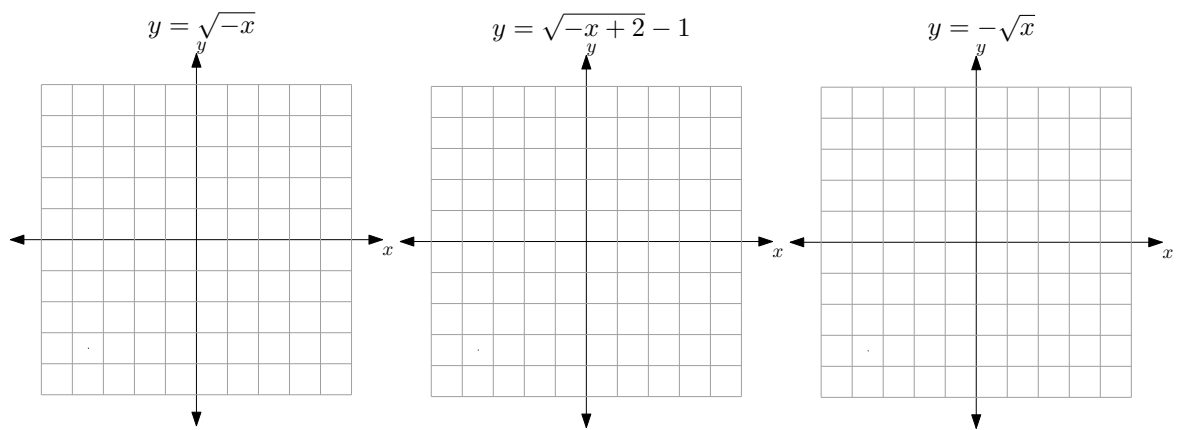
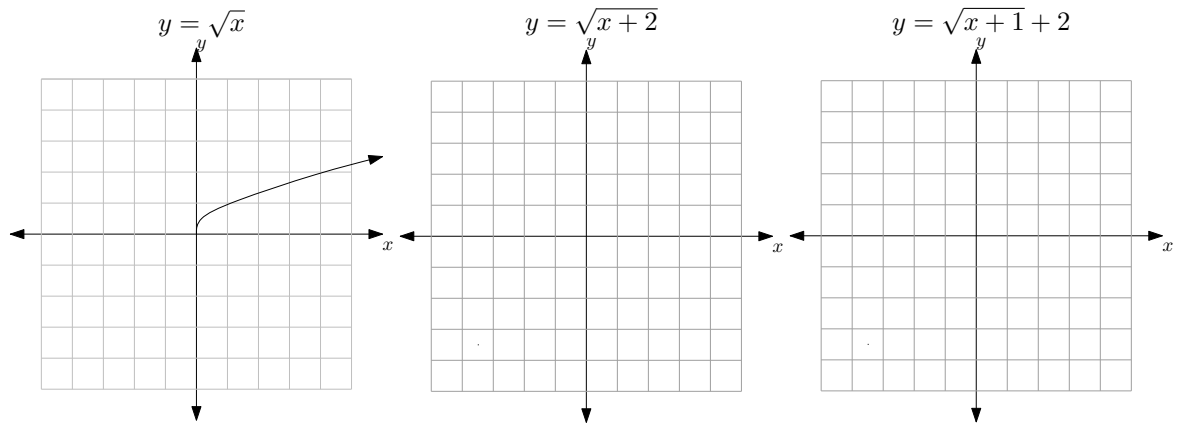
- $y = -f(x)$

Example 0.86. Identify each of the graphs:





Example 0.87. Graph the following translates of $y = \sqrt{x}$:



3.5 - Methods of Combining Functions

$$(f + g)(x) =$$

$$(f - g)(x) =$$

$$(fg)(x) =$$

$$(f/g)(x) = \text{provided}$$

Example 1:

If $f(x) = 1 - x^2$ and $g(x) = 2x + 1$, find

$$(f + g)(x) =$$

$$(f - g)(x) =$$

$$(fg)(x) =$$

$$(f/g)(x) = \text{provided}$$

Composition of Functions

Let $f(x) = 1 - x^2$ and $g(x) = 2x + 1$.

$$f[g(2)] =$$

$$g[f(1)] =$$

$$f[g(x)] =$$

$$g[f(x)] =$$

Given two functions f and g , the function $f \circ g$ is defined by

$$(f \circ g)(x) =$$

The domain of $f \circ g$ consists of the values x in the domain of g for which $g(x)$ is in the domain of f .

Example 2:

Let $f(x) = x - 3$ and $g(x) = x^2 + 1$

$$(f \circ g)(x) =$$

$$(g \circ f)(x) =$$

$$(f \circ f)(x) =$$

$$(g \circ g)(x) =$$

Example 3:

Let $f(x) = \sqrt{x - 3}$ and $g(x) = x^2 - 4$.

Domain of f :

Domain of g :

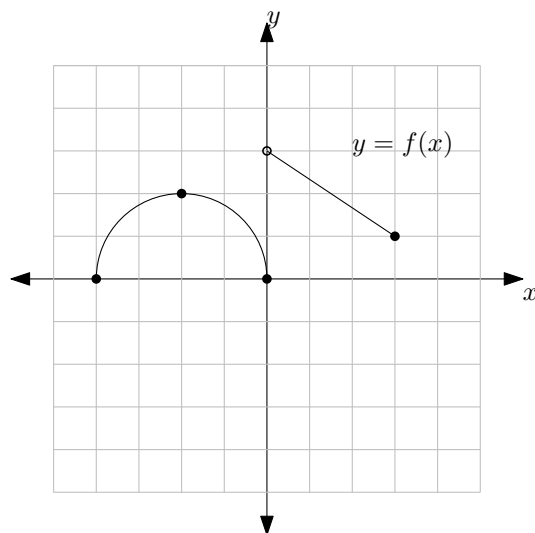
$$(f \circ g)(x) =$$

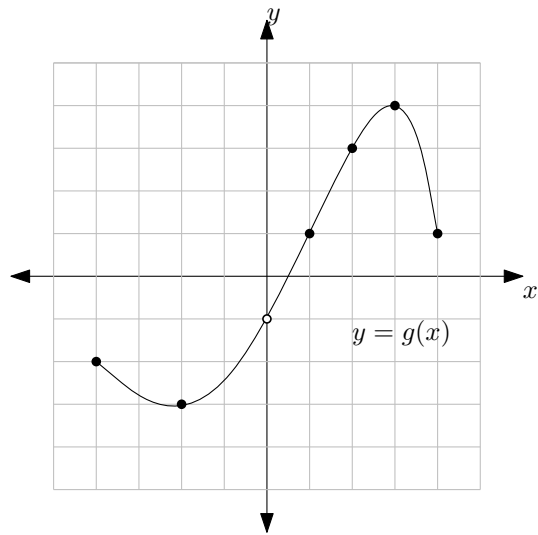
$$(g \circ f)(x) =$$

Domain of $f \circ g$:

Domain of $g \circ f$:

Interpreting Graphs for Compositions.





Find:

$$(f \circ g)(2) =$$

$$(f \circ f)(3) =$$

$$(f \circ g)(-4) =$$

$$(f \circ f)(0) =$$

$$(f + g)(-2) =$$

$$(g^2)(-2) =$$

$$(g \circ f)(3) =$$

$$(g \circ g)(-4) =$$

$$(g \circ f)(-2) =$$

$$(g \circ g)(2) =$$

$$(g \circ f)(-4) =$$

$$(g \circ g)(3) =$$

Applications of Composition of Functions.

Example 1:

Krispy Kreme is making doughnuts at a rate of 60 doughnuts per minute. They sell the doughnuts for \$0.40 each.

$$N(t) =$$

$$R(n) =$$

Find $(R \circ N)(t)$.

Example 2:

A spherical balloon is inflated in such a way that the radius (in cm) after t seconds can be found as

$$r(t) = \frac{1}{2}t + 1$$

The surface area of a sphere is

$$S(r) = 4\pi r^2.$$

Find $(S \circ r)(t)$.

What is the surface area of the balloon after 4 minutes?

Expressing a Function in Terms of Simpler Functions

Let $f(x) = x^2 - 1$, $g(x) = \sqrt{x+1}$, and $h(x) = x + 1$. Then

$$\sqrt{x+2} =$$

$$x^2 =$$

$$x^2 + 2x =$$

$$|x| =$$

3.6 - Inverse Functions!

Two functions f and g are **inverses** of one another provided that

for each x in the domain of g

and

for each x in the domain of f

Say g is the **inverse** of f : $g(x) =$

Say f is the **inverse** of g : $f(x) =$

Note:

This does NOT mean $\frac{1}{f(x)}$ and $\frac{1}{g(x)}$!

Example

Are $f(x) = 4x - 1$ and $g(x) = \frac{1}{4}x + 1$ inverses?

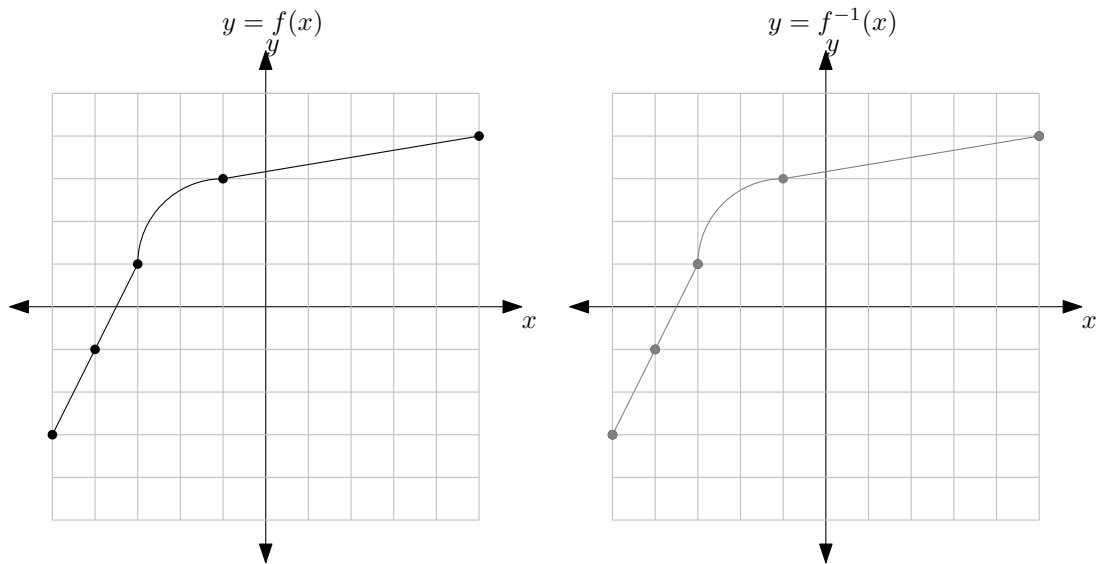
Example

Are $f(x) = 2x + 6$ and $g(x) = \frac{1}{2}x - 3$ inverses?

Example

Suppose that f and g are inverses.

- If $f(2) = 5$, then $g(5) =$
- If $g(7) = -1$, then $f(-1) =$
- If $f(a) = b$, then $g(b) =$
- The domain of g is
- The range of g is



The graph of $y = f^{-1}(x)$ is the graph of $y = f(x)$ reflected over

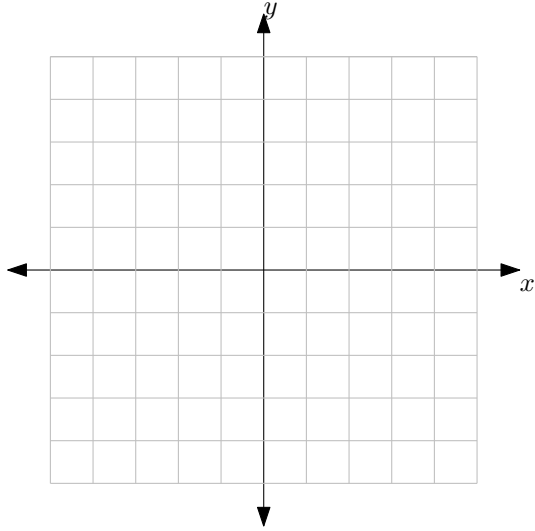
One-to-One Functions

A function is **one-to-one** if

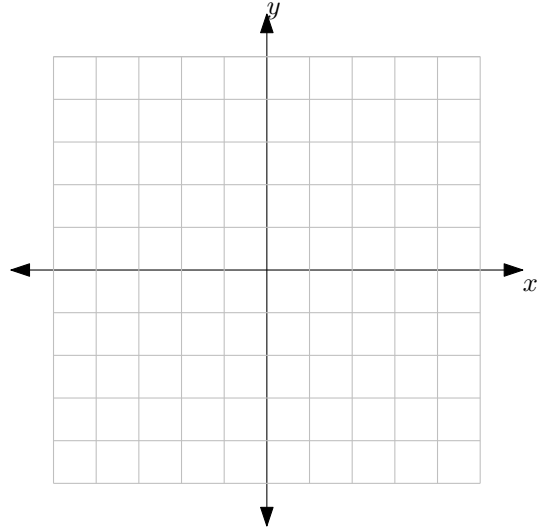
Examples:

Are these functions one-to-one?

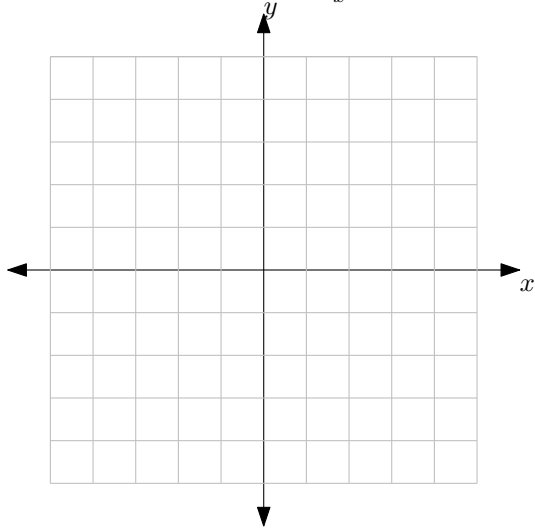
$$y = f(x) = 2x$$



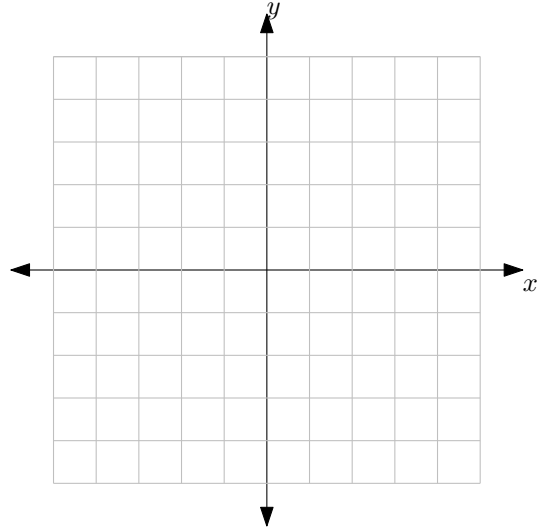
$$y = f(x) = x^2$$



$$y = f(x) = \frac{1}{x}$$



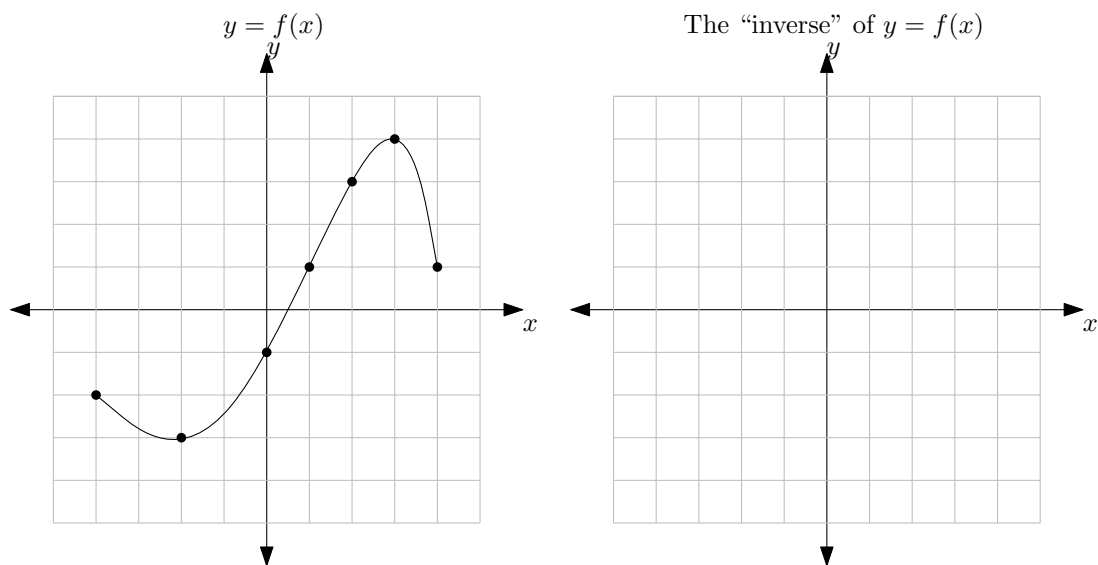
$$y = f(x) = \lceil x \rceil$$



The Horizontal Line Test

: A function f is one-to-one if

For a function f to have an inverse, look at the graph $y = f(x)$ and its “inverse”



The function f has an inverse function when

The graph of the “inverse” must

Finding $f^{-1}(x)$

- $f(x) = 4x - 6$.

- $f(x) = x^2 - 4$

Finding $f^{-1}(x)$ cont.

- $f(x) = x^2 - 4$, domain $x \geq 0$.

- $f(x) = \frac{3x - 2}{x + 1}$.

4.1 - Linear Functions

A **linear function** is a function defined by an equation of the form

where a and b are constants.

Example 0.88. • $f(0) = 2, f(2) = 5$

• $f(2) = 10$, the graph $y = f(x)$ is parallel to $x - 2y = 4$.

• The graph $y = f(x)$ passes through $(1, 5)$ and $(-1, -3)$.

Example 0.89. You can buy a new car (a nice one) for \$30,000, and after 5 years, you can sell it for \$18,000. Assume the depreciation can be modeled by a linear function in terms of time.

- Find the linear function which models the depreciation, with $0 \leq t \leq 5$.

- Find the value of the car after 3 years.

Example 0.91. In 1917, Camp Randall was rebuilt with concrete stands to hold approximately 10,000 people. By 1951, the capacity of Camp Randall was 51,000. Assume that the stadium's growth can be modeled by a linear function of time. Let $C(t)$ be the capacity of the stadium t years after its inauguration in 1917.

Find $C(t)$.

Project how big the stadium would be in 2000.

The current capacity of Camp Randall has 80,321 seats. How good is the prediction?

Cost Functions and Marginal Cost

A function that gives the cost $C(x)$ for producing x units of a commodity is called the

The additional cost to produce one unit is called the

Example 0.92. You open a business making cheesy macaroni art. The initial cost for starting up is \$50 (the foldup table to set up on the corner of the road) and the marginal cost is \$0.10 per artwork for materials.

- Find the linear function $C(m)$, the cost for starting up shop and making m pieces of art.

- What is the cost of making 2000 macaroni pieces?

4.2 - Quadratic Functions

A **quadratic function** is a function of the form

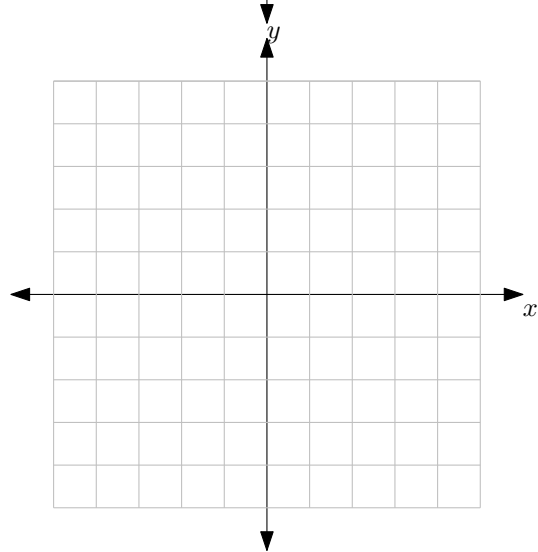
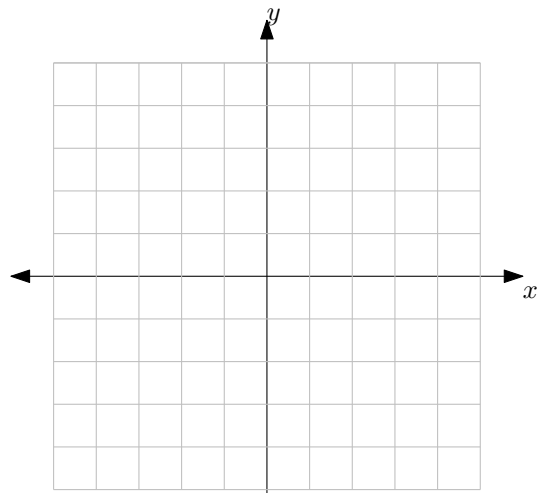
where a , b , and c are constants, and $a \neq 0$.

The graph of a quadratic function is called a parabola.

- When $a > 0$, the parabola opens upwards.
- When $a < 0$, the parabola opens downwards.

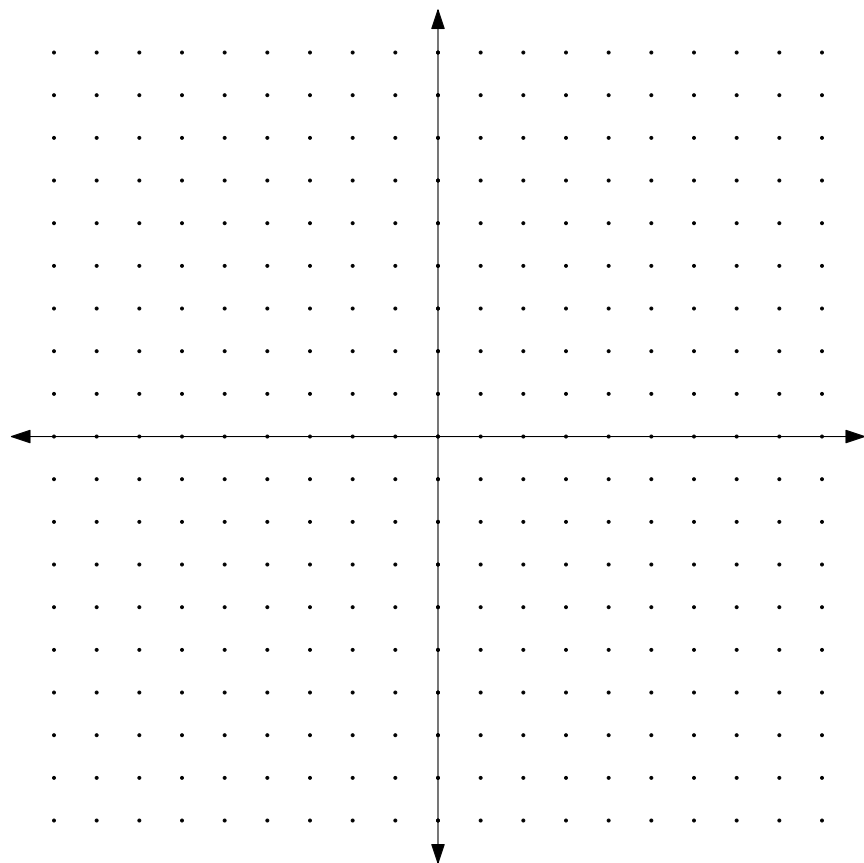
The turning point on the parabola is called the vertex.

The vertical line passing through the vertex is called the axis of symmetry.



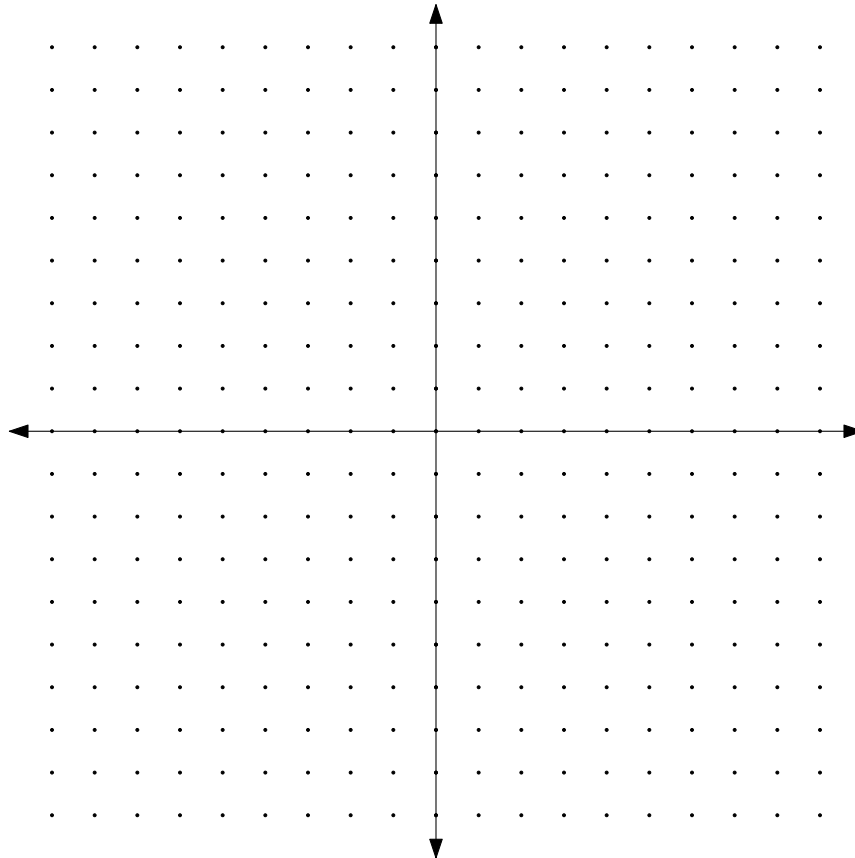
Example 0.94. Graph the function using translation of $y = x^2$. Find the vertex, axis of symmetry, and intercepts.

$$y = x^2 - 6x + 8$$



The Graph of $y = ax^2$

Let $f(x) = \frac{1}{2}x^2$, $g(x) = x^2$, $h(x) = 2x^2$, and $j(x) = -2x^2$.

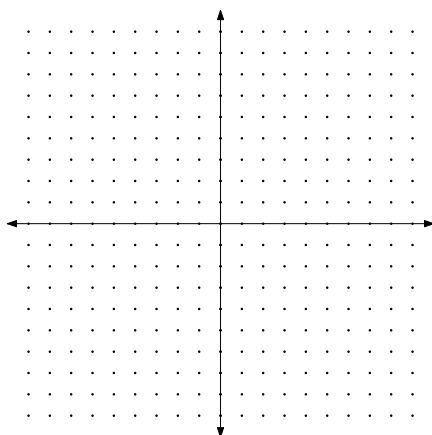


To summarize, $y = ax^2$ is a parabola, similar to $y = x^2$, and

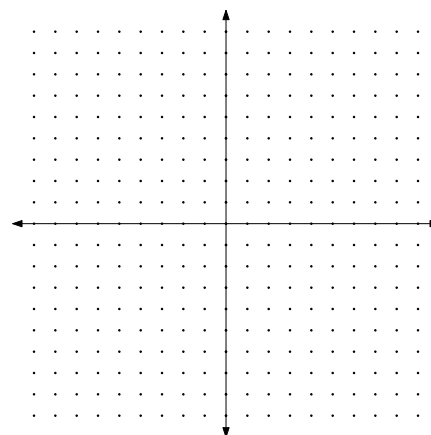
- If $a < 0$, then the graph opens
- If $a > 0$, then the graph opens
- If $|a| > 1$, then the graph opens
- If $|a| < 1$, then the graph opens

Graph the following:

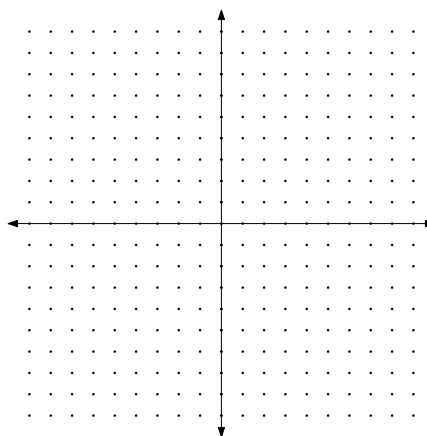
$$y = (x - 4)^2 - 2$$



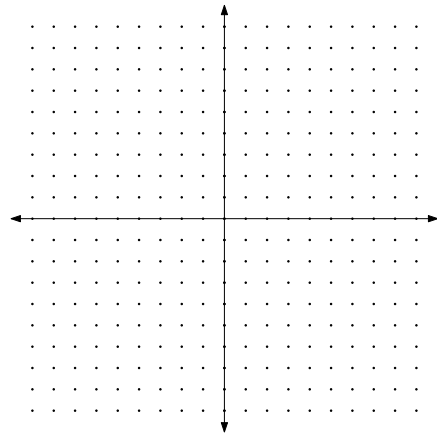
$$y = 3(x + 1)^2 + 4$$



$$y = -2x^2 - 4x + 9$$



$$y = \frac{1}{2}x^2 - 2x + 3$$



Extreme Values

A quadratic function will have a _____ when

A quadratic function will have a _____ when

This will always happen at the

Find the maximum / minimum output for the following functions:

- $f(x) = x^2 - 4x + 3$

- $f(x) = -2x^2 + 6x - 9$

- $f(x) = 4x^2 + 8x + 3$

The Vertex Form of a Quadratic Function

The equation of the parabola $y = ax^2 + bx + c$ can always be rewritten as

where the h is $-\frac{b}{2a}$ and the k is

is

Example 0.95. Find the quadratic function which passes through the point $(-2, 3)$ and has a vertex of $(1, 5)$.

Example 0.96. For what value of c will the minimum value of $f(x) = x^2 - 4x + c$ be -7 ?

Example 0.97. For what value of c will the maximum value of $f(x) = x^2 + 6x + c$ be 12 ?

4.5 - Maximum and Minimum Problems

Example 0.98. If two numbers add to 12, what is the largest their product can be?

The Procedure

1. Express the quantity to be optimized in terms of one variable (4.4)
2. If quadratic, is its parabola opening up or down? Do we have a max or min? Find the vertex.
3. Answer the right question. Do we want x or y ?

Example 0.99. What is the largest possible area of a right triangle if the lengths of the two legs add up to 80 in.?

Example 0.100. A ball is thrown in the air with a velocity of 29 ft/s. Its height is modeled by the function

$$h(t) = -16t^2 + 29t + 6$$

- When does the ball hit the ground?
- When does it reach its maximum height?
- How high does it go?

Example 0.101. Find the point on the curve $y = \sqrt{x}$ which is nearest to $(1, 0)$.

Example 0.102. A farmer wants to erect a fence for cattle using the nearby river as a border. He has 600 feet of fence. What dimensions would maximize the enclosed area?

Example 0.103. The revenue for making x units of methylchloroisothiazolone is modeled by

$$R(x) = 0.1x^2 +$$

4.5 - Maximum and Minimum Problems

Example 0.104. If two numbers add to 12, what is the largest their product can be?

The Procedure

1. Express the quantity to be optimized in terms of one variable (4.4)
2. If quadratic, is its parabola opening up or down? Do we have a max or min? Find the vertex.
3. Answer the right question. Do we want x or y ?

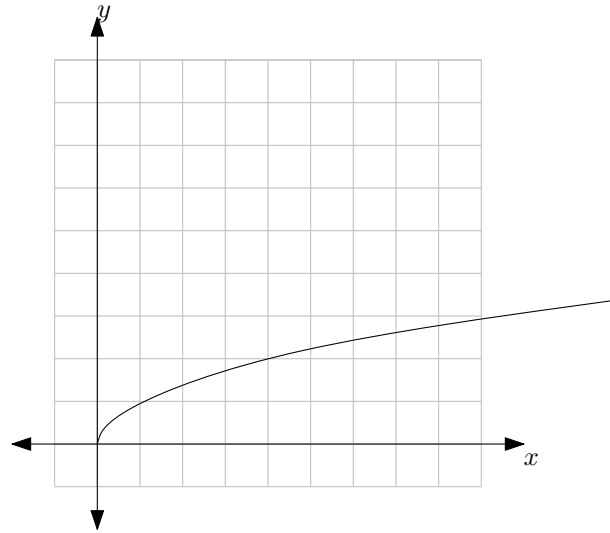
Example 0.105. What is the largest possible area of a right triangle if the lengths of the two legs add up to 80 in.?

Example 0.106. A ball is thrown in the air with a velocity of 29 ft/s. Its height is modeled by the function

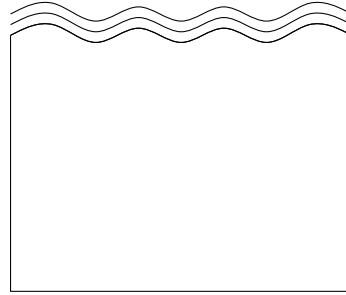
$$h(t) = -16t^2 + 29t + 6$$

- When does the ball hit the ground?
- When does it reach its maximum height?
- How high does it go?

Example 0.107. Find the point on the curve $y = \sqrt{x}$ which is nearest to $(8, 0)$.



Example 0.108. A farmer wants to erect a fence for cattle using the nearby river as a border. He has 600 feet of fence. What dimensions would maximize the enclosed area?

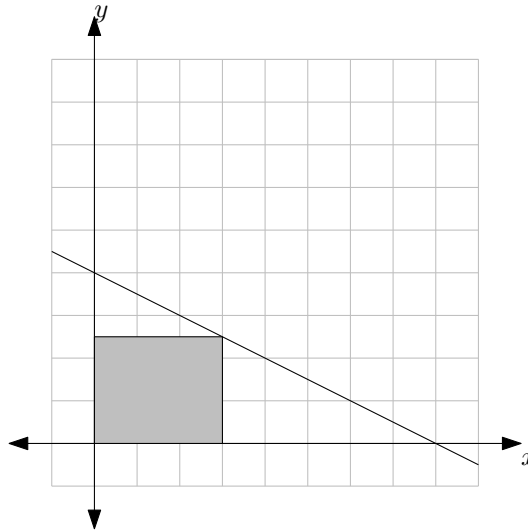


Example 0.109. The revenue for making x units of methylchlorisothiazolone is modeled by

$$R(x) = -0.1x^2 + 20x - 100$$

Find the maximum revenue, assuming you can sell partial units.

Example 0.110. Find the maximum area that a rectangle can have being bordered by the x -axis, the y -axis, and the graph of $y = 4 - \frac{1}{2}x$.



4.6 - Polynomial Functions

A _____ is a function defined by

an equation of the form

where _____ is _____ and _____ are

The value of _____ is called the

Example 0.111. • $f(x) = x^3 - 4x^2 + 1$

• $g(x) = x - x^5$

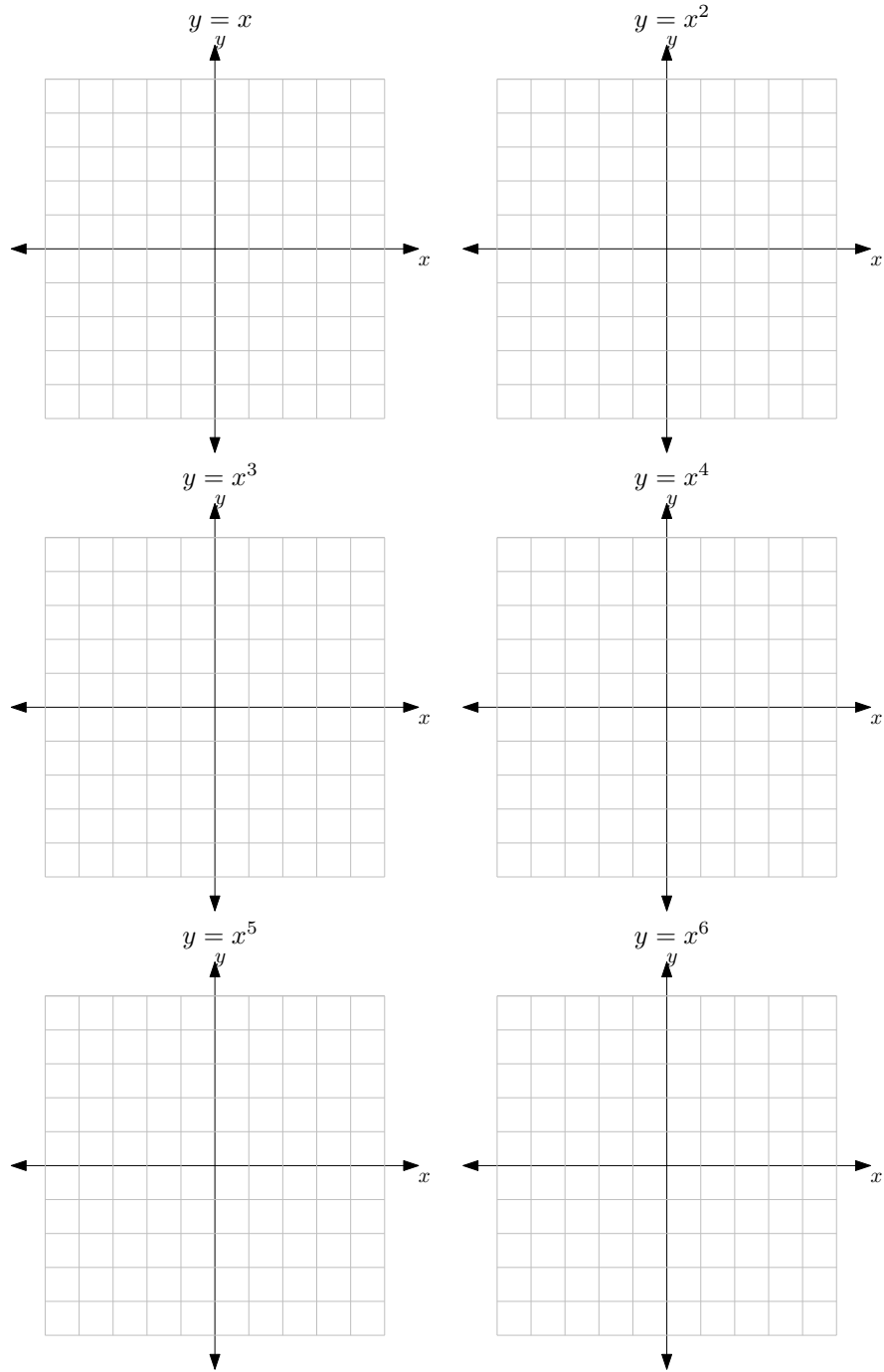
• $h(x) = 4$

• $f(x) = \sqrt{x} + 2$

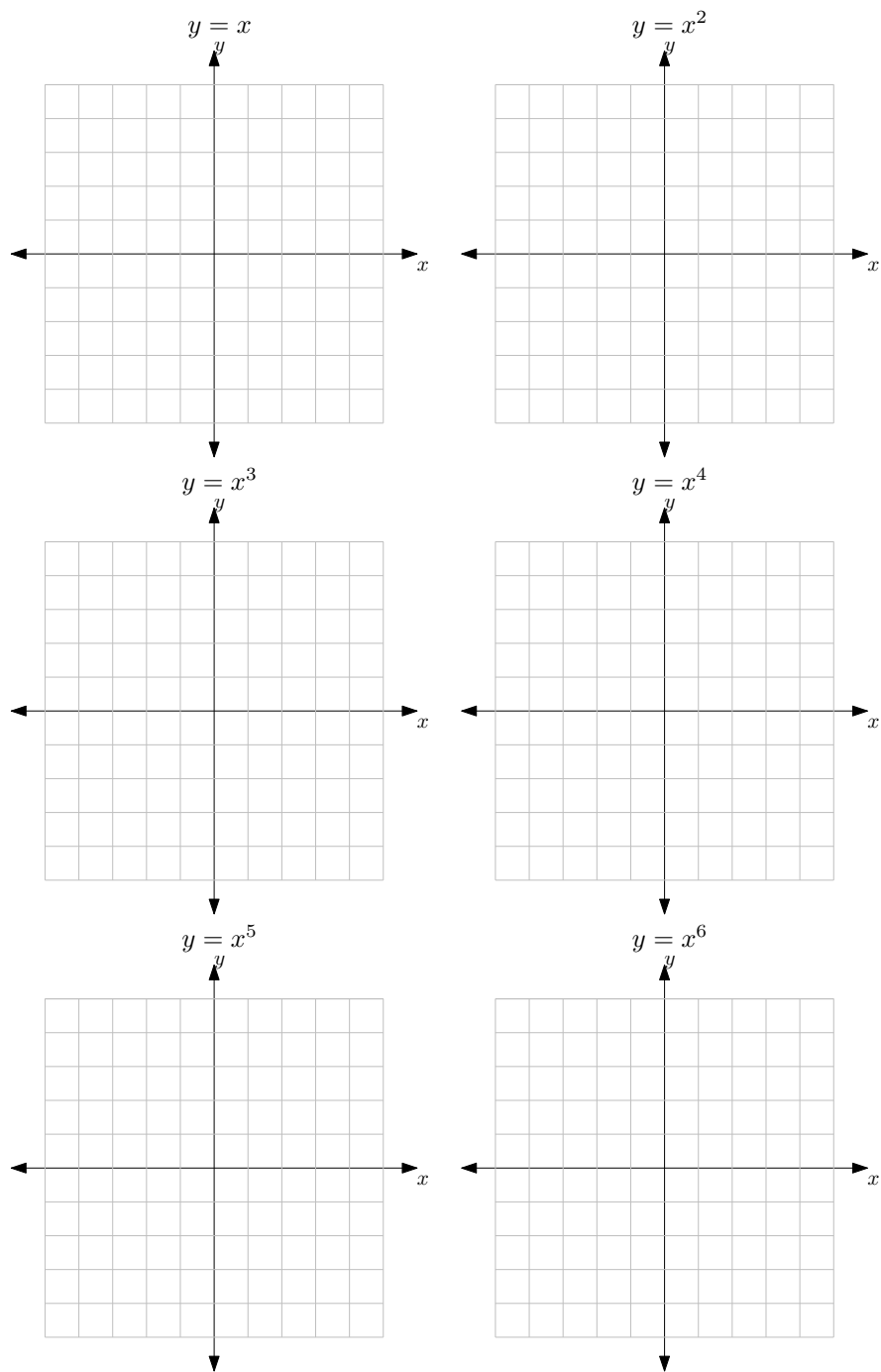
• $g(x) = \frac{1}{x} + x^2 - 2$

A _____ is a function of the form

Large Scale:



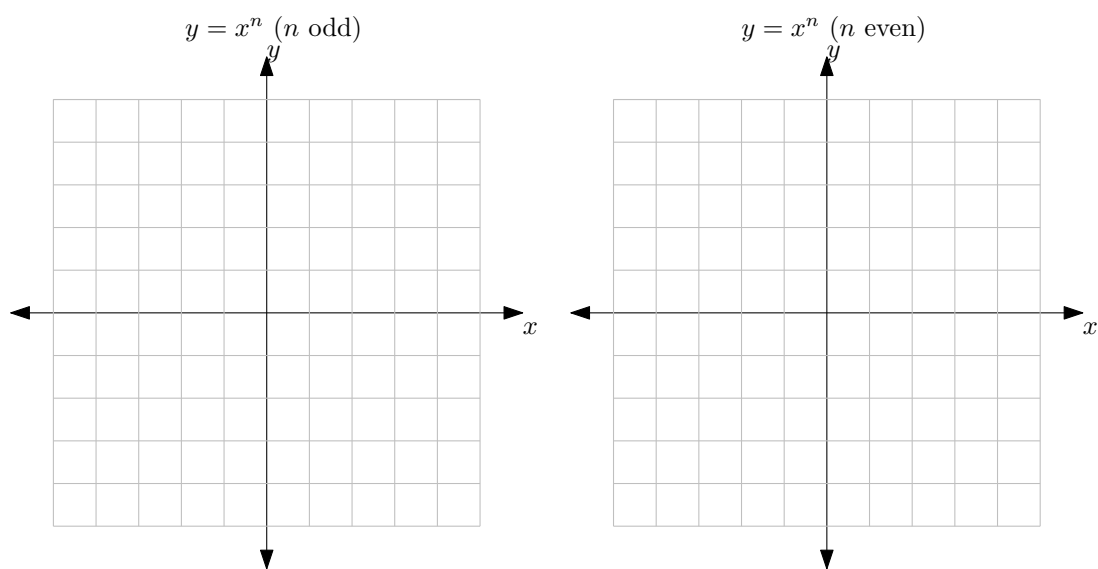
Small Scale:



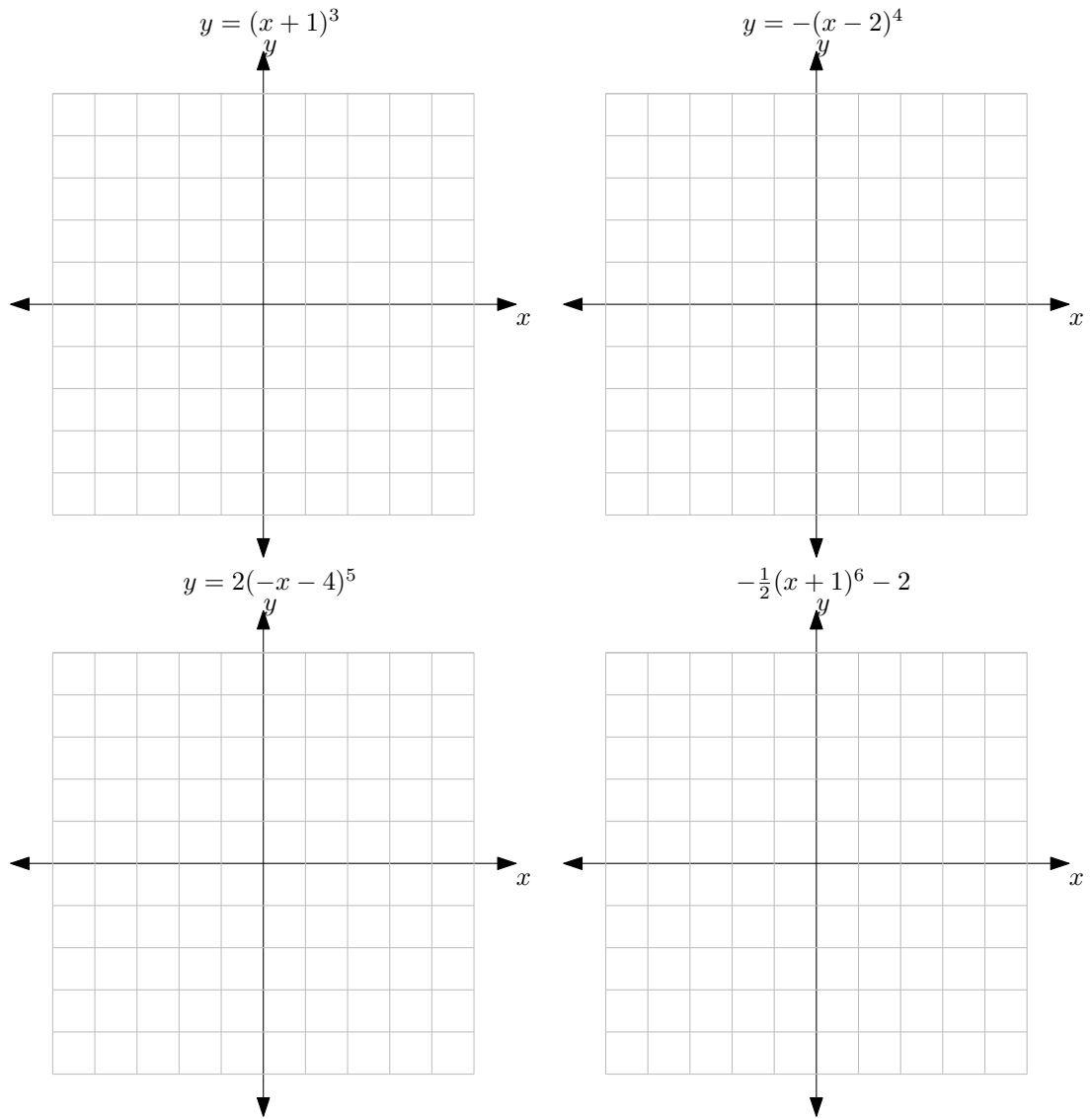
The graph $y = x^n$ goes through the points

when _____, or

when _____



Translating Graphs of Power Functions



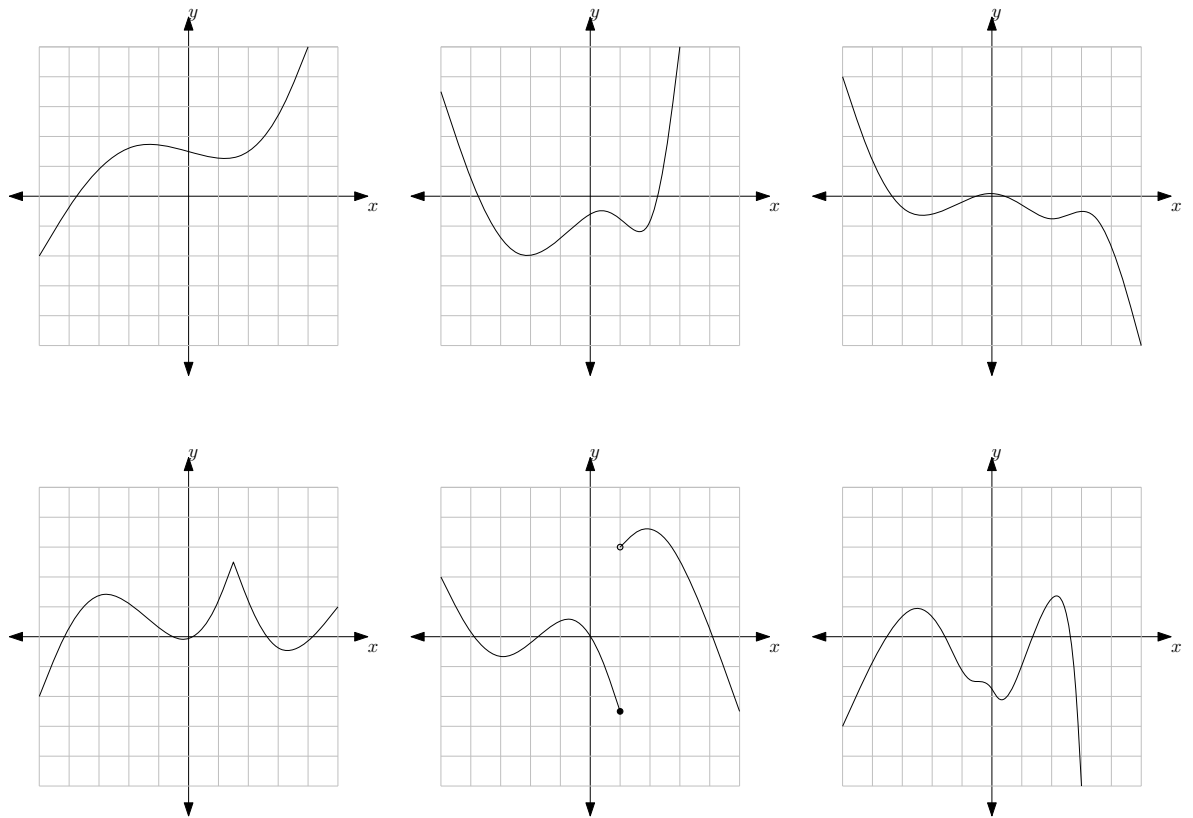
Properties of Graphs of Polynomials

1.

2.

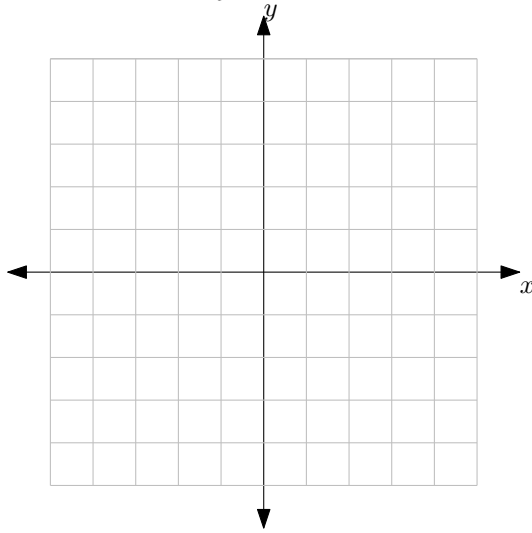
3.

Degree and leading coefficients.

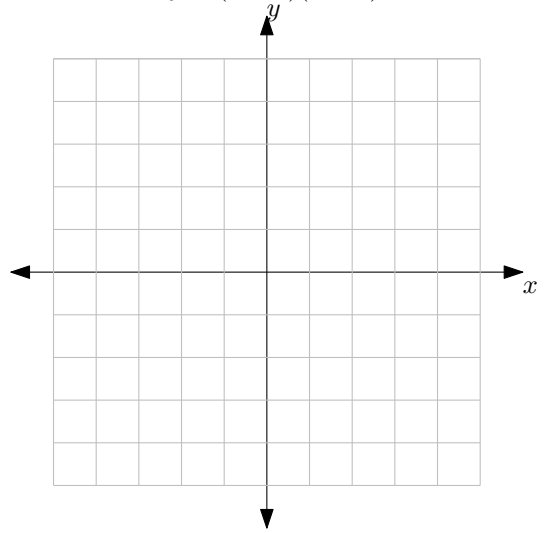


Graphing Polynomials

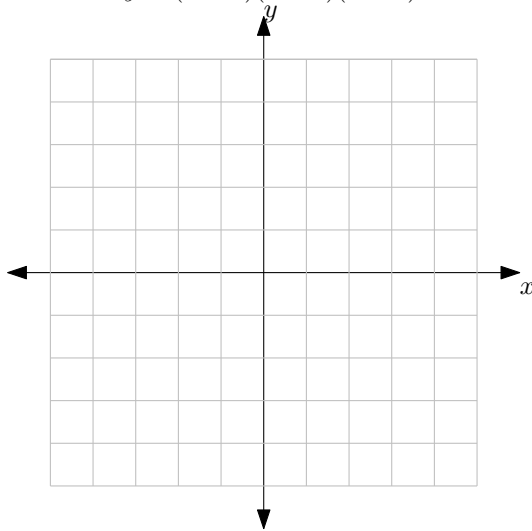
$$y = x - 1$$



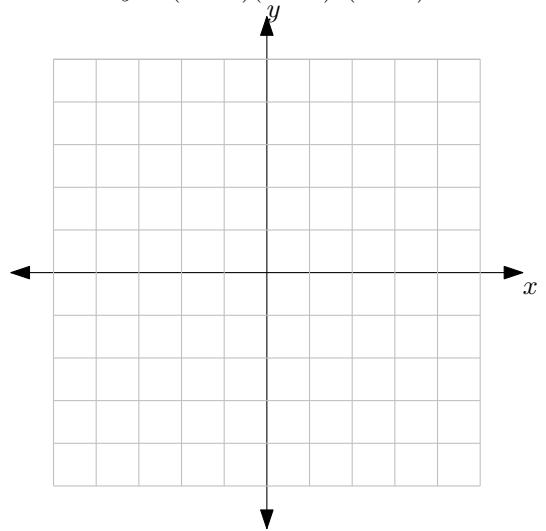
$$y = (x - 1)(x - 3)$$



$$y = (x - 1)(x - 3)(x - 5)$$



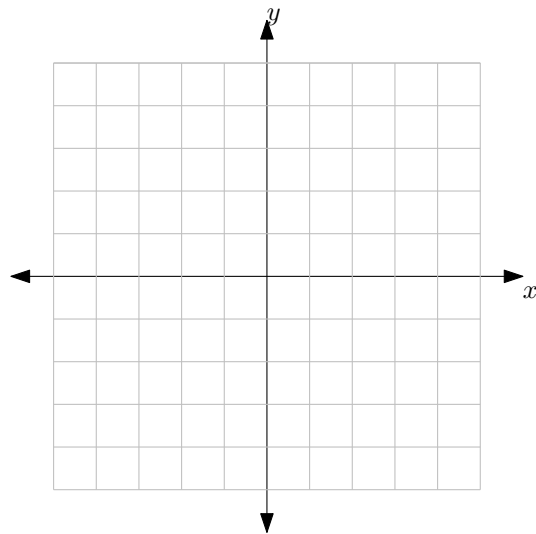
$$y = (x - 1)(x - 3)^2(x - 5)$$

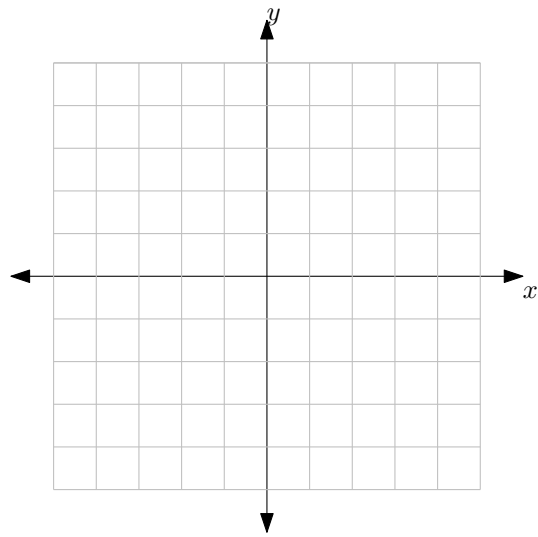


Procedure for graphing polynomials

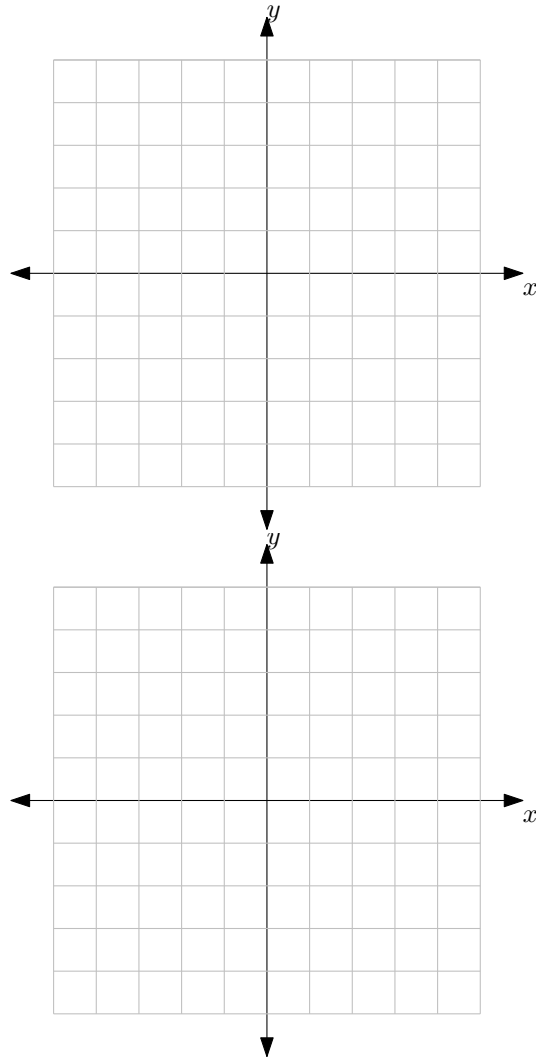
- 1.
- 2.
- 3.
- 4.

Example 0.112. Graph $y = (x - 2)(x - 4)^2$

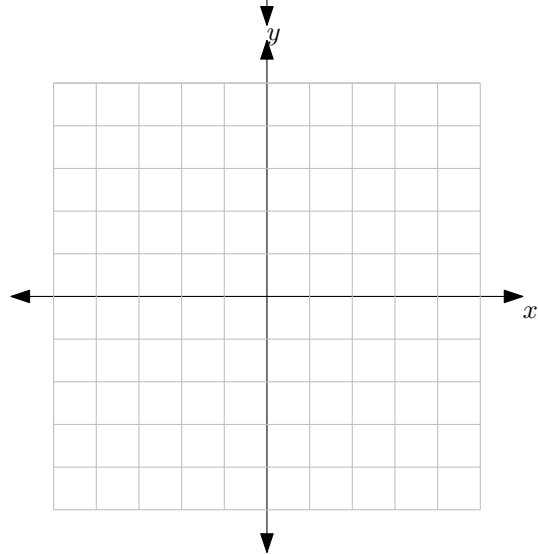
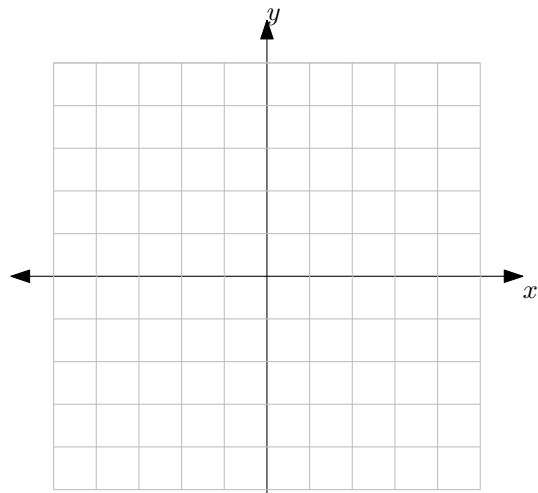




Example 0.113. Graph $y = x^2(x + 2)^2(x - 1)(x - 3)^3$



Example 0.114. Graph $y = -3x(x - 1)(x - 2)(x - 3)$



4.7 - Rational Functions

A **rational function** $r(x)$ is a function of the form

where $p(x)$ and $q(x)$ are

Example 0.115. • $f(x) = \frac{x^2 + 2x - 3}{x^3 - 1}$

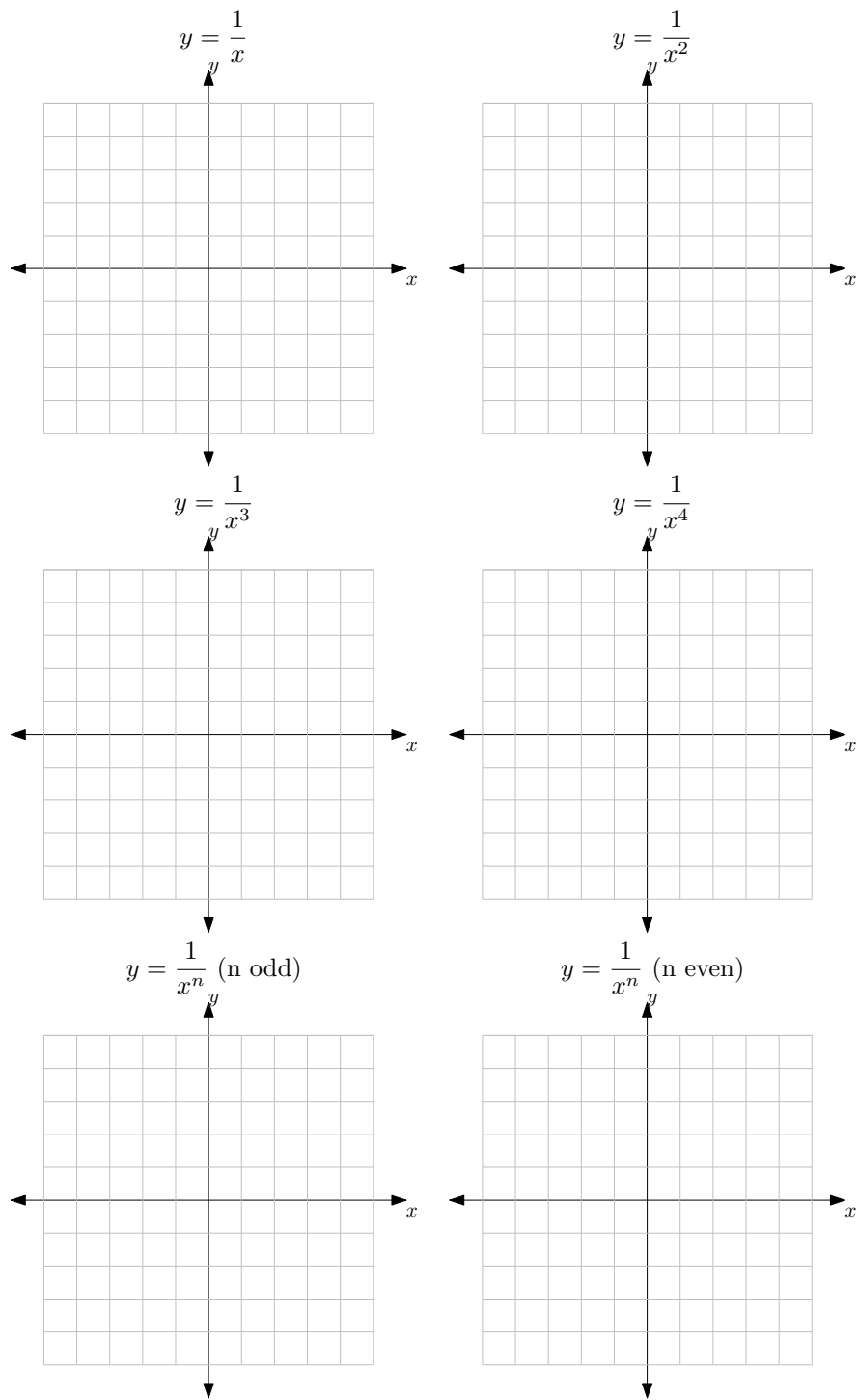
• $g(t) = \frac{t^2 + 1}{t^2 - 1}$

• $h(y) = \frac{1}{y^3 + 1}$

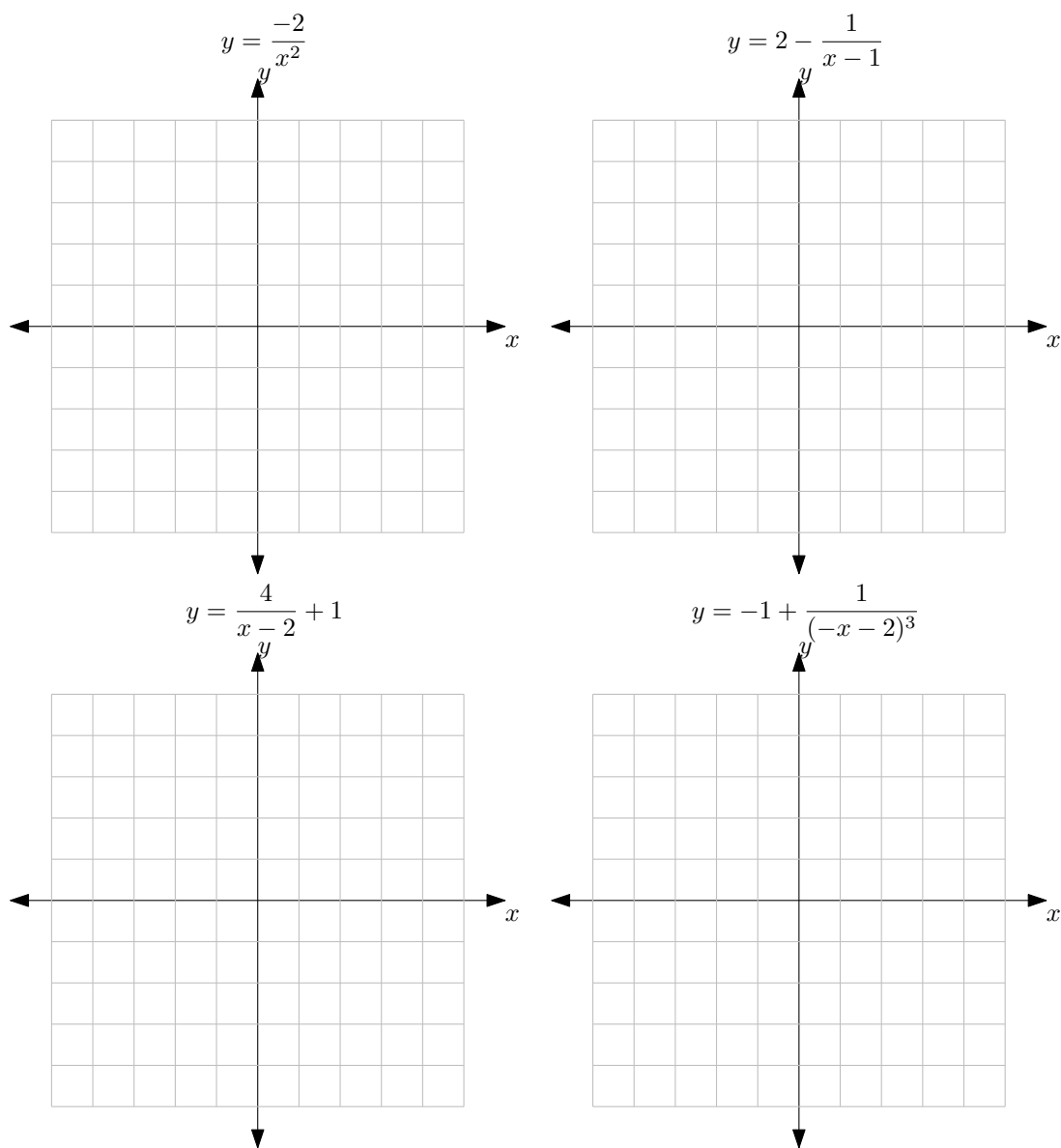
• $k(s) = s^2 - 10s + 7$

An **asymptote** is

Graphing $y = \frac{1}{x^n}$



Graphing Translates and Scales



Finding Asymptotes

A rational function has a vertical asymptote at _____ when _____

A rational function has a horizontal asymptote at _____ when _____

Example 0.116. Find asymptotes for the following:

- $f(x) = \frac{x - 2}{x^2 - 9}$

- $g(t) = \frac{x^2 - 6x}{x^2 - 2x - 3}$

- $f(x) = \frac{x(2 - x)(4x + 3)^3}{4 + x^5}$

- $g(t) = \frac{x^2 + 10000x - 1}{2x^2 - x - 1}$

Graphing Rational Functions

Procedure

1.

2.

3.

Example 0.117. Graph $y = \frac{x - 1}{(x + 1)(x - 3)}$

1. Vertical Asymptote(s)

Horizontal Asymptote

x -Intercept(s)

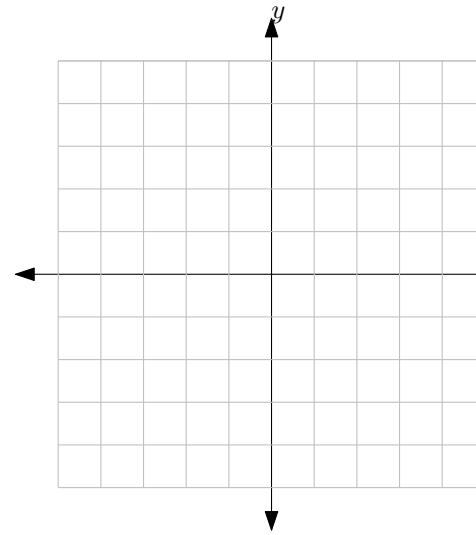
y -Intercept

2. •

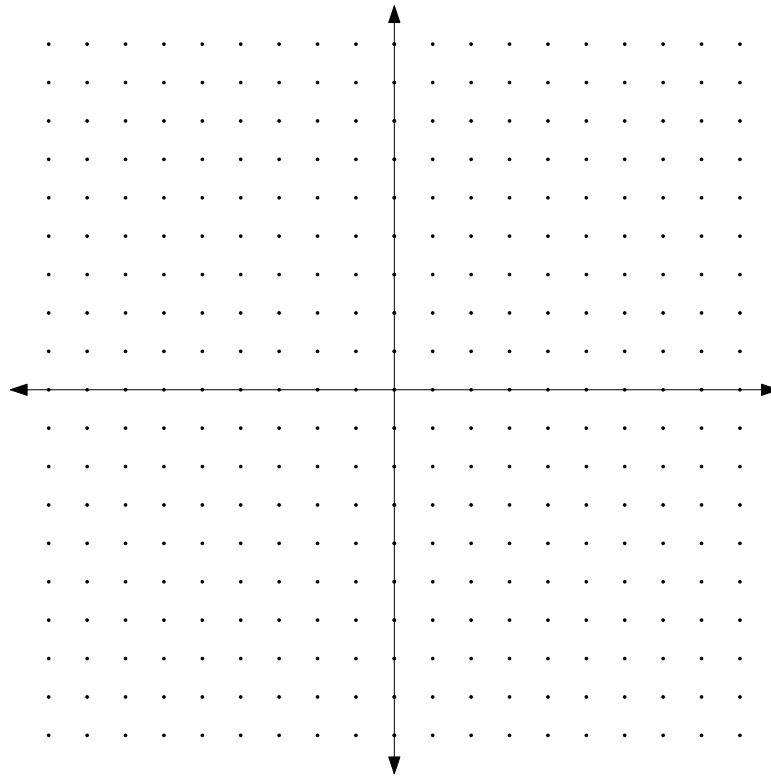
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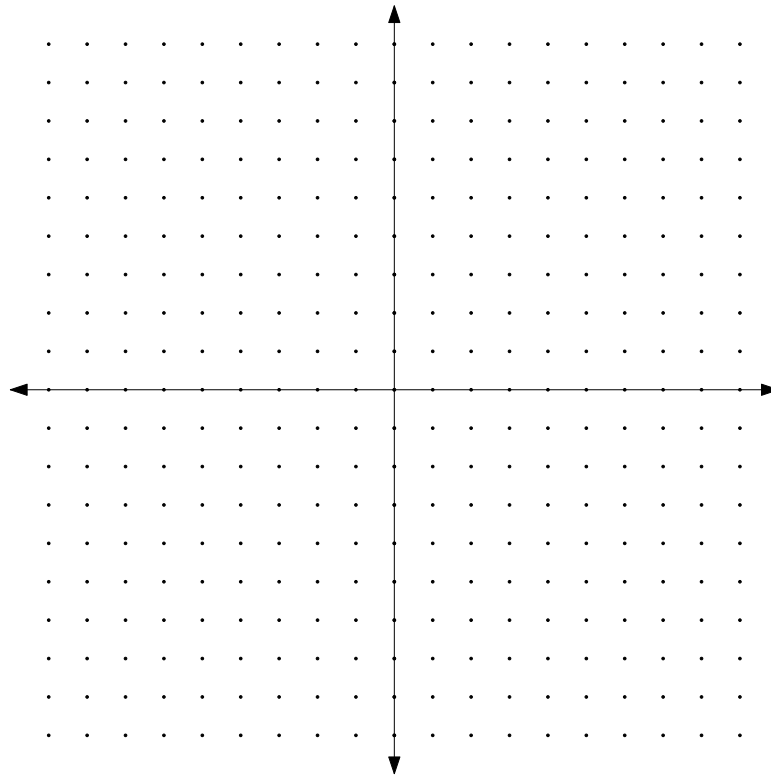
•



Example 0.118. Graph $y = \frac{x^2}{x^2 - 4x - 5}$



Example 0.119. Graph $y = \frac{x(x-2)^2(x+3)}{(x-4)^5(x+6)^3}$



5.1 - Exponential Functions

Properties of Exponents - Assume $a > 0$ and m, n are real numbers.

- $a^m \cdot a^n =$

- $a^0 =$

- $\frac{a^m}{a^n} =$

- $a^1 =$

- $(a^m)^n =$

- $a^{-1} =$

- $(ab)^m =$

- $a^{-n} =$

- $\left(\frac{a}{b}\right)^m =$

- If $a^x = a^y$, then

Example 0.120. Simplify the following expressions.

- $(4^\pi)(4^{2-\pi})$

- $\left(\left(\sqrt{12}\right)^3\right)^{-\frac{1}{2}}$

- $\frac{7^{a+1}}{7^{a-3}}$

Example 0.121. Solve for x :

- $3^x = 81$

- $4^x = \frac{1}{64}$

- $5^{2t} = \left(\frac{1}{25}\right)^{3t-2}$

- $\left(\frac{8}{27}\right)^y = \left(\frac{9}{4}\right)^{y+1}$

Exponential Functions

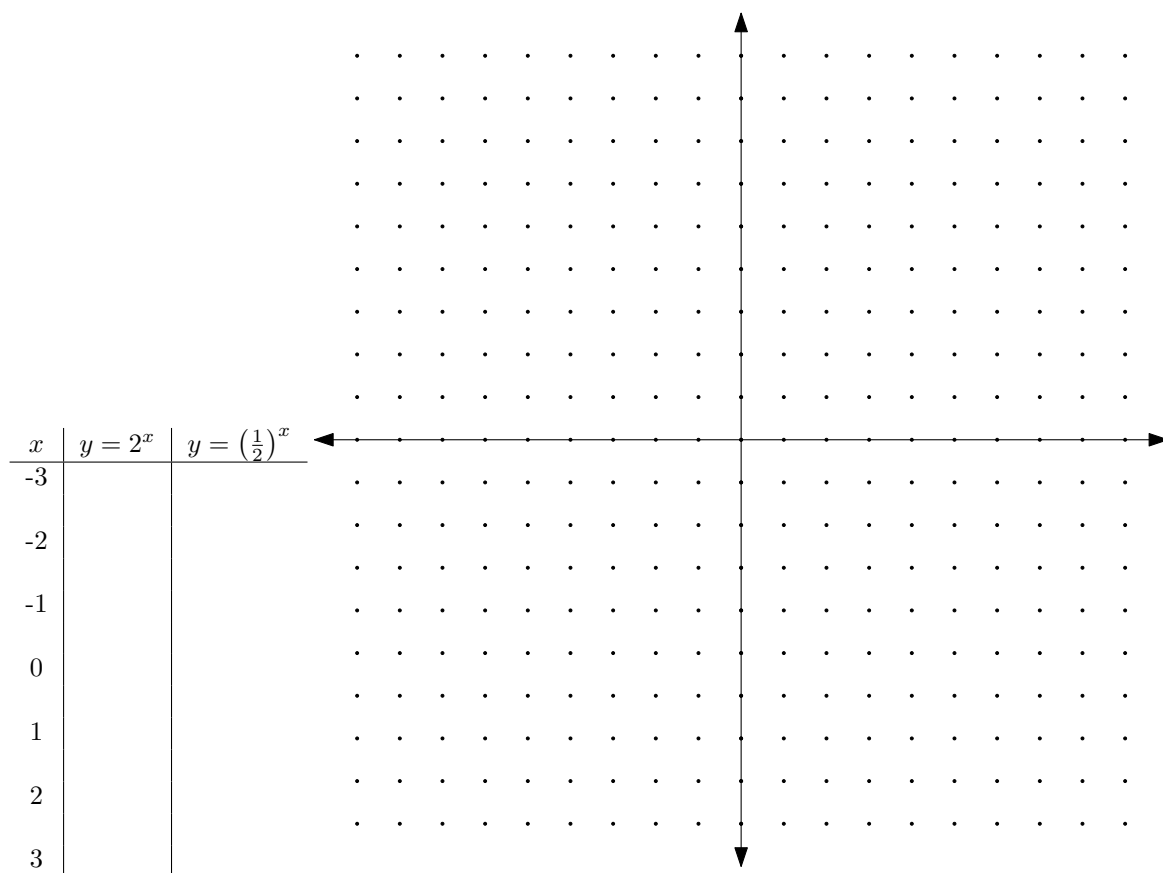
An **exponential function** is a function defined by the equation

where $a > 0$ and $a \neq 1$.

Sometimes, we say an exponential function has the form

where α and β are α and β .

Graphing Exponential Functions



Summary

- Intercept

- Domain

- Asymptote

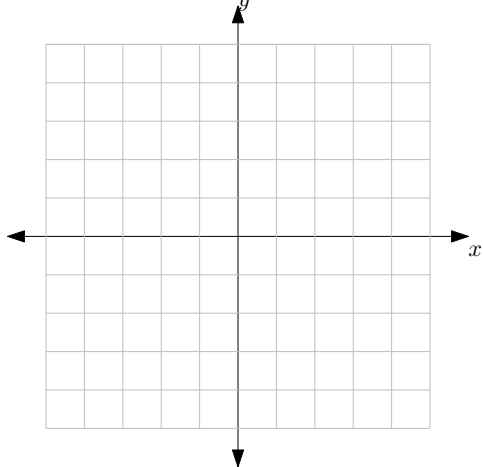
- Range

- Other Points

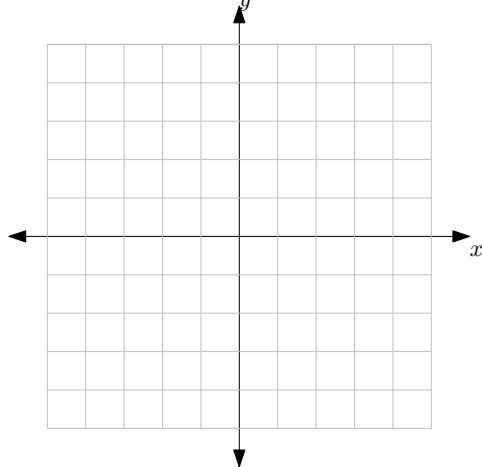
- If $b > 1$, If $b < 1$,

Example 0.122. Graph the following translations.

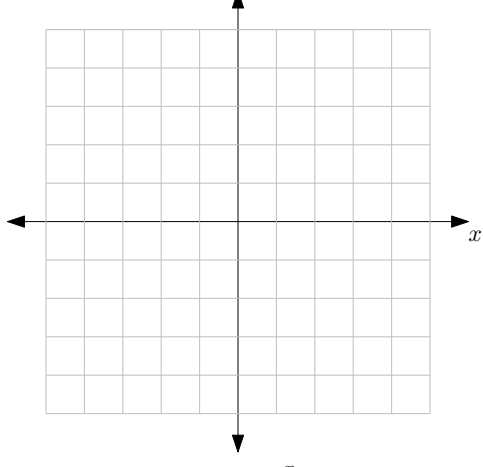
$$y = 3^x$$



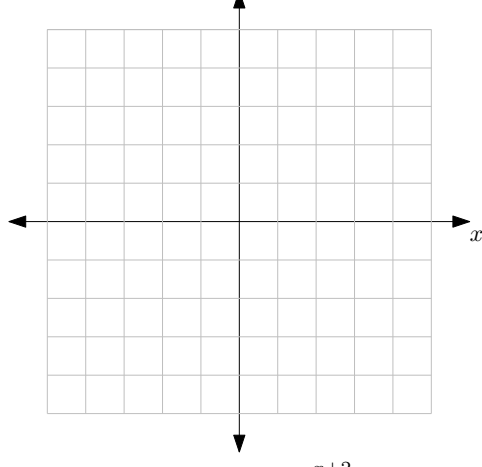
$$y = 1 + 3^{-x}$$



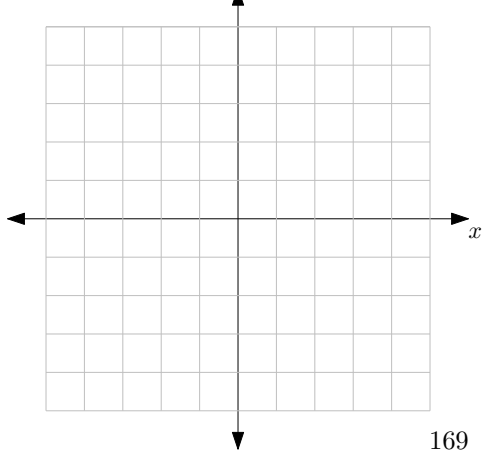
$$y = \left(\frac{1}{2}\right)^x$$



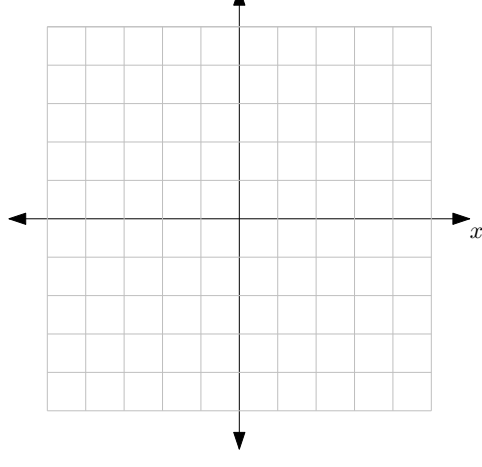
$$y = 1 - \left(\frac{1}{2}\right)^{x-1}$$



$$y = \left(\frac{3}{2}\right)^x$$



$$y = -2 - \left(\frac{3}{2}\right)^{-x+2}$$



Example 0.123. Solve for x :

- $x^2(2^x) - 2^x = 0$

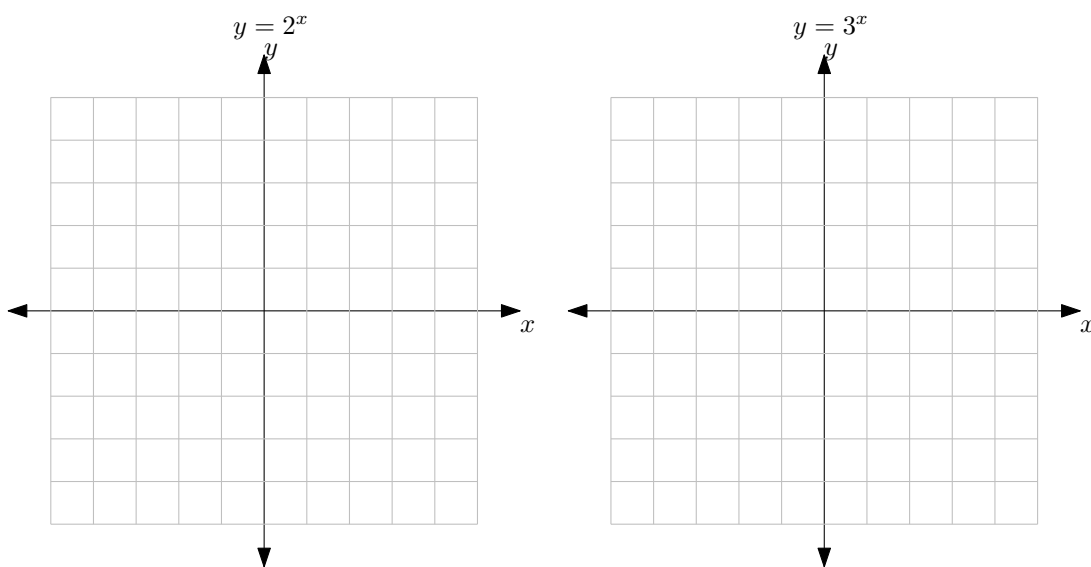
- $x \left(\frac{1}{2}\right)^x = \frac{2x}{(x+1)2^x}$

5.2 - The exponential function $y = e^x$

The value of e is

Why is it interesting? For now, two reasons:

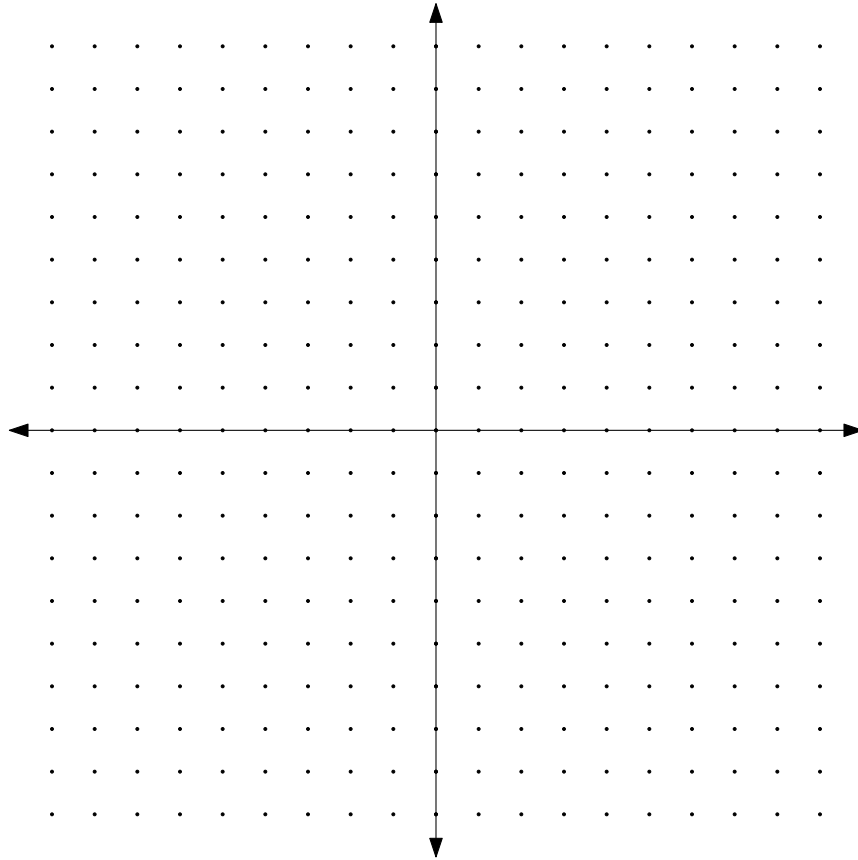
(a) Tangent Lines



(b) Interest Rates An investment of $\$P$ at an interest rate r compounded n times per year for ONE year will be

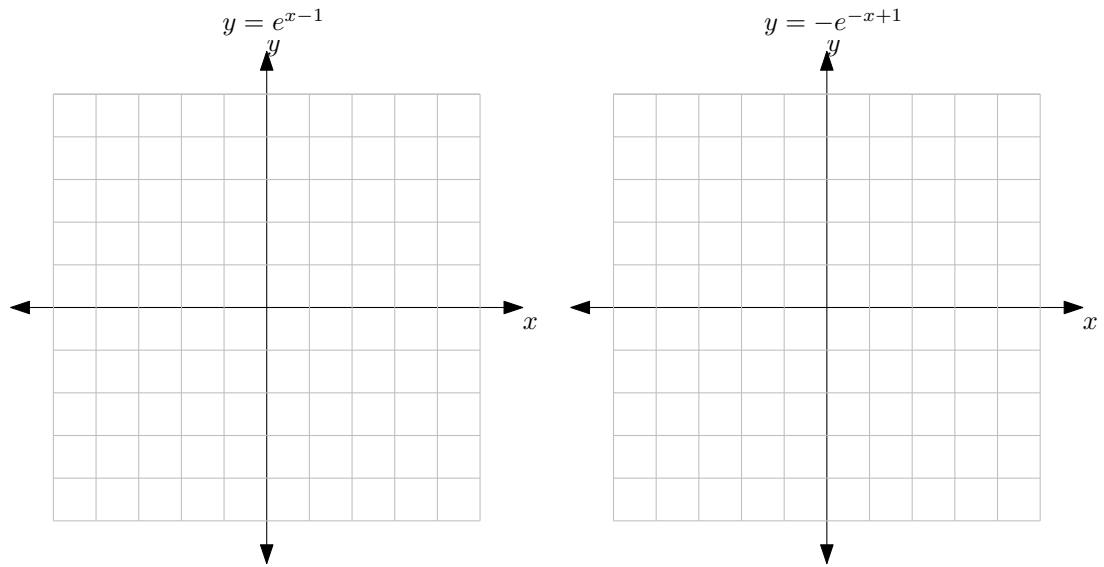
The investment of $\$P$ at an interest rate r compounded continuously is

The Graph of $y = e^x$



- Domain
- Range
- Intercepts
- Asymptotes

Graph these translates.



Which is bigger?

e _____ 2

e^2 _____ 16

\sqrt{e} _____ 1.4

e^4 _____ 81

What is a power function, what is an exponential function?

• $y = e^x$

• $y = 2^x$

• $y = x^2$

• $y = x^e$

• $y = e^2$

• $y = x^x$

5.3 - Logarithmic Functions

A **logarithmic function** is the

of an

Exponential function:

Logarithmic function:

We define $\log_b(x)$ to mean

Example 0.124. Find the following logarithms.

- $\log_2(4)$

- $\log_5(625)$

- $\log_3(27)$

- $\log_{10}(10)$

Example 0.125. Find more logarithms.

- $\log_4(2)$

- $\log_8(4)$

- $\log_{\frac{1}{3}}(\sqrt{27})$

- $\log_2\left(\frac{1}{2}\right)$

- $\log_e\left(\frac{1}{e^2}\right)$

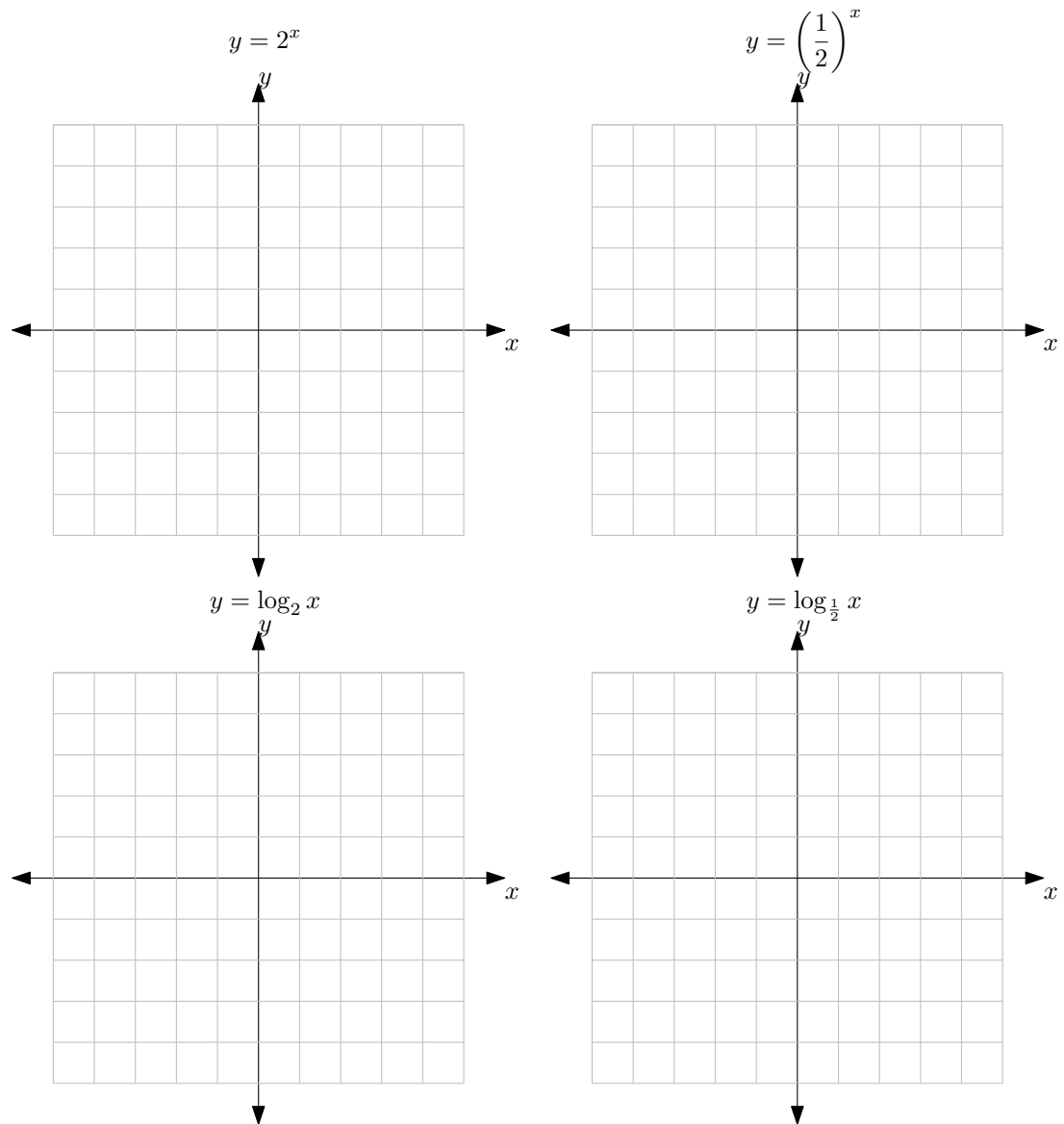
- $\log_{3479}(3479^8)$

Errors to Avoid

- $\frac{\log_2 8}{\log_2 4}$

- $\frac{\log_2 16}{16}$

Graphs of Logarithmic Functions



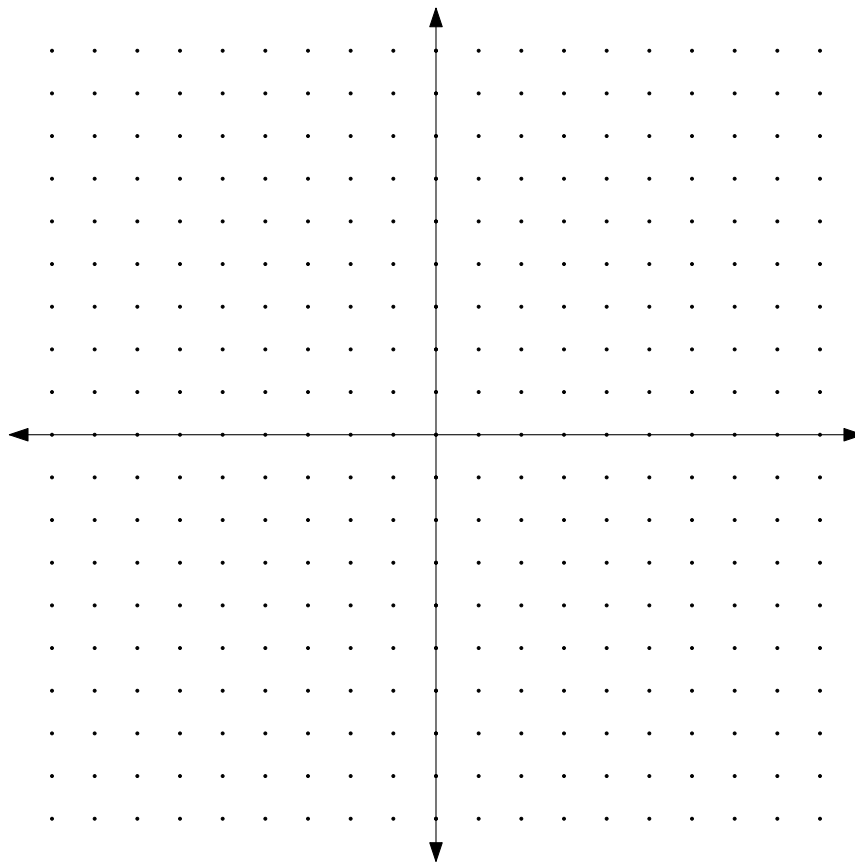
- Domain
- Range
- Intercepts

- Asymptotes

The Natural Logarithm

The natural logarithm is short-hand for .

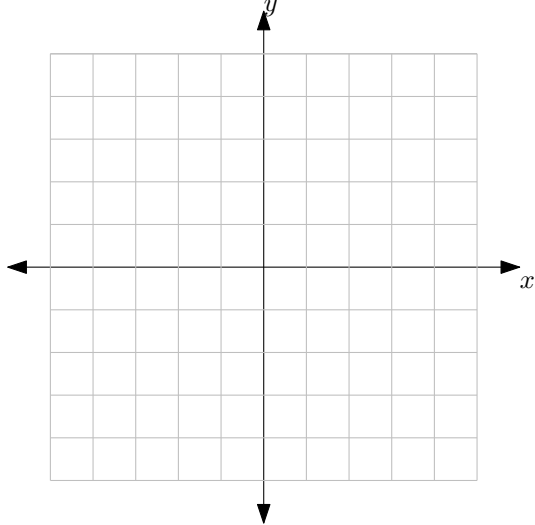
Remember $e =$



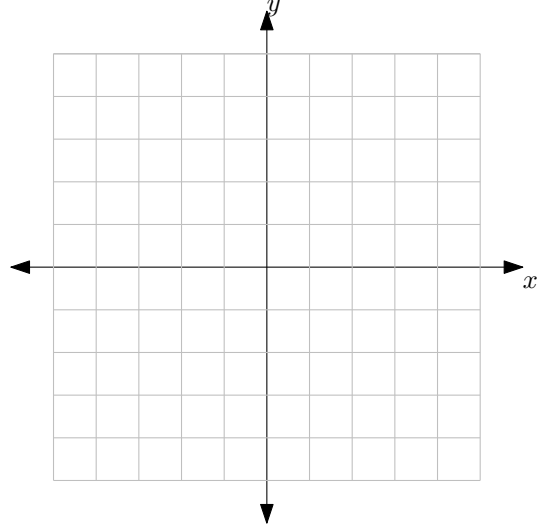
- Domain
- Range
- Intercepts, Points

- Asymptotes

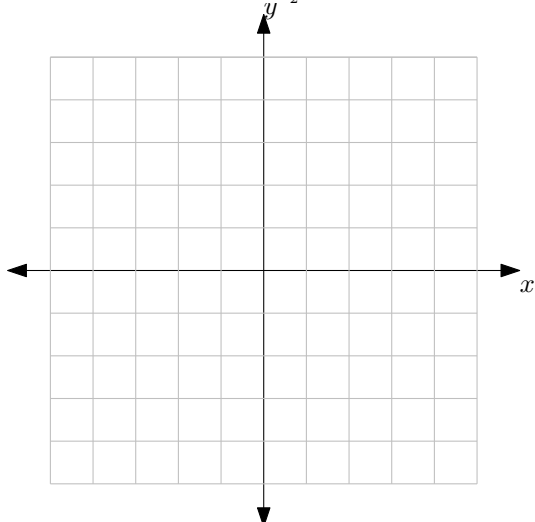
$$y = \log_2(x)$$



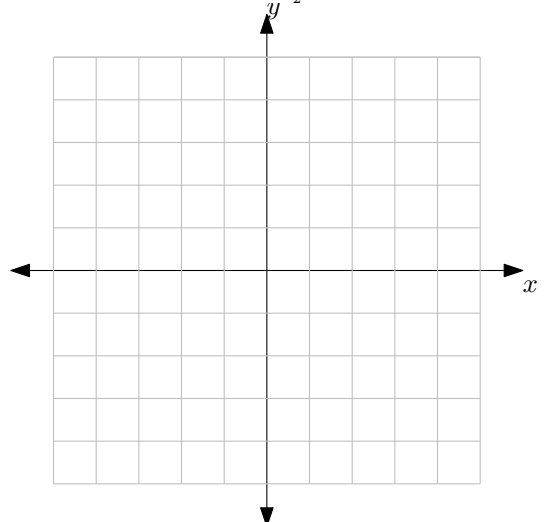
$$y = 2 + \log_2(x - 2)$$



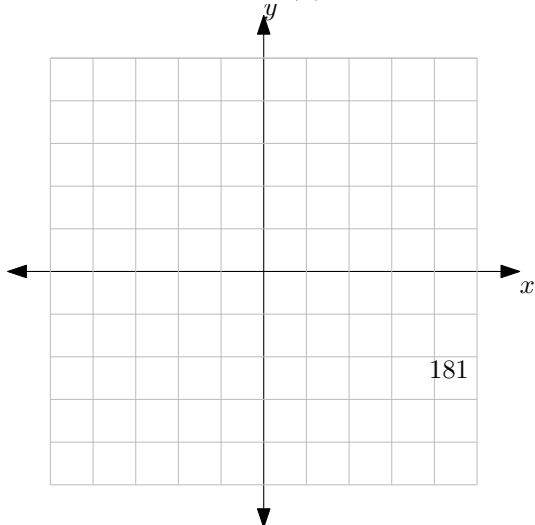
$$y = \log_{\frac{1}{2}}(x)$$



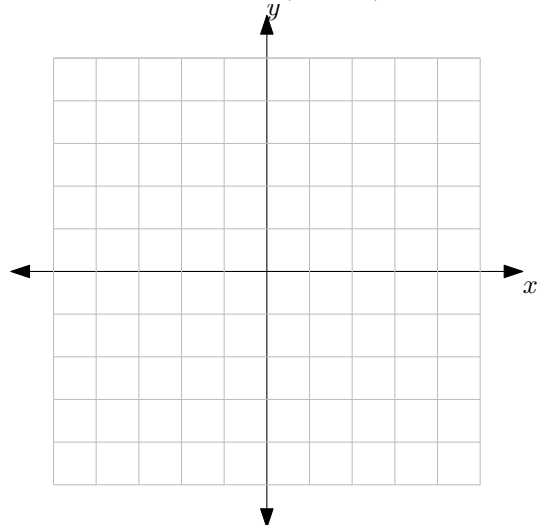
$$y = 2 - \log_{\frac{1}{2}}(x + 1)$$



$$y = \ln(x)$$



$$y = 2 + \ln(-x + 2)$$



Find the domain of the following functions.

- $f(x) = \ln(x - 4)$

- $f(x) = \log_4(1 - 2x)$

- $f(x) = \log_2(x - 4)$

- $f(x) = \log_5(x^2 + 1)$

- $f(x) = \log_{10}(x - 4)$

- $f(x) = \log_{\frac{1}{2}}(x^2 - 1)$

Solve for the appropriate variable.

- $10^x = 200$

- $e^{t-3} = 50$

- $\ln x = 5$

- $\log 3z = 2.5$

The Richter scale is used to measure intensity of earthquakes. The equation used for finding the magnitude of an earthquake is

$$M = \log \left(\frac{A}{A_0} \right),$$

where M is the magnitude, A is the largest output on a seismograph, and A_0 is a constant (determined by the type of seismograph, distance from the earthquake, etc.).

Suppose two earthquakes have magnitudes of 4.4 and 5.8. By what factor is the second more intense than the first?

5.4 - Properties of Logarithms

1. Special Values

(a) $\log_b b =$

(b) $\log_b 1 =$

2. $\log_b PQ =$

3. $\log_b \left(\frac{P}{Q}\right) =$

4. $\log_b P^n =$

5. $b^{\log_b P} =$

Why does $\log_b PQ =$?

Example 0.126. (a) $\log_2 32$

(b) $\log_4 105$

(c) $\ln 2 + \ln 5$

(d) $\log x + \log(x + 1)$

Why does $\log_b \left(\frac{P}{Q} \right) =$

Example 0.127. (a) $\log_2 32 - \log_2 8 =$

(b) $\log_5 \frac{9}{7}$

(c) $\ln \frac{x^2 + 1}{x^2 + 4}$

(d) $\ln x - \ln(x + 1)$

Why does $\log_b P^n =$?

Example 0.128. (a) $\log_2 32$

(b) $\log_4 27$

(c) $\ln e^5$

(d) $\ln(x^3 + 1)^2$

Why does $b^{\log_b P} =$?

Example 0.129. (a) $2^{\log_2 3}$

(b) $5^{\log_5 7}$

(c) $e^{\ln 6}$

(d) $e^{2 \ln 6}$

Simplifying Logarithms

Steps for Putting Together:

1.

2.

3.

4.

Example 0.130. (a) $\ln x + \ln 3 - \ln(x - 2)$

(b) $2 \ln x - 3 \ln(x + 1)$

(c) $\frac{1}{2} \ln(x + 4) + 2 \ln(2x - 3) - 4 \ln x$

Simplifying Logarithms (cont.)

Steps for Splitting apart:

1.

2.

3.

Example 0.131. (a) $\log_7 \left(\frac{x(x+2)}{x^2+1} \right)$

(b) $\log_2 \sqrt{\frac{(x+1)^5}{(x-3)^3}}$

(c) $\log \left(\frac{36\sqrt{x}}{\sqrt[3]{x+1}} \right)$

Example 0.132. • $2^x = 3$

• $5^{2t-3} = 10$

• $10^{2x} - 3 \cdot 10^x - 4 = 0$

Change of Base Formula

For solving general logarithms, we can use that

where $\log_a b = \frac{\log b}{\log a}$, and $\log_a a = 1$.

Why is this useful for finding the value of logarithms?

Example 0.133. Find the following values to 4 decimal places.

(a) $\log_2 5$

(b) $\log_3 8$

(c) $\log_5 24$

(d) $\log_7 64$

5.5 - Equations and Inequalities

1. Special Values

(a) $\log_b b =$

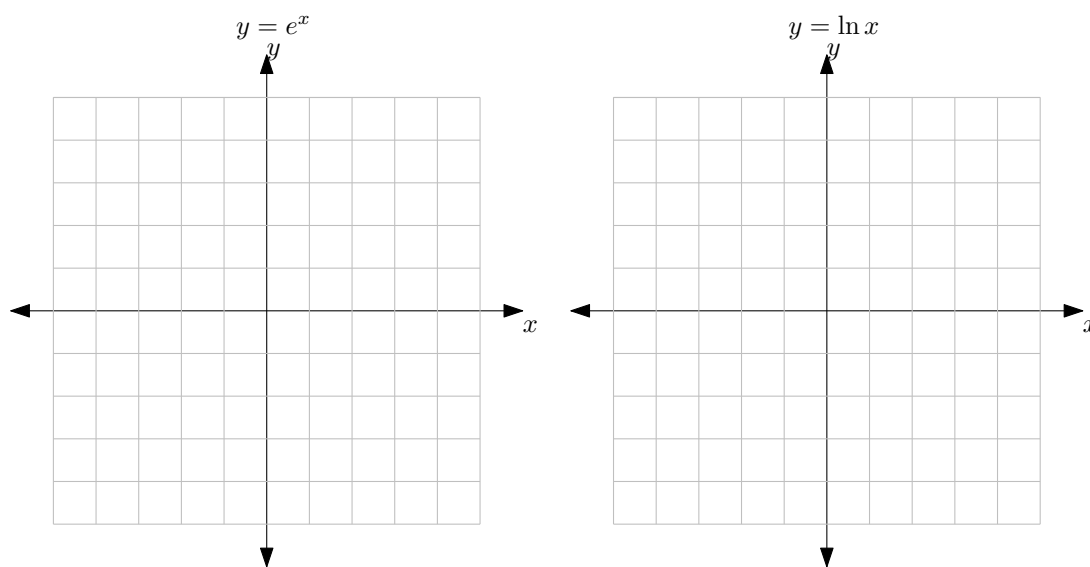
(b) $\log_b 1 =$

3. $\log_b \left(\frac{P}{Q}\right) =$

4. $\log_b P^n =$

2. $\log_b PQ =$

5. $b^{\log_b P} =$



Properties of Inequalities

If $a < b$, then $e^a < e^b$.

If $e^a < e^b$, then $a < b$.

If $a < b$, and $a, b > 0$, then $\ln a < \ln b$.

If $\ln a < \ln b$, then $a < b$.

Example 0.134. Solve for x :

- $\log_3 [\log_2 x] = 1$

- $\log_3 [\log_2 [\log_5 [\ln x]]] = 0$

Example 0.135. Solve for x : (Make sure to check your answers!)

- $\ln x^2 = (\ln x)^2$

- $[\ln x]^2 - 3 \ln x + 2 = 0$

Example 0.136. Solve for x : (Again, check your answers!)

- $\log_4(x^2 + 6x) = 2$

- $\log x + \log(x + 7) = 1$

Example 0.137. Solve for x : (Check your answers!)

- $\log_2(x) + \log_2(x - 2) = 3$

- $2\log x = 1 + \log(x - 1.6)$

Example 0.138. Solve for the appropriate variable.

- $6^{z+2} = 10$

- $2^x = 3^{x-3}$

- $2^y \cdot 3^{-y} \cdot 5^{2-3y} = 10^{y+2}$

Example 0.139. Solve the following inequalities:

- $6^{x-3} < 2$

- $10^x < 2^{x+1}$

Example 0.140. Solve more inequalities:

- $12^{x^2} \geq 12^{3x+4}$

- $\log_2(x - 3) < 4$

Example 0.141. Solve more inequalities (Make sure to check your answers!)

- $\log_2 x + \log_2(x - 3) < 2$

- $\ln x \geq \ln(2x + 4)$

5.6 - Compound Interest

Example 0.142. Suppose we invest \$250 in a savings account which accrues 6% interest annually.

(a) How much interest is made after 1 year?

(b) How much money is in the account after 1 year?

t (years)	Amount in Account
0	
1	
2	
3	
10	
40	

If an account accrues $r\%$ interest every year and started with $\$P$, the amount in the account after t years is

Example 0.143. • If you invest \$500 at 7% for 3 years, how much would you have?

• If you invest \$500 for 3 years and end with \$800, what is the interest rate?

• If you end up with \$1000 dollars after investing for 4 years at 8%, what was the initial investment?

• If you invest \$300 at 8% and yield \$700, how long did you invest?

• If you invest \$700 at 5% for 3 years, how much interest do you earn?

Compounding more than once a year

Example 0.144. An account gets 12% interest compounded quarterly. If you invest \$200, how much is in the account after

- one quarter?

- four quarters (a year)?

- 10 years (40 quarters)?

An account with initial amount $\$P$ which gets $r\%$ interest compounded n times per year will be worth

after t years (or periods).

Compounding Continuously

Compounding continuously means that in a year, interest is computed

Fact that we will not prove (until calculus)

As $n \rightarrow \infty$, we get that

So the formula for continual compounding is

Example 0.145. • If \$150 is invested at 6% compounded continuously for 4 years, what is the yield?

- If you invest in an account compounding continuously, what rate would allow you to double in 8 years?

Effective Interest Rate

Example 0.146. • Find the yield of investing \$200 at 6% compounded monthly for 1 year.

- Find the yield of investing \$200 at 6.16778% compounded annually for 1 year.
- Find the yield of investing \$200 at 6% compounded monthly for 2 years.
- Find the yield of investing \$200 at 6.16778% compounded annually for 2 years.

Given a nominal interest rate, the effective interest rate is

This is used to

Example 0.147. Which is better, investing at 6% compounded monthly, or 5.9% compounded continuously?

5.7 - Exponential Growth and Decay

A population (or amount of substance) has exponential growth or decay when it is modeled by

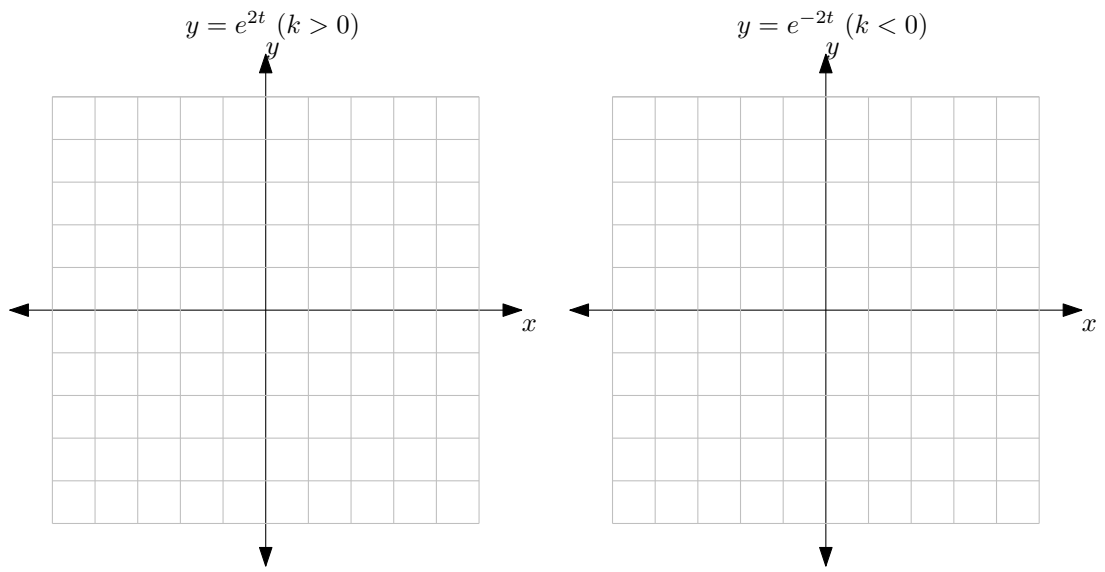
where \mathcal{N}_0 is the _____ ,

k is the _____ ,

and \mathcal{N} is _____ after _____ units of time.

If _____ , then we have

If _____ , then we have



Example 0.148. Suppose a population of bacteria in a Jack O' Lantern t days after Halloween can be modeled by

$$\mathcal{N} = 100e^{\frac{\ln 2}{5} t}$$

- How many bacteria are present on Halloween?
- How many are present 10 days after Halloween?
- How long does it take the bacteria to double its initial amount?

Example 0.150. Suppose the population of American Bald Eagles in Wisconsin follows an exponential model. There were 358 eagles in 1990 and 1065 in 2006.

- Find the growth constant k .

- Find the model for the population t years after 1990.

- How long after 1990 did it take the population to double? How long will it take to quadruple?

6.1 - Systems of Two Linear Equations, Two Unknowns

A **linear equation** in two variables is an equation of the form

where $ax + by = c$ are a and b are

Example 0.153. Which of the following equations are linear?

$$2x - y = 3$$

$$x^2 - 2y = 5$$

$$\sqrt{2}x + \frac{1}{3}y = 10$$

A point (x, y) is a **solution** to $ax + by = c$ if

Example 0.154. Is $(2, 3)$, $(1, 1)$, or both, solutions of

- $x - 3y = -7$

- $2x - y = 1$

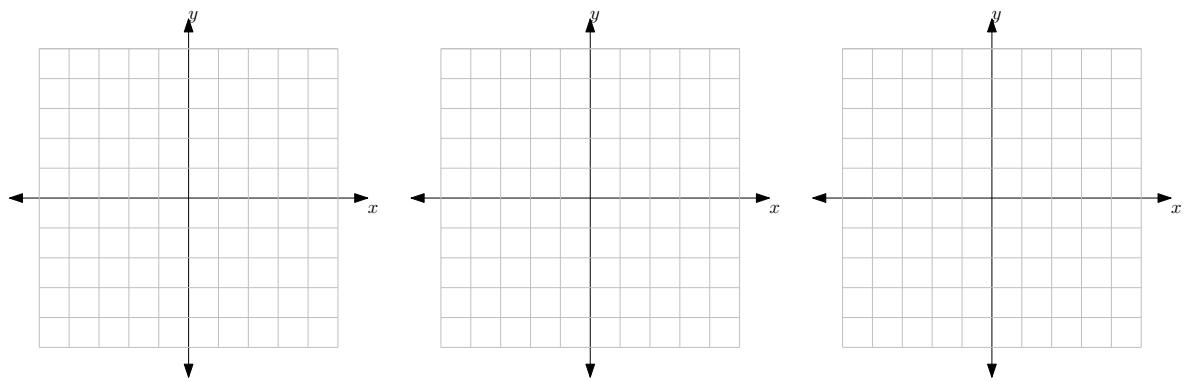
- $3x + y = 4$

A solution to two or more equations is called a

What are the possible solution sets?

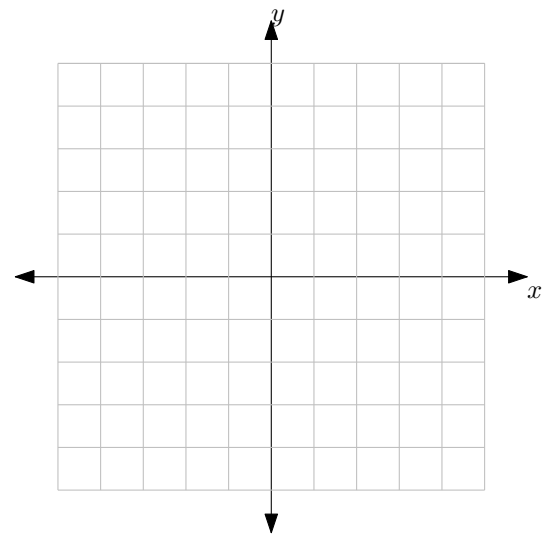
Linear equations define

Solutions to linear equations are where



Example 0.155. Solve the system using substitution.

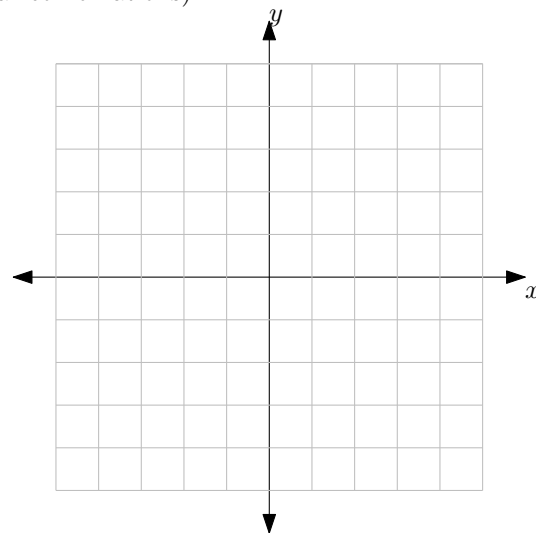
$$\begin{aligned}x - 3y &= -7 \\ 2x - y &= 1\end{aligned}$$



Example 0.156. Solve using addition-subtraction (linear combinations):

$$2x - y = 1$$

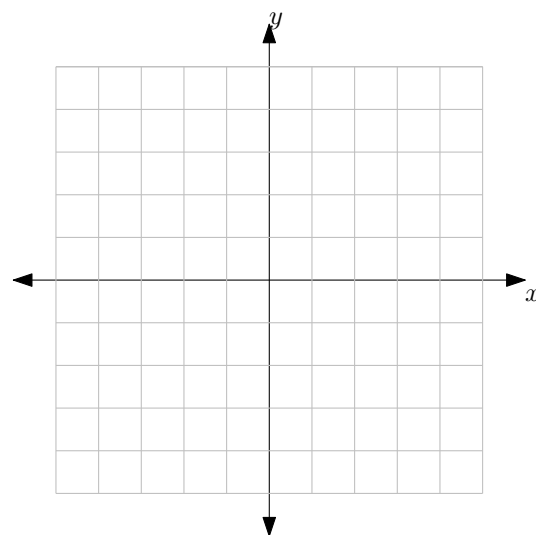
$$3x + y = 4$$



Example 0.157. Solve using linear combinations:

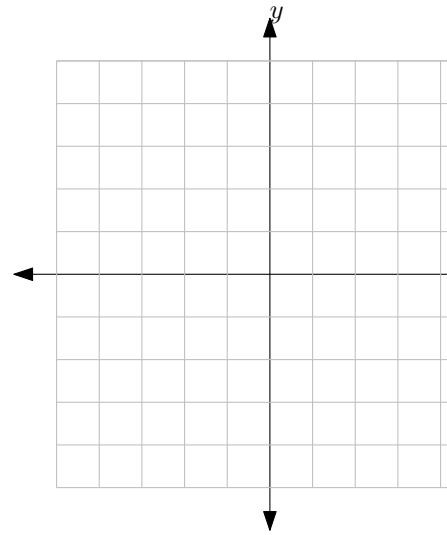
$$2x - 3y = 3$$

$$-4x + 6y = 6$$



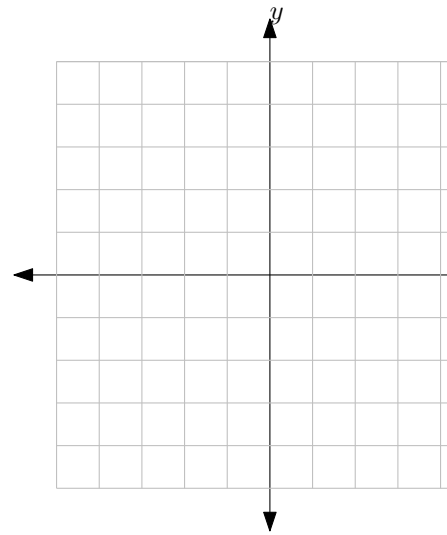
$$\begin{aligned}4x + 6y &= 2 \\6x + 9y &= 3\end{aligned}$$

Example 0.158.



$$\begin{aligned}3x - 4y &= 10 \\7x + 5y &= 9\end{aligned}$$

Example 0.159.



Example 0.160. Suppose a chemist has 10% and 15% acid solutions in stock. How much of each should the chemist mix if 100 mL of a 12% solution is desired?

Example 0.161. Find the equation for a parabola $y = ax^2 + bx - 2$ which goes through $(1, 0)$ and $(2, 6)$

Systems of linear type

Example 0.162. Solve the following system:

$$\begin{aligned}3x^2 - 2y^2 &= 10 \\2x^2 + y^2 &= 11\end{aligned}$$

Example 0.163. Solve the following system:

$$\begin{aligned}3\sqrt{x} + \frac{2}{y} &= 12 \\-\sqrt{x} + \frac{3}{y} &= 7\end{aligned}$$

6.6 - Nonlinear Systems of Equations

Example 0.164. Solve the following system of equations:

$$\begin{aligned}x^2 + y &= 4 \\2x - y &= \end{aligned}$$

6.6 - Nonlinear Systems of Equations

Example 0.165. Solve the following system of equations:

$$\begin{aligned}x^2 + y &= 4 \\2x - y &= 20\end{aligned}$$

Example 0.166. Solve the following system of equations:

$$\begin{aligned}y - x^2 &= 1 \\2x - 3y &= 2\end{aligned}$$

Example 0.167. Where do the following graphs intersect?

$$\begin{aligned}x^2 - y &= 1 \\x^2 + y^2 &= 3\end{aligned}$$

Example 0.168. Solve the following system of equations:

$$\begin{aligned}x^2 - y &= 4 \\x^2 + y^2 &= 4\end{aligned}$$

Example 0.169. Solve the following system of equations:

$$\begin{aligned}xy &= -2 \\y &= 3x + 7\end{aligned}$$

Example 0.170. Solve the following system of equations:

$$\begin{aligned}xy + y &= 3 \\y + 2x &= 1\end{aligned}$$

Example 0.171. Solve the following system of equations:

$$\begin{aligned}x^2 + y^2 &= 64 \\x^2 - y^2 &= 32\end{aligned}$$

Example 0.172. Solve the following system of equations:

$$\begin{aligned}x^2 + y^2 &= 16 \\x^2 - y^2 &= 20\end{aligned}$$

Example 0.173. Solve the following system of equations:

$$\begin{aligned}y &= 2^{x+3} \\y &= 3^{x+2}\end{aligned}$$

Example 0.174. Solve the following system of equations:

$$\begin{aligned}y - 3 \cdot 2^{x+1} &= 0 \\y - 2^{2x} &= 8\end{aligned}$$

Example 0.175. If a right triangle has an area of 60 and a hypotenuse of length 17, what are the lengths of the legs of the triangle?

Example 0.176. If a rectangle has area 384 and has a perimeter of length 80, what are the dimensions for the rectangle?

7.1 - The Complex Number System

We define i as the imaginary unit, namely that

We define a complex number as a number of the form

where a and b are real numbers.

The value of a is the real part of $a + bi$.

The value of b is the imaginary part of $a + bi$.

Two complex numbers $a + bi$ and $c + di$ are equal if

Example 0.177. Find r and s so that $3 - 2i = \log_2 r + \frac{r}{s}i$.

Properties of Complex Numbers

- Addition: $(3 + 2i) + (4 - i) =$
- Subtraction: $(5 + 7i) - (2 - 3i) =$
- Multiplication: $(1 - i) \cdot (2 + 5i) =$

$$(2 + 5i) \cdot (3 + 4i) =$$

The **complex conjugate** of $z = a + bi$ is .

The conjugate of is denoted as .

Example 0.178. For each given z , find \bar{z} and $z\bar{z}$.

- $z = 3 + 4i$

- $z = 4$

- $z = 2 - i$

- $z = a + bi$

- $z = 2i$

Example 0.179. Simplify the following radicals.

- $\sqrt{-4}$

- $\sqrt{-4 \cdot -9}$

- $\sqrt{-4} \cdot \sqrt{-9}$

Example 0.180. Rationalize the denominator.

- $\frac{1 + 3i}{i}$

- $\frac{1 + 3i}{2 - i}$

- $\frac{5 - 2i}{3 + 4i}$

Example 0.181. Simplify the following powers of i .

- i^2

- i^{381}

- i^3

- i^{502}

- i^4

- i^{-1}

- i^5

- i^{-2}

- i^8

- i^{-3}

- i^{100}

- i^{-10}

- i^{2007}

- i^{-15}

- i^{483}

- i^{-207}

Example 0.182. Solve the following quadratic equations:

- $x^2 + 4 = 0$.

- $x^2 - 6x + 13 = 0$.

- $x^2 + 2x + 9 = 0$.

7.2 - Division of Polynomials

Example 0.183. Find the quotient of $\frac{x^2 - 2x + 5}{x - 3}$

The _____ is the polynomial by which you divide.

The _____ is the polynomial which is being divided.

The _____ is the polynomial resulting from the division.

The _____ is the polynomial remaining which cannot be divided.

The solution to the quotient can be expressed as

Example 0.184. Find the quotient of $\frac{x^3 - x + 1}{x + 1}$

Example 0.185. Find the quotient of $\frac{x^4 - 3x^3 + x - 1}{x^2 + 1}$

Example 0.186. Find the quotient of $\frac{x^3 - 3x^2 + 2x + 1}{x^2 - 2x + 3}$

Find the quotient of $\frac{x^2 - 10x + 3}{x - 4}$

Process for Synthetic Division

-

-

-

-

-

Example 0.187. Find the following quotients:

- $\frac{x^2 - 10x + 3}{x - 4}$

- $\frac{x^3 - 2}{x + 2}$

Example 0.188. Find the following quotients:

- $\frac{x^4 - 3x^3 + x^2 - 1}{x - i}$

- $\frac{x^3 - 7x^2 + x - 3}{x^2 + 1}$

- $\frac{x^3 - 7x^2 + x - 3}{x^2 - 1}$

Example 0.189. Find the following quotients:

- $\frac{x^2 - 5x + 2}{2x - 1}$

- $\frac{x^2 - 4}{x - 2}$

- $\frac{x^3 + 1}{x + 1}$

- $\frac{x^4 - 81}{x - 3}$

Example 0.190. Determine when $x - 3$ is a factor of $x^3 + 10x^2 - kx + 3$

Example 0.191. Find the quotient of $\frac{x^4 - 13a^3x + 12a^4}{x - a}$.

7.3 - The Remainder Theorem and the Factor Theorem

Example 0.192. Find the roots for the following problems.

- $x(x - 1) = 0$

- $x^2(x - 1)^3 = 0$

- $(x - 2)^4(x + 1)^5 = 0$

The multiplicity of a solution is

Example 0.193. Find the roots and multiplicities for the following problems.

- $x^3 = 0$

- $(2x - 1)^2(3x + 1)^3 = 0$

- $x(x - 1)(x - 2)^2(x - 3)^3 = 0$

- $x^4 - 2x^2 + 1 = 0$

The Remainder Theorem

When $f(x)$ is divided by $x - r$, the remainder is

Example 0.194. Find the remainder for the following quotients.

- $\frac{x^2 - 2x + 10}{x - 1}$

- $\frac{x^3 - x + 2}{x + 1}$

- $\frac{x^{10} - x^9}{x - 2}$

The Factor Theorem

If $f(r) = 0$, then

If $x - r$ is a factor of $f(x)$, then

Example 0.195. Determine if $d(x)$ is a factor of $p(x)$:

- $p(x) = x^3 - 2x + 1$, $d(x) = x - 1$.

- $p(x) = x^4 + 2x^2 - 7x - 9$, $d(x) = x - 2$.

- $p(x) = x^3 + 4x^2 - 3x - 14$, $d(x) = x + 2$.

Example 0.196. Find all solutions:

- $x^3 - 6x^2 + 11x - 6 = 0$, given that 3 is a root.

- $x^3 - 5x^2 + 2x + 8 = 0$, given that 4 is a root.

- $x^4 - 4x^3 - 3x^2 + 10x + 8 = 0$, given that -1 is a root of multiplicity 2.

Example 0.197. Find an equation which satisfies the following:

- Has roots 2,3,4

- Has roots 2,3, and 4 with multiplicity 3

- all integer coefficients, roots of $\frac{1}{2}$ and $\frac{3}{2}$

- degree 3 with roots 2, and 3 with multiplicity 3.

7.4 - The Fundamental Theorem of Algebra

Every polynomial equation of the form

has at least one root within the complex number system.

What does this mean to us? It means that we can find any polynomial!

The Linear Factors Theorem

If $f(x)$ is a polynomial of

then it can be written as

where a_0, a_1, \dots, a_n are real numbers, and may not be zero.

Note:

If all coefficients of a polynomial are real, and

and if $a + bi$ is a root, then so is $a - bi$.

So, if a polynomial has degree n , then

Example 0.198. Write the following polynomials in the form

$$a(x - r_1)(x - r_2) \cdots (x - r_n) :$$

- $x^2 - 2x + 10$

- $3x^2 - 7x + 2$

- $x^3 - 5x^2 + 16x - 30$, given 3 is a root.

Example 0.199. Write the following polynomials in the form

$$a(x - r_1)(x - r_2) \cdots (x - r_n) :$$

- $x^4 + 5x^2 + 4$

- $x^3 - 5x^2 + 17x - 13$, given that $2 + 3i$ is a root.

Example 0.200. Write the following polynomials in the form

$$a(x - r_1)(x - r_2) \cdots (x - r_n) :$$

- $x^3 - 7x + 6$, given that 1 is a root.

- $x^4 + x^3 - 5x^2 - 3x + 6$, given that 1 and $-\sqrt{3}$ are roots.

Theorem

Every polynomial of degree _____ has _____ ,

when you account for _____ .

Example 0.201. Find a polynomial which meets the given requirements.

- leading coefficient 1, $\frac{\text{Root}}{\text{Multiplicity}} \left| \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 1 & 2 \end{array} \right.$

- leading coefficient 1, $\frac{\text{Root}}{\text{Multiplicity}} \left| \begin{array}{cc} 1 & 2 \\ 3 & 2 \end{array} \right.$

- leading coefficient 1, $\frac{\text{Root}}{\text{Multiplicity}} \left| \begin{array}{ccc} 0 & i & -i \\ 1 & 1 & 1 \end{array} \right.$

- integer coefficients, $\frac{\text{Root}}{\text{Multiplicity}} \left| \begin{array}{ccc} 0 & \frac{1}{2} & -\frac{2}{3} \\ 1 & 1 & 1 \end{array} \right.$

- goes through $(0, 1)$, $\frac{\text{Root}}{\text{Multiplicity}} \left| \begin{array}{ccc} 1 & 3 & -2 \\ 1 & 1 & 1 \end{array} \right.$

9.2 - The Binomial Theorem

Factorials

When n is a non-negative integer, define $n!$ as

By convention, we define

Example 0.202. Simplify

• $1!$

• $3!$

• $5!$

• $7!$

• $\frac{10!}{9!}$

• $\frac{10!}{8!}$

• $\frac{10!}{8!2!}$

• $\frac{6!}{3!3!}$

• $\frac{n!}{(n-1)!}$

• $\frac{n!}{(n-2)!2!}$

• $\frac{(n+2)!}{(n-1)!}$

• $\frac{(n+1)!}{(n-2)!3!}$

Binomial Coefficients

If n and k are positive integers, then

$$\binom{n}{k} =$$

Example 0.203. Simplify

$$\bullet \binom{10}{2}$$

$$\bullet \binom{10}{8}$$

$$\bullet \binom{8}{3}$$

$$\bullet \binom{8}{5}$$

$$\bullet \binom{7}{1}$$

$$\bullet \binom{7}{0}$$

$$\bullet \binom{6}{3}$$

$$\bullet \binom{6}{6}$$

$$\bullet \binom{n}{2}$$

$$\bullet \binom{n}{n-1}$$

Binomials

A _____ is a sum or difference of _____ .

Example 0.204. Which is a binomial?

$$x + y, \quad xy, \quad x^2 + y^2, \quad xyzw + x^2yz, \quad 2xy - y^2$$

What happens when we raise a binomial to a power?

$$(a + b)^0 =$$

$$(a + b)^1 =$$

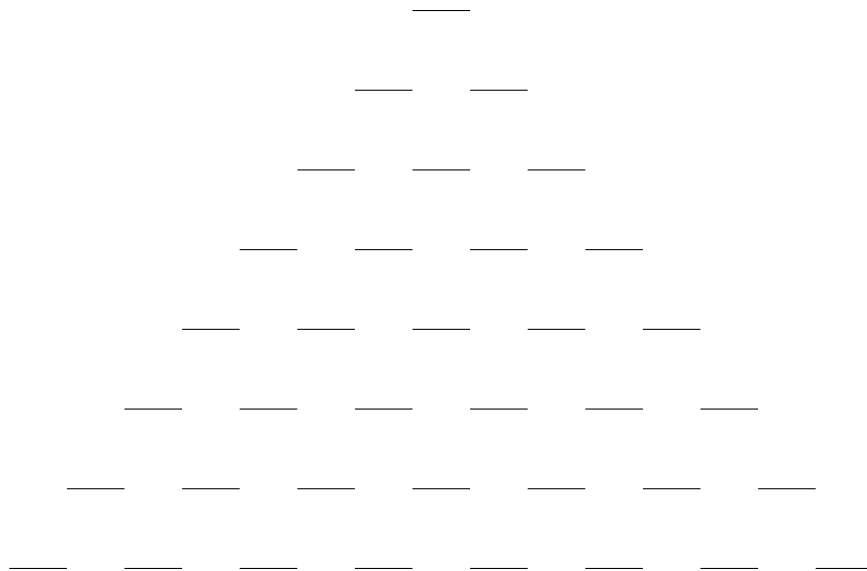
$$(a + b)^2 =$$

$$(a + b)^3 =$$

$$(a + b)^4 =$$

$$(a + b)^8 =$$

Pascal's Triangle



The $\binom{n}{k}$ of Pascal's Triangle give the

Example 0.205. Find

$$(a + b)^6 =$$

$$(a + b)^7 =$$

$$(a + b)^8 =$$

Example 0.206. Expand.

- $(x + y)^4$

- $(x - y)^3$

- $(2x + y)^4$

- $(\frac{1}{2}x - \frac{1}{3}y)^3$

- $(x^2 + y)^5$

- $(2x^2 - 3y^3)^3$

The Binomial Theorem

- The binomial expansion of $(a + b)^n$ is

Example 0.207. Find

- the a^2b^7 term in $(a + b)^9$
- the a^2b^7 term in $(a - b)^9$
- the x^2y^4 term in $(2x - y)^6$
- the x^2y^4 term in $(x^2 + \frac{1}{2}y)^5$
- the 4th term of $(a + b)^9$
- the 3rd term of $(x - y)^{10}$

- the 50th term of $(x^2 - \frac{1}{2}y)^{52}$

9.3 - Intro to Sequences and Series

A _____ is an _____

Examples of numerical sequences

- $2, 4, 8, \frac{2}{3}, -\frac{1}{2}, \dots$
- $2, 3, 5, 7, 11, \dots$
- $1, -1, i, \dots$
- $1, 2, 4, 8, 16, \dots$
- $\{1, 2, 3, \dots\}$
- $2, 1, 0, 0, 0, \dots$
- $1, 3, 5, 7, 9, \dots$
- $1, 3, 6, 10, 15, \dots$
- $1, 4, 1, 5, 9, 2, 6, \dots$
- $7, 1, 8, 2, 8, 1, 8, \dots$

Typically, we label a sequence like _____

where _____ is the _____

Example 0.208. Consider the sequence $2, \frac{1}{2}, -\frac{3}{4}, 10, e, \iota, 12, 16, \dots$

- Find a_3
- Find $a_1 + a_4$
- Find a_5/a_6

Explicit Formulas for sequences

In most cases, a sequence can be given as a

whose domain is either the \mathbb{N} or \mathbb{Z} .

Using the notation before,

Example 0.209. Calculate the first four terms (a_1, a_2, a_3, a_4) of the following sequences:

- $a_n = 3n + 2$
- $a_n = 20 \cdot \left(-\frac{1}{2}\right)^n$
- $a_n = n!$
- $a_n = \frac{1}{2} \cdot n \cdot (n + 1)$

Recursive Definitions

A sequence is a sequence where

- the first term is given, and
- the remaining terms

Example 0.210. Calculate the first four terms (b_1, b_2, b_3, b_4) of the following sequences:

- $$\begin{cases} b_1 = 2 \\ b_n = b_{n-1} + 3, \text{ for } n \geq 2 \end{cases}$$

- $$\begin{cases} b_1 = 10 \\ b_n = -\frac{1}{2}b_{n-1}, \text{ for } n \geq 2 \end{cases}$$

- $$\begin{cases} b_1 = 1 \\ b_n = n \cdot b_{n-1}, \text{ for } n \geq 2 \end{cases}$$

- $$\begin{cases} b_1 = 1 \\ b_n = b_{n-1} + n, \text{ for } n \geq 2 \end{cases}$$

Sigma Notation

is a short-hand method for

The sum

can be expressed as

The variable is called the

Example 0.211. Consider the sequence $a := 2, 10, i, -7, 12$

- Evaluate $\sum_{i=1}^4 a_i$

- Evaluate $\sum_{j=2}^5 a_{j-1}$

- Evaluate $\sum_{k=3}^3 a_k$

- Evaluate $\sum_{p=1}^4 a_p \cdot a_{p+1}$

Example 0.212. Evaluate the following summations:

$$\bullet \sum_{i=1}^5 i$$

$$\bullet \sum_{j=1}^5 j^2$$

$$\bullet \sum_{k=1}^5 2$$

$$\bullet \sum_{p=1}^5 j$$

$$\bullet \sum_{q=-1}^3 q^2 - 1$$

$$\bullet \sum_{r=1}^6 \frac{1}{r} - \frac{1}{r+1}$$

Example 0.213. Express the following sums using sigma (\sum) notation:

- $2 + 3 + 4 + \cdots + 100$

- $1^2 + 2^2 + \cdots + 10^2$

- $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{25}}$

- $\binom{12}{0} + \binom{12}{1} + \cdots + \binom{12}{12}$

- $x + x^2 + x^3 + x^4 + x^5$

- $\binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \cdots + \binom{n}{n}a^0 b^n$

- $\log_{10}(2 \cdot 4 \cdot 6 \cdot 8 \cdots 20)$

- $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{100}$

9.5 - Geometric Sequences and Series

A _____ is a sequence in which _____ .

The value of _____ is called the _____

Example 0.214. Which of the following sequences appear to be geometric?

• $1, 2, 3, 4, \dots$

• $1, 2, 4, 8, \dots$

• $21, 15, 10, 6, \dots$

• $18, 12, 8, \frac{16}{3}, \dots$

• $-3, 2, -\frac{4}{3}, \frac{8}{9}, \dots$

• $9, -6, 4, -3, \dots$

Example 0.215. In a certain geometric sequence, the $a_3 = 4$ and $a_5 = 8$. Find all possible r , and possible values for a_4 .

The formula for geometric sequences

The general form for a geometric sequence is

The recursive definition for a geometric sequence is

$$\begin{cases} a_1 = \\ a_n = \end{cases}$$

The formula for the n th term of the geometric sequence is

Example 0.216. Find a formula for the n th term of the following geometric sequences:

- $1, \frac{1}{2}, \frac{1}{4}, \dots$
- $2, 3, \frac{9}{2}, \frac{27}{4}, \dots$
- $18, -12, 8, -\frac{16}{3}, \dots$
- $2, -6, 18, -54, 162, \dots$

Finite Geometric Series

First, some fun polynomial arithmetic:

$$(1 - x)(1 + x + x^2 + x^3) =$$

$$(1 - x)(1 + x + x^2 + x^3 + x^4) =$$

$$(1 - x)(1 + x + \cdots + x^8 + x^9) =$$

$$(1 - x)(1 + x + \cdots + x^{n-2} + x^{n-1}) =$$

$$\text{So } 1 + x + \cdots + x^{n-2} + x^{n-1} =$$

$$\text{So } S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

=

=

So the sum of the first n terms would be

Example 0.217. Find the sum of the following geometric series:

$$\bullet \sum_{k=1}^{10} 2^{k-1}$$

$$\bullet \sum_{j=2}^6 \left(-\frac{2}{5}\right)^j$$

$$\bullet 6 - 18 + 54 - 162 + \cdots 4374$$

$$\bullet 1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{729}$$

$$\bullet 18 - 12 + 8 - \cdots - \frac{64}{27}$$

Infinite Geometric Series

Example 0.218. Let S_n be the sum of the first n terms of $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$. Find

- S_2

- S_4

- S_{10}

- S_{100}

- All of them added

When _____, then _____

So if _____, the sum of the infinite geometric series

would be _____

Example 0.219. Find the sum of the following geometric series:

- $1 + \frac{4}{5} + \frac{16}{25} + \cdots$

- $6 - 4 + \frac{8}{3} - \cdots$

- $2 + \frac{8}{3} - \frac{32}{9} + \cdots$

- $\frac{2}{3} - \frac{2}{3} + \frac{2}{3} - \frac{2}{3} + \cdots$

- $18 - 12 + 8 - \frac{16}{3} + \cdots$

- $\sum_{i=1}^{\infty} 3 \left(\frac{3}{4}\right)^{i-1}$

Repeating Decimals

Every repeating decimal can be written as

Example 0.220. Rewrite the following decimal expressions as fractions.

- $0.\overline{2}$

- $0.\overline{9}$

- $0.\overline{36}$

- $0.\overline{142857}$

- $0.\overline{86}$

- $0.23\overline{15}$