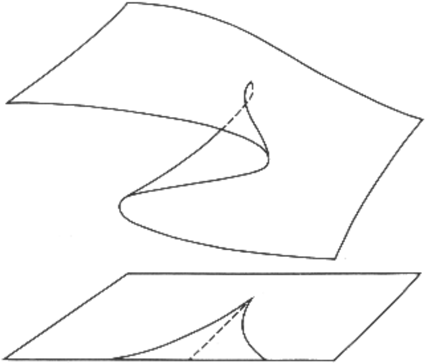
Thom's Catastrophe Theory and Zeeman's model of the Stock Market

Joel W. Robbin

February 19, 2013



The two best things I learned at college.

- Too much of anything is bad; otherwise it wouldn't be too much.

  – Norman Kretzmann

- There are two kinds of people in this world: those who say "There are two kinds of people in this world" and those who don't.

  – William Gelman

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(This has nothing to do with the rest of the talk.)

Catastrophe Theory is ...

Catastrophe theory is a method for describing the evolution of forms in nature. It was invented by René Thom in the 1960's. Thom expounded the philosophy behind the theory in his 1972 book Structural stability and morphogenesis. Catastrophe theory is particularly applicable where gradually changing forces produce sudden effects.

The applications of catastrophe theory in classical physics (or more generally in any subject governed by a 'minimization principle') help us understand what diverse models have in common. The theory has also been applied in the social and biological sciences.

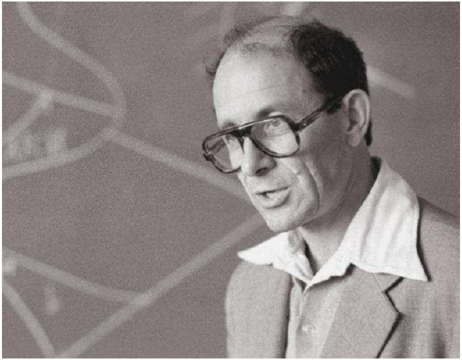
In this talk I will discuss three examples: Zeeman's toy (the "catastrophe machine"), light caustics, and Zeeman's explanation of stock market booms and busts.

René Thom and Christopher Zeeman

Along with his own contributions to the theory and its applications, Christopher Zeeman played St. Paul to Thom's Messiah and roamed the world as a tireless and eloquent expositor.[1] Sir Christopher invented the term "Catastrophe Theory".

[1] From an AMS feature column by Tony Phillips.

Vladimir Igorevich Arnold



Arnold was another major contributor to the subject. He wrote an expository book entitled Catastrophe Theory. It contains a section on the precursors of catastrophe theory: Huygens, de l'Hôpital, Hamilton, Cayley, Jacobi, Poincaré, Andronov and many others.

Of Arnold's *Catastrophe Theory* book Math Reviews said

"This book is named in honour of the theory developed in the 1960s by R. Thom ('the great topologist', to use Arnold's words in the preface) and his followers. The name is taken to include, in present-day terms, singularity theory and bifurcation theory, whether applied to mappings or to dynamical systems, and (very importantly) all the many applications of these disciplines to the world of science. Thus, among the topics treated are bifurcations of equilibrium states, caustics, wavefronts, projections of surfaces, the bypassing of obstacles, symplectic and contact geometry and complex singularities.

...

There is probably no one else in the world who could have written this book. It remains an engrossing summary of a vast body of work which is one of the major achievements of twentieth-century mathematics."

Titles of Some of Zeeman's Catastrophe Theory Papers.

- A catastrophe machine.
- Differential equations for heartbeat and nerve impulse.
- On the unstable behaviour of stock exchanges.
- Duffing's equation in brain modeling.[2]
- Primary and secondary waves in developmental biology.
- A clock and wavefront model for the control of repeated structures during animal morphogenesis (with J. Cooke).
- Euler buckling.
- Stability of ships.
- Conflicting judgements caused by stress.
- A model for institutional disturbances.

(These are all reprinted in E C Zeeman:Catastrophe theory. Selected papers, 1972–1977, Addison-Wesley, 1977. x+675 pp.)

Some Titles of Thom's Works.

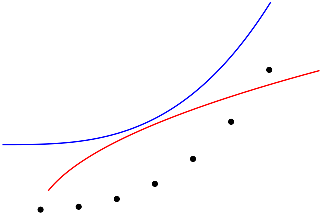
- Structuralism and biology, in *Towards a theoretical biology*, (Edited by C H Waddington) 1975.
- Structural Stability and Morphogenesis, 1975.
- Mathematical Models of Morphogenesis, 1974.
- Semiophysics: A Sketch (Aristotelian Physics and Catastrophe Theory), 1988.

All these books are concerned with biological models of morphogenesis. The last two discuss linguistics.

Catastrophe Theory is Qualitative.

"The use of the term 'qualitative' in science, and above all in physics, has a pejorative ring. It was a physicist who reminded me, not without vehemence, of Rutherford's dictum 'qualitative is nothing but poor quantitative. But consider the following example.' . . ." [3]

- • Experiment
- – Theory A
- – Theory B



Which theory is more accurate ? Better?

[3]René Thom, Structural Stability and Morphogenesis: An outline of a general theory of models (page 4).

Definitions (Informal) from the Theory

Stable Equilibrium (1) A state which depends continuously on the parameters.
(2) An equilibrium such that nearby states remain close as the state evolves.

Catastrophe A sudden change in state.

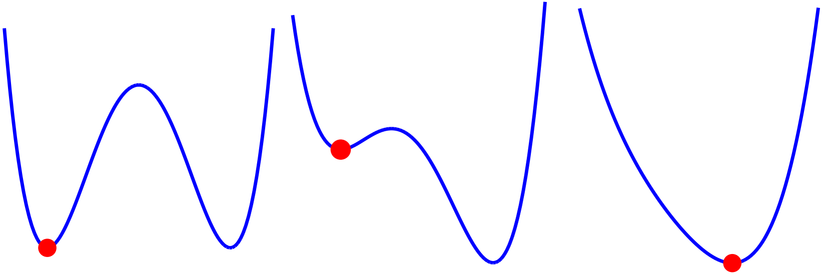
Structural Stability A model is structurally stable if its qualitative behavior is unchanged by small perturbations of the parameters.

Generic A technical term meant to suggest that the property usually holds. For example, the property that a pair (a, b) of real numbers satisfies $4a^3 + 27b^2 \neq 0$ is generic. (The precise definition varies.)

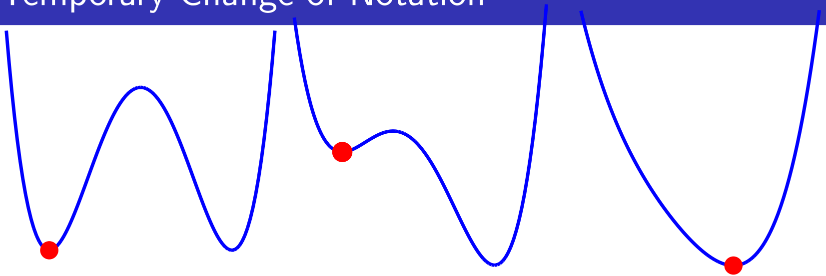
Thom's theory characterizes the catastrophes in structurally stable models and proves that such models are generic. (More later.)

The General Idea

The *state* of a system is described by two kind of variables: *internal variables* $v = (v_1, \ldots, v_m)$ and *control variables* $c = (c_1, \ldots, c_n)$. These are related by a *potential function* $W(c, v)$. When the control variables c have a fixed value the system settles into an equilibrium state where the internal variables v minimize (locally) the function $W(c, \cdot)$. As the control variables vary, a local minimum can disappear and the internal variables jump suddenly to a different equilibrium.



Temporary Change of Notation

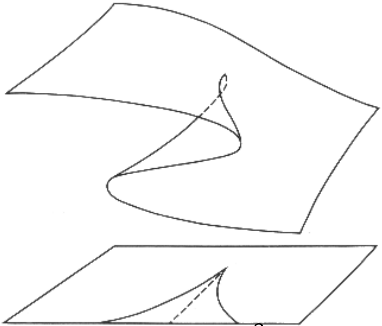


For the rest of this talk we will mostly restrict attention to the case $m = 1$ (one internal variable) and $n = 2$ (two control variables). (The local behavior shown above requires that $m \geq 1$ and $n \geq 2$.)

At the end I will give a more precise statement of some special cases of Thom's theorem. For now we change notation and denote

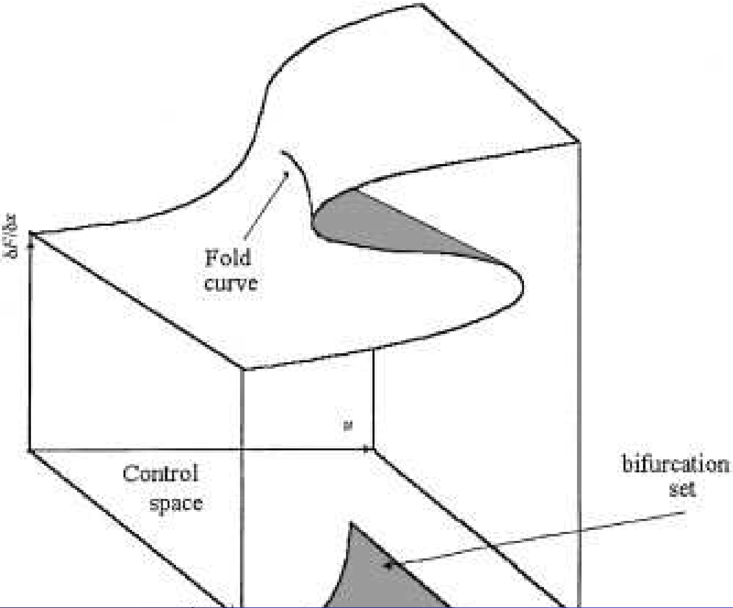
- the internal variable by u (rather than v),
- the control variables by (a, b) (rather than c), and
- the potential function by $V(a, b, u)$ (rather than $W(c, v)$).

The Cusp Catastrophe.



The equilibrium surface has equation $u^3 + au + b = 0$ where (a, b) are coordinates on the control space and the vertical coordinate u is only internal variable. As the control (a, b) varies the state (a, b, u) will be forced to jump to the other sheet when it crosses the fold curve (i.e. the curve over the cusp shown in the next frame).

The Fold Curve and Bifurcation Set (Cusp).



Two python/tkinter Programs.

- (cusp_1.py) In this program the user moves the mouse in the control plane and the corresponding graph of the potential function changes accordingly. As the mouse crosses the cusp curve the shape of the graph changes from two local minima to one (or vice versa).

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- (cusp_2.py) In this program the user changes the position of a point in the control plane by clicking on the buttons. The horizontal line in the control plane represents a line where the coordinate a is constant. The potential function is shown in the middle panel and the right panel represents the slice of the equilibrium surface corresponding to the value of a. The current state of the system is represented in both panels by a red dot.

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Each horizontal slice – except $a = 0$ – is a generic 'one control variable model'. The whole surface is a generic 'two control variable model'.

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Note the hysteresis: crossing the cusp curve in one direction may cause a catastrophe, but immediately reversing direction does not restore the previous state.

Equations for the Cusp Catastrophe.

The potential is

$$V(a, b, u) = \frac{u^4}{4} + \frac{au^2}{2} + bu.$$

The equilibrium surface has equation $\partial V/\partial u = 0$, i.e.

$$u^3 + au + b = 0. \qquad (\dagger)$$

The normal to the surface is vertical when $\partial^2 V/\partial u^2 = 0$ i.e. when

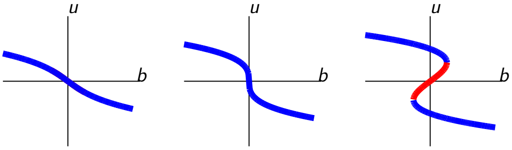
$$3u^2 + a = 0. \qquad (\ddagger)$$

The bifurcation set (cusp) is the critical image of the projection $(a, b, u) \mapsto (a, b)$ from the equilibrium surface onto the control space. The equation of the cusp is

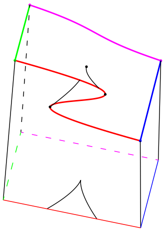
$$4a^3 + 27b^2 = 0.$$

It is obtained by eliminating u from the equations $(\dagger)$ and $(\ddagger)$ for the fold curve.

Three Slices of the Equilibrium Surface $u^3 + au + b = 0$.



$a = 1$ $\qquad$ $a = 0$ $\qquad$ $a = -1$

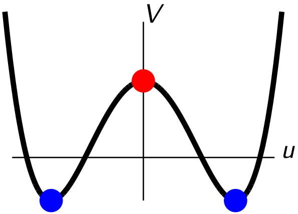


The fold curve (see Frame 14) is the space curve given by the parametric equations

$$u = t, \qquad a = -3t^2, \qquad b = 2t^3.$$

It projects to the bifurcation set (cusp) $4a^3 + 27b^2 = 0$.

The potential $V(a, b, u)$



$$V = \frac{1}{4}u^4 + \frac{1}{2}au^2 + bu$$

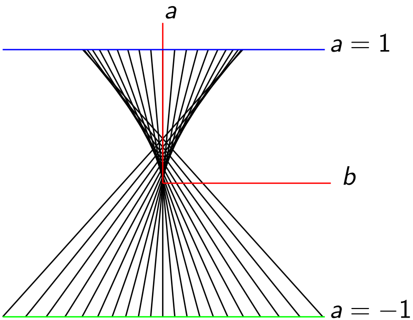
The graph corresponds to the values $a = -2.5$ and $b = 1$. The equilibria are the solutions of $u^3 + au + b = 0$. Since $4a^3 + 27b^2 < 0$ there are three equilibria, two stable and one unstable.

The Cusp is the Envelope of a Family of Lines.

The equilibrium surface $u^3 + au + b = 0$ is ruled. Each real number u determines a line

$$\ell_u = \{(a, b) : u^3 + au + b = 0\}$$

in the (a, b)-plane below a line in the surface. The lines are tangent to the cusp $4a^3 + 27b^2 = 0$.



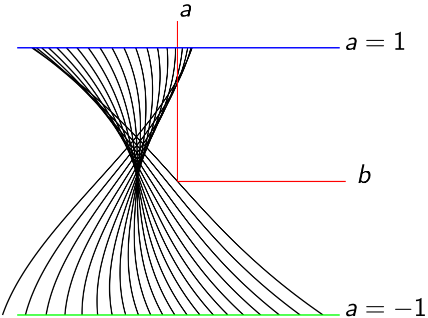
Any Family of Curves has an Envelope.

Let $f(a, b, u)$ be a real valued function of three variables. Each real number u determines a curve

$$\gamma_u = \{(a, b) : f(a, b, u) = 0\}$$

in the (a, b)-plane. The envelope is computed by eliminating u from the equations

$$f(a, b, u) = 0, \qquad \frac{\partial f}{\partial u}(a, b, u) = 0.$$



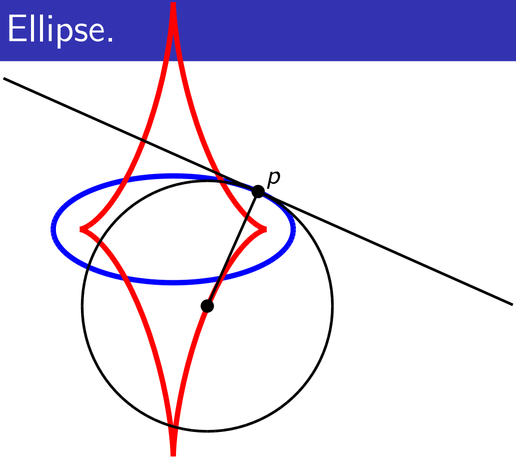
Evolutes Have Cusps. (It all goes back to Huygens)

- The *evolute* of a curve is the locus of its centers of curvature. (The center of curvature of a point on the curve is the center of the circular arc that best approximates the curve near that point.) The perpendicular lines to the curve are tangent to the evolute.
- An *involute* is a curve obtained from a given curve by attaching an imaginary taut string to the given curve and tracing its free end as it is wound onto that given curve.
- The involutes of the evolute of a curve are the parallel curves of the ellipse - including the curve itself.

According to Wikipedia, the notions of the involute and evolute of a curve were introduced by Christiaan Huygens in his work entitled *Horologium oscillatorium sive de motu pendulorum ad horologia aptato demonstrationes geometricae* (1673).

(See http://en.wikipedia.org/wiki/Involute.)

The Evolute of an Ellipse.



The tangent line at the point p is the line which best approximates the curve near p. The circle of curvature at p is the circle which best approximates the curve near p.

See http://www.youtube.com/watch?v=UzDTbKf2I7Q for a movie.

Curvature of an Ellipse.

The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

has parametric equations $x = a \cos t$, $y = b \sin t$. Its curvature function is

$$\kappa = \frac{ab}{(ds/dt)^3}, \qquad \frac{ds}{dt} = (a^2 \sin^2 t + b^2 \cos^2 t)^{1/2}$$

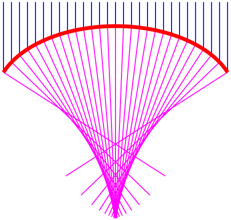
The unit tangent and unit normal are

$$T = \frac{dt}{ds}(-a \sin t, b \cos t), \qquad N = \frac{dt}{ds}(-b \cos t, -a \sin t).$$

The evolute has parametric equations

$$x = \left(\frac{a^2 - b^2}{a}\right) \cos^3 t, \qquad y = \left(\frac{b^2 - a^2}{b}\right) \sin^3 t.$$

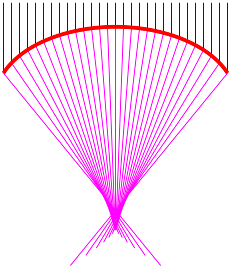
Light Caustics are Cusps.



This cusp is the evolute of an arc of the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$. The vertical rays pass through the arc and leave the arc perpendicular to it. This picture violates Snell's Law because the ratio of the sine of the angle of incidence to the sine of the angle of refraction is not constant.

This Picture Satisfies Both Snell and Fermat.

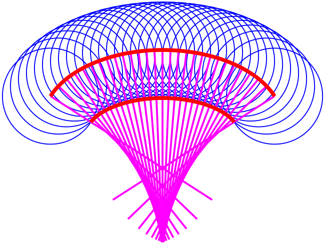
It is qualitatively the same as the last picture.



- Snell's Law: $\dfrac{\sin\theta_1}{\sin\theta_2} = \dfrac{v_1}{v_2} = \dfrac{n_2}{n_1}$
- Fermat's Principle: Light rays between two points (even those passing through the lens) are geodesics: they follow the path of least time.

Snell's Law follows from Fermat's Principle. The index of refraction is $n = c/v$ where c is the speed of light in the vacuum and v is the speed of light in the lens.

The Wave Front.



The involutes of the evolute are the wave fronts. A wave front is the envelope of the circles of a given radius as their centers move along the curve. The light rays might emanate in all directions from the points on the curve but at any instant a point on the corresponding wave front is the end of the light ray that got there first.

Caustics Are Catastrophes.

Let γ be a curve in the plane parameterized by a variable u. For any point $p = (a, b)$ in the (a, b)-plane let

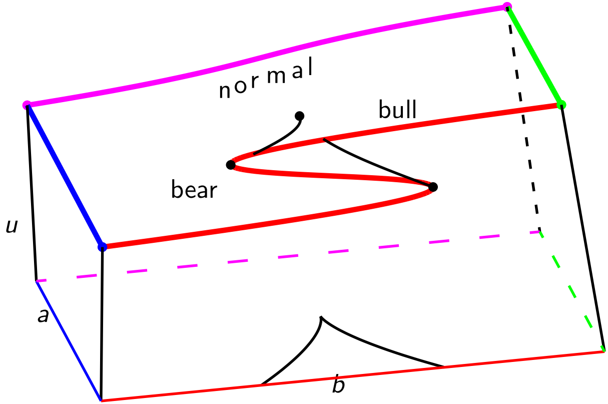
$$V = V(a, b, u)$$

be the distance from p to $\gamma(u)$. If $\gamma(u)$ is the 'closest' point on γ to p then

$$\frac{\partial V}{\partial u}(a, b, u) = 0$$

and the line from p to $\gamma(u)$ is perpendicular to the curve. (The point $\gamma(u)$ need not be closest, but it is 'locally' the closest.)

Zeeman's Model of the Stock Market.



- u = rate of change of Dow Jones Average.
- a = speculative content (as measured by shares held by elves).
- b = excess demand for stock.

Zeeman argues that a feedback mechanism explains why crashes are more common then frenzies.

The Van der Pol Equation.

In some of his publications Zeeman added a system of differential equations to model this feedback. (See the articles on brain modeling and heart beat as well as the one on the stock market.) The equation in control space is a variant of the Van der Pol – Lienard equation.

The Van der Pol equation is the dynamical system

$$\dot{x} = y - c(x^2 - 1), \qquad \dot{y} = -x$$

where $c > 0$. It is the special case $F(x) = c(x^3/3 - x)$ and $g(x) = x$ of the Lienard Equation

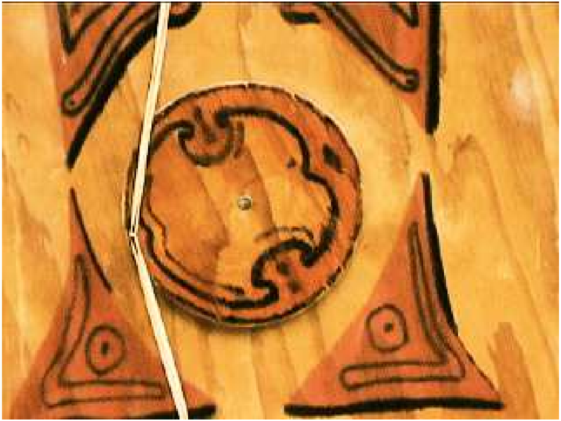
$$\dot{x} = y - F'(x), \qquad \dot{y} = -g(x).$$

Lienard proved that under suitable hypotheses the system has a (necessarily unique) attracting limit cycle. If you add a periodic forcing term you get the equation studied by Dame Mary Cartwright and J.E. Littlewood where they proved it was chaotic.

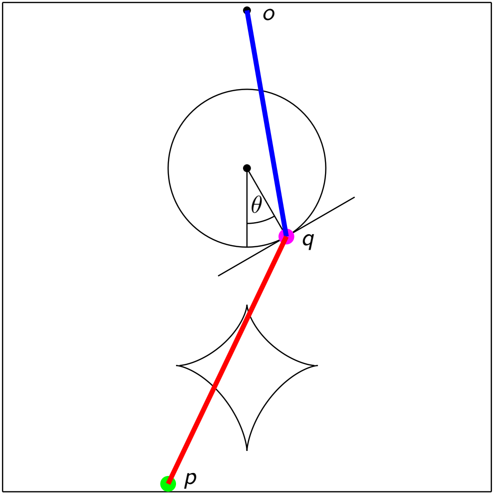
Zeeman's Catastrophe Machine.

This device, invented by Christopher Zeeman consists of a wheel which is tethered by an elastic to a fixed point in its plane. The control input to the system is another elastic attached to the same point as the first and roughly of the same length. The other end of the elastic can be moved about an area diametrically opposite to the fixed point. The particular instantiation of the concept shown in the next frame is about one meter high. The detail shows the way the two elastics are attached to the wheel.

Dr. Zeeman's Original Catastrophe Machine.



The Potential $V(a, b, \theta)$ for the Catastrophe Machine.



$$o = (0, h),$$

$$q = (\sin\theta, -\cos\theta),$$

$$p = (b, a),$$

$$\begin{aligned} V &= V(a, b, \theta) \\ &= V_o(\ell_{oq}) + V_p(\ell_{pq}), \end{aligned}$$

$$V_o,\ V_p\ = ?$$

The potential energy V is the sum of the potential energies $V_o(\ell_{oq})$ and $V_p(\ell_{pq})$ in the two elastics. These energies depend only on the respective extended lengths ℓ_{oq} and ℓ_{pq} of the two elastics.

A Simplified Model of the Zeeman Catastrophe Machine.

I don't know good formulas for the potentials V_o and V_q. However, the form of the potential guarantees that the tangential component of the force vanishes at equilibrium. Postulate that

$$V(a, b, \theta) = \frac{\ell_{oq}^2}{2} + \frac{\ell_{pq}^2}{2} + F(\theta).$$

where F is independent of a and b. If F is nonzero, this destroys the condition that V depends only on the lengths ℓ_{oq} and ℓ_{pq} but now the equipotential curves are straight lines. Postulate further that the derivative $F'(\theta)$ satisfies

$$F'(\theta) \approx (\theta - \theta_0)^3$$

for $\theta \approx \theta_0$ and $\theta_0 = 0, \pi/2, \pi/3\pi/4$. ([4])

[4] I actually used $F'(\theta) = -2\sin(2\theta) \approx -4\theta + 8\theta^3/3$ to draw the picture in the next frame.

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The method of "postulating" what we want has many advantages; they are the same as the advantages of theft over honest toil.

–Bertrand Russell

Introduction to Mathematical Philosophy 1919, p 71.

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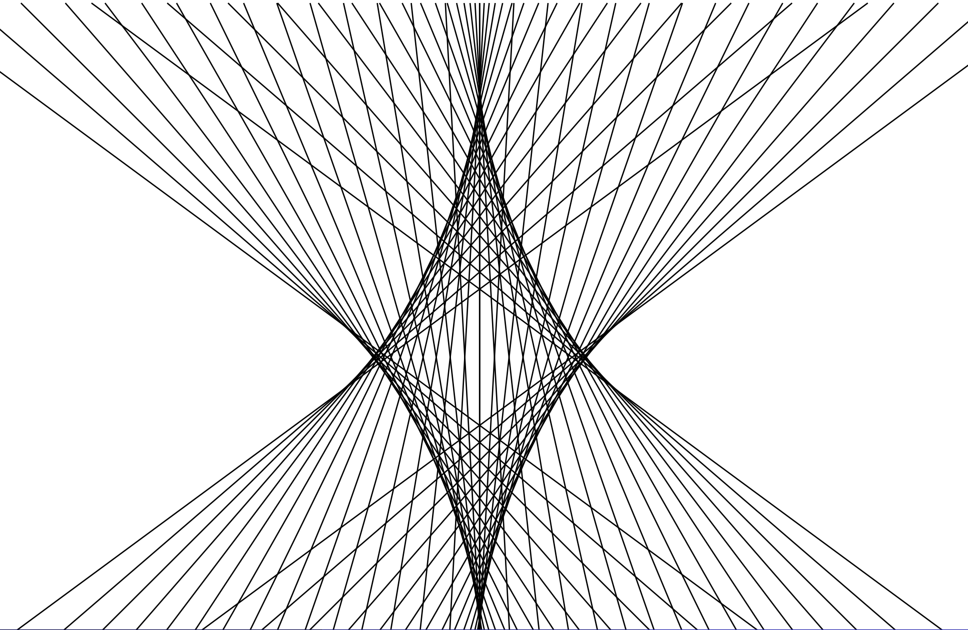
Here's the Bifurcation Set for the Simplified Machine.



A More Precise Formulation of Thom's Theorem.

Now we'll state Thom's theorem in the special case where the number n of control variables is 2. We will assume that the number m of internal variables is arbitrary. We must give mathematically precise definitions of "qualitative behavior" and "generic". (We gave informal definitions in Frame 10.)

The Definition of "Qualitative Behavior is Unchanged".

Recall from Frame 11 that c represents a point in the control space, that v is represents the internal state, and that $W(c, v)$ is the potential.

Definition

Two potential functions $W = W(c, v)$ and $W' = W'(c', v')$ are said to be **equivalent**, if there is a smooth change of variables

$$c' = \gamma(c), \qquad v' = \phi(c, v)$$

and a real valued function $c' = \kappa(c)$ such that

$$W'(c', v') = W(c, v) + \kappa(c).$$

The Definition of Generic.

Think of a potential function $W = W(c, v)$ as a function of c whose value is a function of v. The vector

$$(j^3 W)(c, v) = \left(c, \partial_v^1 V(c, v), \partial_v^2 V(c, v), \partial_v^3 V(c, v)\right)$$

is called the 3-jet of V at (c, v). For example, if $c = (a, b)$, $v = u$, and $V(a, b, u) = \frac{u^4}{4} + \frac{au^2}{2} + bu$, then

$$(j^3 V)(a, b, u) = (a, b, u^3 + au + b, 3u^2 + a, 6u).$$

The space of 3-jets (a, b, p, q, r) has a natural stratification. The strata are the orbits of the jet group. When the number m of internal variables is one, the stratification is

$$S_3 = \{p = q = r = 0\} \subset S_2 = \{p = q = 0\} \subset S_1 = \{p = 0\}.$$

Definition

The potential W is said be **generic** if the map $j^3 W$ is transverse to each stratum of this stratification.

Thom's Theorem (for $n = 2$).

- In a neighborhood of an equilibrium point a generic potential is equivalent to one of the following three forms:

(Nondegenerate) $$W(c, v) = \sum_{j=1}^{m} \pm v_j^2$$

(Fold) $$W(c, v) = \frac{v_1^3}{3} + c_1 v_1 + \sum_{j=2}^{m} \pm v_j^2$$

(Cusp) $$W(c, v) = \frac{v_1^4}{4} + \frac{c_2 v_1^2}{2} + c_1 v_1 + \sum_{j=3}^{m} \pm v_j^2$$

- The generic potentials are **open** (any potential sufficiently close to a generic potential is itself generic) and **dense** (any potential may be approximated by a generic potential).
- A generic potential is **structurally stable** (any potential sufficiently close to the generic potential is equivalent to it).

The Zeeman Catastrophe Machine Again.

Recall that I said that the exact form of the potential in the ZCM doesn't matter too much. (See Frame 32.) Here's why. Let V_0 and V_1 be two models for the ZCM (say defined by different pairs (V_o, V_p) of tension functions) and let

$$V_t = V_0 + t(V_1 - V_0).$$

Then, if each F_t is generic, then V_0 and V_1 are equivalent. (This is an immediate consequence of the definitions, Thom's Theorem, and the compactness of the unit interval.)

What Does It All Mean?

- Is Rutherford ("qualitative is just bad quantitative") right? The geometric optics model can be shown to be a limit of the quantum mechanical model.
- Does a heuristic model have any value?
- In particular, does a heuristic of the stock market have predictive value? (Two problems: What do you measure and the market adjusts.)
- Is science about making models?

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Web Resources on Catastrophe Theory

- The Catastrophe Machine, by Tony Phillips of Stony Brook, (one of the AMS Monthly Feature Columns).
- A set of transparencies from a 1995, San Antonio lecture by E. C. Zeeman himself.
- The Catastrophe Teacher a website of Lucien Dujardin in Lille complete with ingenious applets illustrating several "experiments" with the phenomena, including Zeeman's Catastrophe Machine.
- A lecture on the Lienard Equation by O. Knill of Harvard.
- Edward O. Thorp, Beat the Market: A Scientific Stock Market System, 1967, ISBN 0-394-42439-5 (online version)

Also

- Large portions of this talk were shamelessly cribbed (using cut and paste) from various wikipedia articles.
- The web page for this talk is at http://www.math.wisc.edu/ robbin/catastrophe/. The interactive programs cusp_1.py and cusp_2.py. from Frame 15 may be downloaded there. They use Python3.2 and Tcl downloaded from ActiveState.