# Heegard Splittings and Floer Homology

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## 1 Acknowledgement

I am describing joint work [7, 8, 9] with **Dietmar Salamon** and **Ralf Gautschi**. The definition of Combinatorial Floer homology is due to **Vin De Silva** [17]. The (higher dimensional, analytic) Floer homology described at the end of the talk was introduced by Floer (see [4] and studied by many authors, e.g. [13]). We undertook this project in an attempt to understand **Wu Chung Hsiang**'s speculation [10] on the Poincaré conjecture and Floer homology. **Yasha Eliashberg** explained this speculation to Dietmar and I first heard about it in a seminar talk at the University of Wisconsin in 1999.

## 2 Summary of Talk

I will describe three theorems (A,B, and C) which summarize what survives in three dimensional of Smale's proof of the higher dimensional Poincaré conjecture. These theorems are well known to the experts. The proofs require (a) a slightly improved version of Smale's Cancellation Lemma and (b) a theorem asserting the existence of a "lune".

Floer homology will be used to count lunes. The number of lunes joining two given points will be a coefficient in the boundary operator of a homology theory. We will define three kinds of lune - smooth, combinatorial, and holomorphic - but the number of lunes joining two given points is the same for each kind. In our application the Floer homology will be invariant under isotopy and not just Hamiltonian isotopy as in Floer's original theory.

# 3 Intersection Numbers

For transverse embedded closed curves $\alpha, \beta$ in a orientable 2-manifold $\Sigma$ there are three ways we can count the number of points in their intersection:

**1.** The **numerical intersection number** $\mathrm{num}\,(\alpha, \beta)$ is the actual number of intersection points.

**2.** The **geometric intersection number** $\mathrm{geo}\,(\alpha, \beta)$ is defined as the minimum of the numbers $\mathrm{num}\,(\alpha, \beta')$ over all embedded loops $\beta'$ that are transverse to $\alpha$ and isotopic to $\beta$.

**3.** The **algebraic intersection number** $\mathrm{alg}\,(\alpha, \beta)$ is the absolute value $\mathrm{alg}\,(\alpha, \beta) = |\alpha \cdot \beta|$ of the sum $\alpha \cdot \beta = \sum_{x \in \alpha \cap \beta} \pm 1$ where the plus sign is chosen iff the two orientations of $T_x\Sigma = T_x\alpha \oplus T_x\beta$ match. This definition is independent of the choice of orientations of $\alpha$, $\beta$, and $\Sigma$.

The inequalities

$$\mathrm{alg}\,(\alpha, \beta) \leq \mathrm{geo}\,(\alpha, \beta) \leq \mathrm{num}\,(\alpha, \beta)$$

are immediate.

**Remark 3.1.** A theorem of Epstein [2] says that two embedded loops in $\Sigma$ are homotopic if and only if they are isotopic, i.e. if, in the definition of geometric intersection number, the word *isotopic* is replaced by word *homotopic*, the value of $\mathrm{geo}\,(\alpha, \beta)$ is unchanged.

# 4 Morse–Smale Systems

A **Morse–Smale vector field** on a manifold $M$ is a vector field $\xi$ having only hyperbolic rest points with stable and unstable manifolds intersecting transversally and admitting a smooth function $h : M \to \mathbb{R}$ (called a **height function** for $\xi$) such that $dh(z)\xi(z) < 0$ if $z$ is not a rest point. It is not hard to prove that $\xi$ admits a **self indexing** height function, i.e. one which satisfies $h(p) = k$ for $p$ a rest point of index $k$. Our terminology is non standard in that for us a Morse–Smale system has no periodic orbits.

## 5 HMS Structures

**Notation 5.1.** *Throughout $Y$ is a* **closed** *(i.e. compact and without boundary) smooth oriented connected 3-manifold.*

**Definition 5.2.** An **HMS structure** on $Y$ is a quadruple $(Y_0, Y_1, \Sigma, \xi)$ consisting of a Morse–Smale vector field $\xi$ on $Y$ and a decomposition $Y = Y_0 \cup Y_1$ of $Y$ into two 3-submanifolds intersecting in their common boundary

$$Y = Y_0 \cup Y_1, \qquad \Sigma = Y_0 \cap Y_1 = \partial Y_0 = \partial Y_1, \tag{1}$$

such that

**(i)** $\xi$ has one rest point $p_0$ of index zero, one rest point $q_0$ of index three, $g$ rest points $p_1, \dots, p_g$ of index one, and $g$ rest points $q_1, \dots, q_g$ of index two;

**(ii)** $p_0, p_1, \dots, p_g \in Y_0$ and $q_0, q_1, \dots, q_g \in Y_1$;

**(iii)** $\xi$ is transverse to $\Sigma$.

A **Heegard splitting** of $Y$ is a decomposition as in (1) which arises from some HMS structure. An HMS structure determines embedded curves

$$\alpha_i := W^s(p_i) \cap \Sigma, \qquad \beta_j := W^u(q_j) \cap \Sigma, \qquad i, j = 1, \dots, g. \tag{2}$$

The curves $\alpha_1, \dots, \alpha_g$ are pairwise disjoint as are the curves $\beta_1, \dots, \beta_g$ but each connecting orbit from $q_j$ to $p_i$ intersects $\Sigma$ in an intersection point of $\alpha_i$ and $\beta_j$. The pair

$$\alpha = \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_g, \qquad \beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_g$$

of 1-submanifolds is called the **trace** of the HMS structure in $\Sigma$. It is easy to see that the number $g$ is the genus of $\Sigma$; we call it the **genus** of the HMS structure.

**Remarks 5.3. 1)** Given a Morse–Smale vector field satisfying (i) we may construct an HMS structure using a self-indexing height function $h$; simply take

$$Y_0 = h^{-1}([0, 3/2]), \qquad \Sigma = h^{-1}(3/2), \qquad Y_1 = h^{-1}([3/2, 3]).$$

**2)** If a Morse–Smale vector field on $Y$ has exactly one critical point of index zero and exactly one critical point of index three, then (by Morse Theory) the number of critical points of index one must equal the number of critical points of index two.

**3)** By definition of "Morse–Smale vector field" $W^s(p_i)$ intersects $W^u(q_j)$ transversally in $Y$; it follows from item (iii) above that the curves $\alpha_i$ and $\beta_j$ intersect transversally in $\Sigma$.

**Definition 5.4.** We say that an HMS structure is

$$\left\{ \begin{array}{l} \textbf{algebraically} \\ \textbf{geometrically} \\ \textbf{numerically} \end{array} \right\} \textbf{reduced} \text{ iff } \left\{ \begin{array}{l} \mathrm{alg}\,(\alpha_i, \beta_j) = \delta_{ij} \\ \mathrm{geo}\,(\alpha_i, \beta_j) = \delta_{ij} \\ \mathrm{num}\,(\alpha_i, \beta_j) = \delta_{ij} \end{array} \right\}$$

for $i, j = 1, \ldots, g$.

**Theorem A.** *The 3-manifold $Y$ admits an HMS structure.*

**Theorem B.** *The* 3*-manifold $Y$ is an integral homology* 3*-sphere if and only if it admits an algebraically reduced HMS structure.*

**Theorem C.** *The following are equivalent.*

**(i)** *$Y$ is diffeomorphic to the* 3*-sphere.*

**(ii)** *$Y$ admits an HMS structure of genus zero.*

**(iii)** *$Y$ admits a numerically reduced HMS structure.*

**(iv)** *$Y$ admits a geometrically reduced HMS structure.*

These theorems are proved in [7]. Except for the implication (iv) $\Longrightarrow$ (iii) the proofs of these theorems are not much different from the higher dimensional case treated in Smale's original paper [18]. (The standard exposition is [12].) Theorem A is explicitly stated in [18]. Its proof uses the Cancellation Theorem and the "Morse homology theory" described below. Theorem B also uses this Morse homology theory and a "handle sliding argument". The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) of Theorem C are trivial. The implication (ii) $\Longrightarrow$ (i) is a smooth version of Reeb's Theorem [15]; it follows easily from Smale's theorem [19] that $\mathrm{Diff}_+(S^2)$ is connected. The implication (iii) $\implies$ (ii) is very similar to the higher dimensional case; it requires an Improved Cancellation Lemma (7.1 below) which assures that the cancellation of critical points does not introduce new (index difference one) connecting orbits between the remaining critical points. The connection with Floer Homology comes in the proof of (iv) $\implies$ (iii) and will be explained below.

Theorems A-C fail to prove the Poincaré conjecture because there is an algebraically reduced HMS structure on $S^3$ which is not geometrically reduced. See Example 9.13 below.

# 6 Morse Homology

Let $\xi$ be a Morse–Smale vector field on an oriented manifold $M$, $P_k$ be the set of rest points $\xi$ of index $k$, and $C_k$ be the free abelian group generated by $P_k$. Orient each $W^u(p)$ arbitrarily and orient $W^s(p)$ so that the orientation of $T_pM = T_pW^u(p) \oplus T_pW^s(p)$ is the orientation of $T_pM$. Let $\nu(q,p)$ be the algebraic intersection number of $W^u(q) \cap h^{-1}(k+\frac{1}{2})$ with $W^s(p) \cap h^{-1}(k+\frac{1}{2})$ for $q \in P_{k+1}$ and $p \in P_k$ where $h$ is a self-indexing height function. Define $\partial : C_{*+1} \to C_*$ by

$$C_k = \bigoplus_{p\in P_k} \mathbb{Z}p, \qquad \partial q = \sum_{p\in P_k} \nu(q,p)p, \quad q \in P_{k+1}. \tag{3}$$

This chain complex is usually ascribed to Witten [21] and Floer [5], but the following theorem is older: a proof may be found in [11].

**Theorem 6.1.** *The operator $\partial$ defined in equation (3) satisfies $\partial \circ \partial = 0$ and its homology is isomorphic to the usual (singular) homology of $M$:*

$$\frac{\mathrm{Kernel}\,(\partial : C_k \to C_{k-1})}{\mathrm{Image}\,(\partial : C_{k+1} \to C_k)} \cong H_k(M,\mathbb{Z}).$$

*Proof of Theorem B.* Take $M = Y$ and $\xi$ the vector field of an HMS structure. Then equation (3) is

$$\partial q_0 = 0, \qquad \partial q_j = \sum_{i=1}^{g} (\alpha_i \cdot \beta_j) p_i, \qquad \partial p_i = 0.$$

Thus $Y$ is an integral homology sphere if and only if the intersection matrix $(\alpha_i \cdot \beta_j)$ is unimodular. This is certainly the case if the HMS structure is algebraically reduced. For the converse transform the matrix $(n_{ij})$ to the identity matrix using elementary operations: reversing the sign of a column, interchanging two rows or two columns, adding one column to another. Each operation may be realized by a corresponding operation on the HMS structure: reversing the sign of the $j$th column corresponds to reversing the orientation of $W^u(q_j)$, interchanging rows or columns corresponds to relabeling, and adding the $k$th column to the $\ell$th column corresponds to replacing $\xi$ by $\xi'$ where $\alpha_i' = \alpha_i$, $\beta_j' = \beta_j$ for $j \neq \ell$ and $\beta_\ell' = \beta_\ell \# \beta_k$ a connected sum of $\beta_\ell$ and $\beta_k$. (To construct $\xi'$ use handle sliding as in [12].) $\square$

# 7 The Cancellation Lemma

For a Morse–Smale vector field $\xi$ on a compact manifold $M$ let $P(\xi)$ denote the set of rest points of $\xi$ and for $p, q \in P(\xi)$ let $n(q, p, \xi)$ denote the number of connecting orbits from $q$ to $p$; define $n(q, p, \xi) = 0$ if the index of $q$ is not equal to one more than the index of $p$. When $\xi$ is as in Definition 5.2 we have

$$n(q_j, p_i) = \mathrm{num}\,(\alpha_i, \beta_j)$$

for $i, j = 1, 2, \ldots, g$. The following is an improved form of Smale's Cancellation Lemma with essentially the same proof. See [7].

**Theorem 7.1 (Cancellation Lemma).** *Suppose that $\xi$ is a Morse–Smale vector field on $M$ and let $\bar{p}, \bar{q} \in P(\xi)$ be such that*

$$n(\bar{q}, \bar{p}; \xi) = 1.$$

*Let $\Gamma$ denote the closure of the connecting orbit. Then, for every neighborhood $U$ of $\Gamma$, there exists a Morse–Smale vector field $\eta$ on $M$ which agrees with $\xi$ on the complement of $U$ and satisfies*

$$P(\eta) = P(\xi) \setminus \{\bar{p}, \bar{q}\}, \tag{4}$$

*and*

$$n(q, p; \eta) = n(q, p; \xi) + n(q, \bar{p}; \xi) n(\bar{q}, p; \xi) \tag{5}$$

*for $p, q \in P(\eta)$.*

*Proof of Theorem A.* By transversality $Y$ admits a Morse–Smale vector field $\xi$. For $q \in P_1(\xi)$ and $p \in P_0(\xi)$ we have $n(q, p) = 0, 1, 2$ and $\nu(q, p) = 0$ if

$n(q,p) = 0, 1$. Hence by Theorem 6.1 there must be a pair with $n(q,p) = 1$ if $P_0(\xi)$ has more than one element. Then by Theorem 7.1 we may find another Morse–Smale vector field $\eta$ with $P_0(\eta)$ of smaller size than $P_0(\xi)$. The same argument works to reduce $P_3(\xi)$. □

*Proof of Theorem C (iii) $\Longrightarrow$ (ii).* Theorem 7.1 says that we can modify a numerically reduced HMS structure so as to produce another numerically reduced HMS structure of genus one less. (Then use induction.) □

# 8 Isotopy

**Lemma 8.1 (Isotopy Lemma).** *Let $(Y_0, Y_1, \Sigma, \xi)$ be an HMS structure on $Y$ with trace*

$$\alpha = \alpha_1 \cup \cdots \cup \alpha_g, \qquad \beta = \beta_1 \cup \cdots \cup \beta_g.$$

*Suppose that $f : \Sigma \to \Sigma$ is a diffeomorphism isotopic to the identity such that $f(\beta)$ is transverse to $\alpha$. Then there is an HMS structure $(Y_0, Y_1, \Sigma, \xi')$ on $Y$ with trace*

$$\alpha = \alpha_1 \cup \cdots \cup \alpha_g, \qquad f(\beta) = f(\beta_1) \cup \cdots \cup f(\beta_g).$$

*Proof.* Use the graph of the isotopy to modify the flow. □

Lemma 8.1 does not suffice to prove (iv) $\Longrightarrow$ (iii) in Theorem C. If the HMS structure is geometrically reduced but not algebraically reduced there is a pair of indices $(i_0, j_0)$ and a diffeomorphism $f$ isotopic to the identity with

$$\delta_{i_0, j_0} = \mathrm{geo}\,(\alpha_{i_0}, \beta_{j_0}) = \mathrm{num}\,(\alpha_{i_0}, f(\beta_{j_0})) < \mathrm{num}\,(\alpha_{i_0}, \beta_{j_0}); \tag{6}$$

This does not prove (iv) $\Longrightarrow$ (iii) because we do not know that

$$\mathrm{num}\,(\alpha_i, f(\beta_j)) \le \mathrm{num}\,(\alpha_i, \beta_j) \tag{7}$$

for all $i, j = 1, 2, \ldots, g$. We need to choose $f$ more carefully. The following lemma is a consequence of the smooth Floer homology theory explained below.

**Lemma 8.2.** *Denote the standard half disk by*

$$\mathbb{D} := \{z \in \mathbb{C} \,|\, \mathrm{Im}\, z \ge 0,\ |z| \le 1\}$$

*and let $\alpha, \beta \subset \Sigma$ be two embedded circles in a closed 2-manifold $\Sigma$. Assume that*

$$\mathrm{geo}\,(\alpha, \beta) < \mathrm{num}\,(\alpha, \beta).$$

*Then there is a smooth embedding $u : \mathbb{D} \to \Sigma$ such that*

$$u(\mathbb{D} \cap \mathbb{R}) \subset \alpha, \qquad u(\mathbb{D} \cap S^1) \subset \beta.$$

*Proof of Theorem C (iv) $\Longrightarrow$ (iii).* Let $(Y_0, Y_1, \Sigma, \xi)$ be a geometrically reduced HMS structure on $Y$ with trace

$$\alpha = \alpha_1 \cup \cdots \cup \alpha_g, \qquad \beta = \beta_1 \cup \cdots \cup \beta_g.$$

Assume that this HMS structure is not numerically reduced so that

$$\mathrm{geo}\,(\alpha_{i_0}, \beta_{j_0}) < \mathrm{num}\,(\alpha_{i_0}, \beta_{j_0})$$

for some pair $(i_0, j_0)$. By Lemma 8.2 there is a smooth embedding $u : \mathbb{D} \to \Sigma$ with $u(\mathbb{D} \cap \mathbb{R}) \subset \alpha_{i_0}$ and $u(\mathbb{D} \cap S^1) \subset \beta_{j_0}$. Using $u$ we construct a diffeomorphism $f$ isotopic to the identity which moves $u(\mathbb{D})$ to a small strip just below $u(\mathbb{D} \cap \mathbb{R})$ and is the identity off a small neighborhood of $u(\mathbb{D})$. This eliminates the intersection points $u(\pm 1)$ so that (6) holds. Any intersection points in $u(\mathbb{D} \cap S^1) \cap \alpha$ are removed and no others are changed so that (7) holds. Now use induction. □

# 9 Smooth Floer homology

To define an operator as in equation (3) we require only a set of "critical points", a notion of "connecting orbit of index (difference) one", and a way of counting these connecting orbits. We now describe a theory where the critical points are the intersection points $\alpha \cap \beta$ of two transverse embedded circles $\alpha$ and $\beta$ in a closed orientable 2-manifold $\Sigma$ and the connecting orbits are objects called "lunes". We eventually define three kinds of lune - smooth, combinatorial, and holomorphic - and we prove that for $x, y \in \alpha \cap \beta$ the number $n(x, y)$ of lunes from $x$ to $y$ is the same in all three cases.

**Definition 9.1.** Throughout $\alpha$ and $\beta$ are transverse embedded circles in a closed orientable 2-manifold $\Sigma$. A **smooth** $(\alpha, \beta)$**-lune** is an equivalence class of orientation preserving immersions $u : \mathbb{D} \to \Sigma$ such that

$$u(\mathbb{D} \cap \mathbb{R}) \subset \alpha, \qquad u(\mathbb{D} \cap S^1) \subset \beta.$$

The equivalence relation is defined by

$$[u] = [u']$$

iff there is an orientation preserving diffeomorphism $\phi : \mathbb{D} \to \mathbb{D}$ such that

$$\phi(-1) = -1, \qquad \phi(1) = 1, \qquad u' = u \circ \phi.$$

That $u$ is an immersion means that $u$ is smooth and $du$ is injective in all of $\mathbb{D}$, even at the corners $\pm 1$. The **endpoints** of the lune are intersection points

$$u(-1), u(1) \in \alpha \cap \beta$$

of $\alpha$ and $\beta$. When $x = u(-1)$ and $y = u(1)$ we say the lune is **from** $x$ **to** $y$. A smooth lune is called **embedded** if the map $u$ is injective. These notions

Figure 1: Three lunes.

are clearly independent of the choice of the immersion $u$ representing the smooth lune.

**Theorem 9.2.** *Define a chain complex* $\partial : \mathrm{CF}(\alpha, \beta) \to \mathrm{CF}(\alpha, \beta)$ *by*

$$\mathrm{CF}(\alpha, \beta) = \bigoplus_{x \in \alpha \cap \beta} \mathbb{Z}_2 x, \qquad \partial x = \sum_y \big( n(x, y) \bmod 2 \big)\, y. \tag{8}$$

*Then*

$$\partial \circ \partial = 0. \tag{9}$$

*The homology group*

$$\mathrm{HF}(\alpha, \beta) := \ker \partial / \mathrm{im} \partial$$

*of this chain complex is called the* **Floer homology** *of the pair* $(\alpha, \beta)$*.*

**Proposition 9.3.** *Assume that neither* $\alpha$ *nor* $\beta$ *is contractible and that they are not isotopic to each other. (This can only happen if* $\Sigma$ *has positive genus.) For* $x, y \in \alpha \cap \beta$ *let* $n(x, y)$ *denote the number of smooth* $(\alpha, \beta)$*-lunes from* $x$ *to* $y$*. Then*

$$n(x, y) \in \{0, 1\}.$$

*Proof.* It is not hard to prove (see [8]) that a lune $[u]$ is determined by its boundary $A \cup B$ where $A = u(\mathbb{R} \cap \mathbb{D})$ and $B = u(S^1 \cap \mathbb{D})$. As there are only two choices for $A$ (the two arcs in $\alpha$ from $x$ to $y$) and two for $B$ there are at most four lunes from $x$ to $y$. However two can be excluded because $u(-1) = x$ and $u(1) = y$ and $u$ is orientation preserving. If there were two lunes we could construct an isotopy between $\alpha$ and $\beta$. □

**Theorem 9.4.** *Under the hypothesis of Proposition 9.3, smooth Floer homology is invariant under isotopy: If $\alpha', \beta' \subset \Sigma$ are transverse embedded loops such that $\alpha$ is isotopic to $\alpha'$ and $\beta$ is isotopic to $\beta'$ then*

$$\mathrm{HF}(\alpha, \beta) \cong \mathrm{HF}(\alpha', \beta').$$

**Corollary 9.5.** $\dim \mathrm{HF}(\alpha, \beta) = \mathrm{geo}\,(\alpha, \beta)$.

**Remark 9.6.** It is not hard to prove (see [8]) that if there is an $(\alpha, \beta)$-lune then there is an embedded $(\alpha, \beta)$-lune.

*Proof of Corollary 9.5.* By Theorem 9.4 we may assume that $\mathrm{num}\,(\alpha, \beta) = \mathrm{geo}\,(\alpha, \beta)$. Suppose that there exists an $(\alpha, \beta)$-lune. Then by 9.6 there is an embedded $(\alpha, \beta)$-lune and hence there exists an embedded curve $\beta'$ that is isotopic to $\beta$ and satisfies $\mathrm{num}\,(\alpha, \beta') < \mathrm{num}\,(\alpha, \beta)$. This contradicts our assumption. Hence there exists no $(\alpha, \beta)$-lune, hence the Floer boundary operator is zero, and hence the dimension of $\mathrm{HF}(\alpha, \beta) \cong \mathrm{CF}(\alpha, \beta)$ is the geometric intersection number $\mathrm{geo}\,(\alpha, \beta)$. □

*Proof of Lemma 8.2.* By Corollary 9.5, the Floer homology group $\mathrm{HF}(\alpha, \beta)$ has dimension $\mathrm{geo}\,(\alpha, \beta)$. Since the Floer chain complex $\mathrm{CF}(\alpha, \beta)$ has dimension $\mathrm{num}\,(\alpha, \beta)$ it follows that the Floer boundary operator is nonzero. Hence there exists a smooth $(\alpha, \beta)$-lune and hence, by 9.6 there exists an embedded $(\alpha, \beta)$-lune. □

**Remark 9.7.** The hypothesis that $\alpha$ and $\beta$ not be isotopic is crucial. Let $\Sigma = T^*S^1$, $\alpha$ be the zero section and $\beta$ be the graph of $df$ where $f : S^1 \to \mathbb{R}$ is a Morse function with two critical points $x$ and $y$. Then there are two lunes from (say) $x$ to $y$ and no lunes from $y$ to $x$ so $\dim \mathrm{HF}(\alpha, \beta) = 1$. But $\beta$ is isotopic to $\beta'$ where $\alpha \cap \beta' = 0$ and hence $\dim \mathrm{HF}(\alpha, \beta') = 0$. By joining the ends we can make an example where $\Sigma$ is a torus. (The Floer homology of [4] is invariant under Hamiltonian isotopy; the curves $\beta$ and $\beta'$ are not Hamiltonian isotopic.)

**Remark 9.8.** The hypothesis that $\alpha$ and $\beta$ are not contractible is also crucial. For example if $\beta$ is the boundary of a small disk intersecting $\alpha$ in exactly two points $x$ and $y$, then $\partial x = y$ and $\partial y = x$ so $\partial \circ \partial \neq 0$.

We now sketch the proofs of Theorems 9.2 and 9.4. For more details see [8].

**Definition 9.9.** Let $x, z \in \alpha \cap \beta$. A **broken** $(\alpha, \beta)$**-heart** from $x$ to $z$ is a triple

$$h = ([u], y, [v])$$

such that $y \in \alpha \cap \beta$, $[u]$ is a smooth $(\alpha, \beta)$-lune from $x$ to $y$, and $[v]$ is a smooth $(\alpha, \beta)$-lune from $y$ to $z$. The point $y$ is called the **midpoint** of the heart.

**Proposition 9.10.** *Let $h = ([u], y, [v])$ be a broken $(\alpha, \beta)$-heart from $x$ to $z$ and abbreviate $A_{xy} = u(\mathbb{R} \cap \mathbb{D})$, $A_{yz} = v(\mathbb{R} \cap \mathbb{D})$, $B_{xy} = u(S^1 \cap \mathbb{D})$, $B_{yz} = v(S^1 \cap \mathbb{D})$. Then exactly one of the following four alternatives (see Figure 2) holds:*

**(a)** $A_{xy} \cap A_{yz} = \{y\}$, $B_{yz} \subsetneq B_{xy}$. **(b)** $A_{xy} \cap B_{yz} = \{y\}$, $B_{xy} \subsetneq B_{yz}$.

**(c)** $B_{xy} \cap B_{yz} = \{y\}$, $A_{yz} \subsetneq A_{xy}$. **(d)** $B_{xy} \cap B_{yz} = \{y\}$, $A_{xy} \subsetneq A_{yz}$.

Figure 2: Four broken hearts.

Let $N \subset \mathbb{C}$ be an embedded convex half disk such that

$$[0,1] \cup i[0,\varepsilon) \cup (1 + i[0,\varepsilon)) \subset \partial N, \qquad N \subset [0,1] + i[0,1]$$

for some $\varepsilon > 0$ and define

$$H := ([0,1] + i[0,1]) \cup (i + N) \cup (1 + i - iN).$$

(See Figure 3.) The boundary of $H$ decomposes as

$$\partial H = \partial_0 H \cup \partial_1 H$$

where $\partial_0 H$ denotes the boundary arc from 0 to $1+i$ that contains the horizontal interval $[0,1]$ and $\partial_1 H$ denotes the arc from 0 to $1+i$ that contains the vertical interval $i[0,1]$.

Figure 3: The domains $N$ and $H$.

**Definition 9.11.** Let $x, z \in \alpha \cap \beta$. A **smooth** $(\alpha, \beta)$**-heart of type (ac) from** $x$ **to** $z$ is an equivalence class of orientation preserving immersions $w : H \to \Sigma$ that satisfy

$$w(0) = x, \qquad w(1+i) = z, \qquad w(\partial_0 H) \subsetneq \alpha, \qquad w(\partial_1 H) \subsetneq \beta. \qquad (ac)$$

The equivalence relation is defined by $[w] = [w']$ iff there exists an orientation preserving diffeomorphism $\chi : H \to H$ such that

$$\chi(0) = 0, \qquad \chi(1+i) = 1+i, \qquad w' = w \circ \chi.$$

A **smooth** $(\alpha, \beta)$**-heart of type (bd) from** $x$ **to** $z$ is a smooth $(\beta, \alpha)$-heart of type (ac) from $z$ to $x$. Let $[w]$ be a smooth $(\alpha, \beta)$-heart of type (ac) from $x$ to $y$ and $h = ([u], y, [v])$ be a broken $(\alpha, \beta)$-heart from $x$ to $y$ of type (a) or (c). The broken heart $h$ is called **compatible** with the smooth heart $[w]$ if there exist orientation preserving embeddings $\phi : \mathbb{D} \to H$ and $\psi : \mathbb{D} \to H$ such that

$$\phi(-1) = 0, \qquad \psi(1) = 1+i, \tag{10}$$

$$H = \phi(\mathbb{D}) \cup \psi(\mathbb{D}), \qquad \phi(\mathbb{D}) \cap \psi(\mathbb{D}) = \phi(\partial D) \cap \psi(\partial D), \tag{11}$$

$$u = w \circ \phi, \qquad v = w \circ \psi. \tag{12}$$

**Proposition 9.12. (i)** *Let $h = ([u], y, [v])$ be a broken $(\alpha, \beta)$-heart of type (a) or (c) from $x$ to $z$. Then there exists a unique smooth $(\alpha, \beta)$-heart $[w]$ of type (ac) from $x$ to $z$ that is compatible with $h$.*

**(ii)** *Let $[w]$ be a smooth $(\alpha, \beta)$-heart of type (ac) from $x$ to $z$ . Then there exists precisely one broken $(\alpha, \beta)$-heart of type (a) from $x$ to $z$ that is compatible with $[w]$, and precisely one broken $(\alpha, \beta)$-heart of type (c) from $x$ to $z$ that is compatible with $[w]$.*

Figure 4: Breaking a heart.

*Proof of Theorem 9.2.* The square of the boundary operator is given by

$$\partial\partial x = \sum_{z\in\alpha\cap\beta} n_H(x,z)z,$$

where

$$n_H(x,z) := \sum_{y\in\alpha\cap\beta} n(x,y)n(y,z)$$

is the number of broken hearts from $x$ to $z$. By Proposition 9.12 broken hearts from $x$ to $z$ occur in pairs so their number $n_H(x,z)$ is even for all $x$ and $z$ and hence $\partial\circ\partial = 0$. □

*Sketch of proof of Theorem 9.4:* By composing with a suitable ambient isotopy assume without loss of generality that $\alpha = \alpha'$. Furthermore assume the isotopy $\{\beta_t\}_{0\le t\le 1}$ with $\beta_0 = \beta$ and $\beta_1 = \beta'$ is generic in the following sense. There exists a finite sequence of pairs $(t_i, z_i) \in [0,1]\times\Sigma$ such that

$$0 < t_1 < t_2 < \cdots < t_m < 1,$$

$\alpha \pitchfork_z \beta_t$ unless $(t,z) = (t_i, z_i)$ for some $i$, and for each $i$ there exists a coordinate chart $U_i \to \mathbb{R}^2 : z \mapsto (\xi,\eta)$ at $z_i$ such that

$$\alpha\cap U_i = \{\eta = 0\}, \qquad \beta_t\cap U_i = \left\{\eta = -\xi^2 \pm (t-t_i)\right\} \tag{13}$$

for $t$ near $t_i$. It is enough to consider two cases. Case 1 is $m = 0$. In this case there exists an ambient isotopy $\phi_t$ such that $\phi_t(\alpha) = \alpha$ and $\phi_t(\beta) = \beta_t$. It follows that the map $\mathrm{CF}(\alpha,\beta) \to \mathrm{CF}(\alpha,\beta')$ induced by $\phi_1 : \alpha\cap\beta \to \alpha\cap\beta'$ is a chain isomorphism that identifies the boundary maps. In Case 2 we have $m = 1$, the isotopy is supported near $U_1$, and (13) holds with the minus sign. This means that there are two intersection points in $U_1$ for $t < t_1$ and none for $t > t_1$. One proves

$$n'(x',y') = n(x',y') + n(x',y)n(x,y') \tag{14}$$

for $x', y' \in \alpha\cap\beta'\setminus\{x,y\}$, where $n(x',y')$ denotes the number of $(\alpha,\beta)$-lunes from $x'$ to $y'$ and $n'(x',y')$ denotes the number of $(\alpha,\beta')$-lunes from $x'$ to $y'$. (Note the similarity with the Cancellation Lemma 7.1.) That $\mathrm{HF}(\alpha,\beta)$ and $\mathrm{HF}(\alpha,\beta')$ are isomorphic follows by the arguments of [5]; see [8] for more details.

Figure 5: Three HMS structures

**Example 9.13.** Let $\Sigma = \partial Y_0 = \partial Y_1$ have genus two and let the embedded loops $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$ form a standard basis of $H_1(\Sigma)$. The embedded loop $\gamma \subset \Sigma$ is homologous to zero in $\Sigma$ and contractible in both handlebodies $Y_0$ and $Y_1$ (see Figure 5). Hence the Dehn twist $\phi : \Sigma \to \Sigma$ along $\gamma$ extends to a diffeomorphism of $Y_1$ and hence the pair $(\alpha, \beta')$ is a trace of the same Heegard splitting of $S^3$. It is algebraically reduced, but not geometrically reduced. Replacing $\phi$ by a diffeomorphism which rotates $\Sigma$ by a half turn on one side of $\gamma$ (i.e. a square root of $\phi$) we obtain a trace $(\alpha, \beta'')$ of the same Heegard splitting, which is not algebraically reduced. (Francois Laudenbach and Denis Auroux showed this example to Dietmar Salamon. It comes from [3].)

# 10 Combinatorial Floer Homology

Let $\alpha$ and $\beta$ be transverse embedded circles in a closed orientable 2-manifold $\Sigma$. Assume that $\Sigma$ has positive genus and that $\alpha$ and $\beta$ are not isotopic and that neither is contractible. Denote by

$$\pi : \mathbb{R}^2 \to \Sigma$$

the universal cover of $\Sigma$.

**Definition 10.1.** An $(\alpha, \beta)$**-pre-lune** is a quadruple

$$\Lambda = (x, y, A, B) \tag{15}$$

that satisfies the following conditions.

**(i)** $x$ and $y$ are distinct intersection points of $\alpha$ and $\beta$.

**(ii)** $A$ is an arc in $\alpha$ from $x$ to $y$ and $B$ is an arc in $\beta$ from $x$ to $y$.

**(iii)** $A$ is homotopic to $B$ with endpoints fixed.

**Proposition 10.2.** *For a smooth* $(\alpha, \beta)$*-lune* $[u]$ *the quadruple*

$$\Lambda_u = (u(-1), u(1), u(\mathbb{R} \cap \mathbb{D}), u(S^1 \cap \mathbb{D})) \tag{16}$$

*is a* $(\alpha, \beta)$*-pre-lune. It is called the* **pre-lune determined by** $[u]$.

*Proof.* As $u$ is an immersion so are $u|\mathbb{R} \cap \mathbb{D}$ and $u|S^1 \cap \mathbb{D}$. One must show that they are injective. See [8]. $\square$

Let $[u]$ be a smooth $(\alpha, \beta)$-lune $[u]$ from $x$ to $y$ and $\tilde{x} \in \pi^{-1}(x)$. Then there is a unique lift $\tilde{u} : \mathbb{D} \to \mathbb{C}$ (i.e. $\pi \circ \tilde{u} = u$) such that $\tilde{u}(-1) = \tilde{x}$. Similarly the $(\alpha, \beta)$-pre-lune (eq:pre-lune) has a lift

$$\tilde{\Lambda} = (\tilde{x}, \tilde{y}, \tilde{A}, \tilde{B})$$

determined uniquely by the choice of $\tilde{x} \in \pi^{-1}(x)$: $\tilde{A}$ and $\tilde{B}$ of $A$ and $B$ which start at $\tilde{x}$. By (iii) these lifts must end at the same point $\tilde{y} \in \pi^{-1}(y)$. For $\tilde{z} \in \mathbb{C} \setminus (\tilde{A} \cup \tilde{B})$ let $\mathrm{w}_{\tilde{\Lambda}}(\tilde{z})$ denote the winding number of the loop $\tilde{A} - \tilde{B}$ about $\tilde{z}$. The **winding number** of the pre-lune (eq:pre-lune) is the locally constant function

$$\mathrm{w}_\Lambda : \Sigma \setminus (A \cup B) \to \mathbb{Z}$$

defined by

$$\mathrm{w}_\Lambda(z) = \sum_{\tilde{z} \in \pi^{-1}(z)} \mathrm{w}_{\tilde{\Lambda}}(\tilde{z}) \tag{17}$$

For $z \in \Sigma \setminus (A \cup B)$. As an immediate consequence of Rouché's Theorem we have

**Proposition 10.3.** *Let $u$ be a smooth $(\alpha, \beta)$-lune. Then the winding number of $\Lambda_u$ counts the number of points in the preimage in the sense that*

$$\mathrm{w}_{\Lambda_u}(z) = \# u^{-1}(z) \tag{18}$$

*for every $z \in \Sigma \setminus (A \cup B)$.*

**Remark 10.4.** The winding number satisfies

$$\mathrm{w}_\Lambda(\gamma(1)) - \mathrm{w}_\Lambda(\gamma(0)) = A \cdot \gamma - B \cdot \gamma. \tag{19}$$

for every smooth curve $\gamma : [0, 1] \to \Sigma$ such that $\gamma(0), \gamma(1) \notin A \cup B$ and $x, y \notin \gamma([0, 1])$. The function $\mathrm{w}_\Lambda$ is uniquely determined by equation (19) and its value at one point. To see this think of the arcs $A$ and $B$ as train tracks. Crossing $A$ increases $\mathrm{w}_\Lambda$ by one if the train comes from the left, and decreases it by one if the train comes from the right. Crossing $B$ decreases $\mathrm{w}_\Lambda$ by one if the train comes from the left and increases it by one if the train comes from the right. Moreover, at each point of $A \cap B \setminus \{x, y\}$, the function $\mathrm{w}_\Lambda$ takes the values $k$, $k+1$, $k$, $k-1$ as we march counterclockwise along a small circle surrounding the intersection point.

Figure 6: Pre-lunes which are not lunes.

**Definition 10.5.** A **combinatorial** $(\alpha, \beta)$**-lune** is an $(\alpha, \beta)$-pre-lune one (and hence all) of whose lifts satisfies the following conditions.

**(I)** The intersection number of $\tilde{A}$ and $\tilde{B}$ at $\tilde{x}$ is $+1$ and at $\tilde{y}$ is $-1$.

**(II)** $\mathrm{w}_{\tilde{\Lambda}}(z) \in \{0, 1\}$ for $z$ sufficiently close to $x$ or $y$.

**(III)** $\mathrm{w}_{\tilde{\Lambda}}(z) \geq 0$ for every $z \in \Sigma \setminus (A \cup B)$.

Note that condition (I) says that the angle from $\tilde{A}$ to $\tilde{B}$ at $\tilde{x}$ is between zero and $\pi$ and the angle from $\tilde{B}$ to $\tilde{A}$ at $\tilde{y}$ is also between zero and $\pi$.

The following theorem provides a solution of a special case of the Picard-Loewner problem; see for example [6] and the references cited therein, e.g. [20, 1, 14]. Our result is a special case because no critical points are allowed, the source is a disc, and the prescribed boundary circle decomposes into two embedded arcs.

**Theorem 10.6.** *If $[u]$ is a smooth $(\alpha, \beta)$-lune, then $\Lambda_u$ is a combinatorial $(\alpha, \beta)$-lune. The map $[u] \mapsto \Lambda_u$ defines a bijection between the set of smooth $(\alpha, \beta)$-lunes and the set of combinatorial $(\alpha, \beta)$-lunes. In particular, for $x, y \in \alpha \cap \beta$, the number (zero or one) of smooth $(\alpha, \beta)$-lunes from $x$ to $y$ is the same as the number of combinatorial $(\alpha, \beta)$-lunes from $x$ to $y$.*

The proof will be found in [8].

# 11 Analytic Floer Homology

Let $\alpha$ and $\beta$ be transverse embedded circles in a closed Riemann surface $\Sigma$ and suppose that $x, y \in \alpha \cap \beta$. An $(\alpha, \beta)$**-strip** from $x$ to $y$ is a holomorphic map $u : \mathbb{S} \to \Sigma$ where

$$\mathbb{S} = \mathbb{R} + i[0, 1]$$

is the infinite strip,

$$u(\mathbb{R}) \subset \alpha, \qquad u(\mathbb{R} + i) \subset \beta,$$

and

$$\lim_{s\to-\infty} u(s+it) = x, \qquad \lim_{s\to\infty} u(s+it) = y,$$

both limits being uniform in $t$.

Let $u : \mathbb{S} \to \Sigma$ be a $(\alpha, \beta)$-strip, let $\mathcal{E}_u = C^\infty(u^*T\Sigma)$ be the set of smooth sections of the pull back bundle, and $\mathcal{B}_u$ be the set of smooth sections $\xi \in C^\infty(u^*T\Sigma)$ which satisfy the boundary conditions

$$\xi(s) \in T_{u(s)}\alpha, \qquad \xi(s+i) \in T_{u(s+i)}\beta$$

for all $s \in \mathbb{R}$. The linear operator $D_u : \mathcal{B}_u \to \mathcal{E}_u$ defined by

$$D_u\xi = \partial_s\xi + j \cdot \partial_t\xi$$

is Fredholm. (Here $j$ is the complex structure on $\Sigma$.)

**Definition 11.1.** A **holomorphic** $(\alpha, \beta)$**-lune** is an equivalence class of $(\alpha, \beta)$-strips of Fredholm index one; two strips being equivalent if they differ by a time shift.

**Theorem 11.2.** *The holomorphic $(\alpha, \beta)$-lunes from $x$ to $y$ is the same as the number of smooth $(\alpha, \beta)$-lunes from $x$ to $y$.*

The proof will be found in [9].

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