Hamiltonian Toric Manifolds

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August 26, 2001

1 Notation Throughout T is a torus, $T_{\mathbf{C}}$ is its complexification,

$$V = L(T)$$

is its Lie algebra, and $\Lambda \subset V$ is the kernel of the exponential map so that

$$T = V/\Lambda, \qquad T_{\mathbf{C}} = V \otimes C/\Lambda.$$

The **standard n-torus** T^n has $V = \mathbf{R}^n$ and $\Lambda = \mathbf{Z}^n$ and any torus is isomorphic to a standard one. Denote by $L(T)^*$ the dual space of $L(T)$ (as a vector space over $\mathbf{R}$).

2 Let (M, ω) a symplectic manifold, and $H : M \to L(T)^*$ a smooth map. Each element $u \in L(T)$ determines a Hamiltonian function $H_u : M \to \mathbf{R}$ via the formula

$$H_u(x) = \langle H(x), u \rangle .$$

Let $X_u \in \mathcal{X}(M)$ be the corresponding Hamiltonian vectorfield on M. A **Hamiltonian T-manifold** is a triple (M, ω, H) such that the map

$$L(T) \to \mathcal{X}(M) : u \mapsto X_u$$

is the derivative of a (necessarily symplectic) group action $T \to \mathrm{Diff}(M)$. The map H is called the **moment map**.

3 Example. Consider the $2n$ dimensional manifold

$$M_0 = T^k \times \mathbf{R}^k \times \mathbf{C}^{n-k}$$

and define the coordinates by

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in T^k, \qquad a = (a_1, a_2 \ldots, a_k) \in \mathbf{R}^k,$$

$$z = (z_{k+1}, z_{k+2}, \ldots, z_n) \in \mathbf{C}^{n-k}.$$

(The α_i are only defined modulo one.) Introduce the symplectic form

$$\omega_0 = \sum_{j=1}^{k} d\alpha_j \wedge da_j + \frac{i}{2} \sum_{j=k+1}^{n} dz_j \wedge d\bar{z}_j.$$

Define $H_0 : M \to \mathbf{R}^n$ by

$$H_0((\alpha, a), z, w) = \left((a_1, \ldots, a_k), \frac{|z_{k+1}|^2}{2}, \ldots, \frac{|z_n|^2}{2} \right).$$

This is the moment map for an action of T^n on M_0 given by

$$\theta \cdot ((\alpha, a), z) = ((\alpha + \theta', a), \exp(\theta'')z)$$

where

$$\theta' = (\theta_1, \theta_2, \ldots, \theta_k), \qquad \theta'' = (\theta_{k+1}, \theta_{k+2}, \ldots, \theta_n)$$

for $\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in T^n$. The image of moment map is

$$H_0(M_0) = \{0\} \times \mathbf{R}^k \times [0, \infty)^{n-k}$$

and the isotropy group of the point $((\alpha, a), z)$ is $\{0\} \times T^I \subset T^k \times T^{n-k}$ where

$$I = \{i : z_i = 0\}, \qquad T^I = \{\theta'' \in T^{n-k} : \theta_i = 0 \text{ for } i \notin I\}.$$

4 (Local Normal Form) Let (M, ω, H) be a Hamiltonian T manifold and $o \in M$. Then there is a neigborhood U of the T orbit To of o in M, a neighborhood U_0 of $(T^k \times 0) \times 0$ in M_0, a diffeomorphism $f : U_0 \to U$ carrying $((0, 0), 0)$ to o, a homomorphism $\eta : T \to T^n$, and a constant $c \in L(T)^*$ such that

$$\omega_0 = f^*\omega, \qquad H \circ f = (\eta^* H_0) + c.$$

Here we denoted by $\eta : L(T) \to L(T^n)$ the derivative of $\eta : T \to T^n$ and by $\eta^* : L(T^n)^* \to L(T)^*$ its adjoint. If the action is effective then η is injective, and for a toric manifold (see below) η is an isomorphism.

5 (AGS Convexity Theorem) For a compact connected Hamiltonian T^n manifold we have

(A_n) The preimage $H^{-1}(p)$ of a point $p \in H(M)$ is connected;

(B_n) The image $H(M)$ is a convex polytope whose vertices are the images under H of the fixed points of the action.

6 A **polytope** in V^* is a compact set $\Delta \subset \mathbf{R}^n$ of form

$$\Delta = \bigcap_{i=1}^{d} \{x \in V^* : \langle x, u_i \rangle \geq c_i\}.$$

The vectors $u_i \in L(T)$ are the normal vectors to the faces of the polytope. A polytope $\Delta \subset V$ is called **Delzant** iff at each vertex the normal vectors of the faces through the vertex may be chosen to be **Z**-basis for Λ. (Thus there are exactly $n = \dim(T)$ faces at each vertex.) A **Hamiltonian toric Manifold** is a compact connected Hamiltonian T manifold M with $\dim(M) = 2\dim(T)$ and such that the T action is effective.

7 Theorem (Delzant) There is a bijective correspondence between Delzant polytopes and Hamiltonian toric manifolds as follows:

(I) If (M, ω, H) is a Hamiltonian toric manifold, the image $H(M)$ of the moment map is a Delzant polytope.

(II) The construction given below assigns to every Delzant polytope Δ a compact connected Hamiltonian toric manifold

$$(M_\Delta, \omega_\Delta, H_\Delta)$$

such that $H_\Delta(M_\Delta) = \Delta$.

(III) If (M, ω, H) is a compact Hamiltonian toric manifold and $\Delta = H(M)$. then is (M, ω, H) isomorphic to $(M_\Delta, \omega_\Delta, H_\Delta)$ in the sense that there is a diffeomorphism $f : M \to M_\Delta$ such that

$$\omega_\Delta = f^*\omega \qquad \text{and} \qquad H_\Delta = H \circ f.$$

8 A Delzant polytope $\Delta \subset L(T)^*$ determines an exact sequence

$$0 \to N \xrightarrow{\iota} T^d \xrightarrow{\pi} T \to 0$$

by the condition that the derivative $\pi : L(T^d) \to L(T)$ satisfies

$$\pi(e_i) = u_i$$

for $i = 1, 2, \ldots, d$ where $e_1, e_2, \ldots, e_d$ are the standard basis for $\mathbf{R}^d = L(T^d)$. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d$ denote the dual basis for $L(T^d)^*$ and $H_0 : \mathbf{C}^d \to L(T^d)^*$ the map given by

$$H_0(z) = \sum_{i=1}^{d} \left(\frac{|z_i|^2}{2} + c_i \right) \varepsilon_i$$

so that H_0 is a moment map for the standard multiplicative action of T^d on $\mathbf{C}^d$. Define

$$H_N = \iota^* \circ H_0, \qquad Z = H_N^{-1}(0).$$

Then $H_N : \mathbf{C}^d \to L(N)^*$ is a moment map for the restriction of the T^d action to $N \subset T^d$.

9 Theorem (Delzant)

(1) *N acts freely on Z.*

(2) *0 is a regular value of H_N so Z is a manifold.*

(3) **(Marsden Weinstein Reduction)** *There is a unique Hamiltonian T^n space $(M_\Delta, \omega_\Delta, H_\Delta)$ such that $M_\Delta = Z/N$ is the orbit space, $\omega_0|Z = q^*\omega_\Delta$ where $q : Z \to M_\Delta$ is the quotient map, and $\pi^* \circ H_\Delta = H_0 \circ q$.*

(4) *The image $H_\Delta(M_\Delta)$ is Δ.*

(5) *M_Δ is compact.*

10 Projective space. Let $d = n+1$ and $e_0, e_1, \ldots, e_n$ be the standard basis for $\mathbf{R}^{n+1}$. Let

$$u_0 = e_0 - e_n, \qquad u_i = e_i - e_{i-1}$$

for $i = 1, 2, \ldots, n$. The $\mathbf{R}$-span of $u_0, u_1, \ldots, u_n$ is the hyperplane $L(T) = \nu^\perp$ where $\nu = e_0 + e_1 + \cdots + e_n$. The vectors $u_0, u_1, \ldots, u_n$ generate a lattice Λ and deleting any u_i gives a $\mathbf{Z}$-basis for Λ. Take $c_0 = -1$ and $c_1 = c_2 = \cdots = c_n = 0$. The inclusion $\iota : N = \mathbf{R}/\mathbf{Z} \to \mathbf{R}^{n+1}/\mathbf{Z}^{n+1}$ sends $1 \in \mathbf{R}$ to $\nu \in \mathbf{R}^{n+1}$. Then $Z = S^{2n+1} \subset \mathbf{C}^{n+1}$, N acts via scalar multiplication, Δ is the standard n-simplex $0 \le x_1 \le x_2 \le \cdots \le x_n \le 1$, and T on acts $M_\Delta = \mathbf{C}P^n$ via diagonal unitary matrices of determinant one. The form ω_Δ is (a multiple of) the Fubini Study form.

11 Each subset $I \subset \{1, 2, \ldots, d\}$ determines a set

$$\mathbf{C}^d_I = \{z \in \mathbf{C}^d : z_i = 0 \iff i \in I\}.$$

The sets $\mathbf{C}^d_I$ are precisely the $T^d_{\mathbf{C}}$ orbits of the standard action. Define

$$F_I = \{x \in \Delta : \langle x, u_i \rangle = c_i \iff i \in I\}.$$

If nonempty, F_I is a face of Δ and it is an open subset of its linear span. Let

$$\mathbf{C}_\Delta = \bigcup_{F_I \neq \emptyset} \mathbf{C}^d_I.$$

12 Theorem (Delzant) *$\mathbf{C}_\Delta$ is an open subset of $\mathbf{C}^d$ on which $N_{\mathbf{C}}$ acts freely. $Z \subset \mathbf{C}_\Delta$ and every $N_{\mathbf{C}}$ orbit in $\mathbf{C}_\Delta$ intersects Z in a N orbit. Hence there is a natural isomorphism*

$$M_\Delta = Z/N \simeq \mathbf{C}_\Delta / N_\Delta.$$

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