The dynamical zeta function

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1. An invertible integer matrix $A \in \mathrm{GL}_n(\mathbb{Z})$ generates a toral automorphism $f : \mathbb{T}^n \to \mathbb{T}^n$ via the formula

$$f \circ \pi = \pi \circ A, \qquad \pi : \mathbb{R}^n \to \mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$$

The set

$$\mathrm{Fix}(f) := \{x \in \mathbb{T}^n : f(x) = x\}$$

of fixed points of f is the kernel of the homomorphism $1 - f : \mathbb{T}^n \to \mathbb{T}^n$. Now $df(x) = A$ for $x \in \mathbb{T}^n$ and, if $1 - df(x)$ is invertible whenever $f(x) = x$, then the fixed points of f are isolated and hence finite in number. In fact

Theorem 2. *If* $\det(I - A) \neq 0$ *then* $\#\mathrm{Fix}(f) = \big|\det(I - A)\big|$.

Lemma 3 (Smith Normal Form[1])**.** *If* $B \in \mathbb{Z}^{m\times n}$ *there exist* $P \in \mathrm{GL}_m(\mathbb{Z})$ *and* $Q \in \mathrm{GL}_n(\mathbb{Z})$ *such that* $B = PDQ$ *where* D *is diagonal, (i.e.* $D_{ij} = 0$ *for* $i \neq j$*) and, where* $d_i := D_{ii}$ *we have* $d_i \geq 0$ *and* d_i *divides* d_{i+1} *for* $i = 1, \ldots, \min(m, n)$*.*

Proof. Do row and column operations to get the gcd of the B_{ij} in the $(1, 1)$ position and zero elsewhere in the first row and column and then induct on the size of the matrix. □

Proof of Theorem 2. By Smith $I - A = PDQ$ so $1 - f = \phi \circ \delta \circ \psi$ where ϕ and ψ are automorphisms and

$$\delta(\pi(x)) = \pi(d_1 x, d_2 x, \ldots, d_n x),$$

[1] This is essentially the fundamental theorem for abelian groups.

Hence $\det(I-A) = \det(P)\det(D)\det(Q) = \pm d_1 \cdots d_n$. Now

$$x \in \ker(\delta) \iff x = \pi\left(\frac{k_1}{d_1}, \ldots, \frac{k_n}{d_n}\right)$$

for some integers k_i with $k_1 = 0, \ldots, k_i - 1$. Because ϕ and ψ are automorphisms, $\#\ker(1-f) = \#\ker(\delta)$. Hence

$$\bigl|\det(I-A)\bigr| = d_1 \cdots d_n = \#\ker(\delta) = \#\ker(1-f) = \#\mathrm{Fix}(f) \qquad \square$$

Definition 4. Let $f : X \to X$ have the property that

$$N_k := \#\mathrm{Fix}(f^k)$$

is finite for all $k = 1, 2, \ldots$. Then the **dynamical zeta function** for f is the formal power series

$$\zeta_f(t) := \exp\left(\sum_{k=1}^{\infty} \frac{N_k t^k}{k}\right).$$

Note that each orbit of f or period k consists of k fixed points of f^k so that $\log \zeta_f(t)$ is the generating function for the number N_k/k of periodic orbits of period k.

Theorem 5. *Let $A \in \{0,1\}^{n\times n}$ and let $\sigma_A : \Sigma_A \to \Sigma_A$ be the corresponding subshift of finite type. Then*

$$\zeta_{\sigma_A}(t) = \det(I - tA)^{-1}.$$

Proof. The (i,j) in A^k is the number of paths in the incidence graph from i to j Hence $N_k = \mathrm{tr}(A^k)$ is the number of closed paths in the incidence graph, i.e. the number if distinct elements of Σ_A which are periodic of period k. Thus

$$\begin{aligned}\zeta(t) &= \exp\left(\sum_{k=1}^{\infty} \frac{\mathrm{tr}(A^k)t^k}{k}\right)\\ &= \exp \mathrm{tr}\left(-\log(I - tA)\right)\\ &= \det\exp(-\log(I-tA))\\ &= \det(I-tA)^{-1}\end{aligned}$$

where we have used the formula $\exp\mathrm{tr}(B) = \det\exp(B)$ (the exponential of the sum of the eigenvalues is the product of the exponentials of the eigenvalues). $\square$

Theorem 6. *Assume that* $N_k = \det(I - A^k)$ *and define*

$$\zeta(t) = \sum_{k=1}^{\infty} \frac{N_k t^k}{k}$$

as above. Then

$$\zeta(t) = \prod_{j=0}^{n} \det\left(I - t\Lambda^j A\right)^{(-1)^{j+1}}.$$

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Proof. For any matrix $B \in \mathbb{R}^{n\times n}$ we have

$$\det(I - B) = \sum_{j=0}^{n} (-1)^j \mathrm{tr}(\Lambda^j B)$$

where $\Lambda^j B : \Lambda^j \mathbb{R}^n \to \Lambda^j \mathbb{R}^n$ is the map on the jth exterior power induced by $B : \mathbb{R}^n \to \mathbb{R}^n$. Note that Λ^j is functorial: $\Lambda^j(B_1 \circ B_2) = (\Lambda^j B_1) \circ (\Lambda^j B_2)$. Now

$$\begin{aligned}
\log \zeta(t) &= \sum_{k=1}^{\infty} \frac{t^k N_k}{k} \\
&= \sum_{k=1}^{\infty} \frac{t^k}{k} \det(I - A^k) \\
&= \sum_{k=1}^{\infty} \frac{t^k}{k} \sum_{j=0}^{n} (-1)^j \mathrm{tr}(\Lambda^j A^k) \\
&= \sum_{j=0}^{n} (-1)^j \mathrm{tr}\left(\sum_{k=1}^{\infty} \frac{t^k (\Lambda^j A)^k}{k}\right) \\
&= \sum_{j=0}^{n} (-1)^j \mathrm{tr}\big(-\log(I - t\Lambda^j A)\big) \\
&= \sum_{j=0}^{n} \mathrm{tr} \log\left(\left(I - t\Lambda^j A\right)^{(-1)^{j+1}}\right) \\
&= \mathrm{tr} \sum_{j=0}^{n} \log\left(\left(I - t\Lambda^j A\right)^{(-1)^{j+1}}\right)
\end{aligned}$$

so exponentiating

$$\zeta(t) = \prod_{j=0}^{n} \det \exp \log \left(I - t\Lambda^j A\right)^{(-1)^{j+1}} = \prod_{j=0}^{n} \det \left(I - t\Lambda^j A)\right)^{(-1)^{j+1}}$$

□

7. Assume $f : M \to M$ is a smooth map of a compact oriented manifold of dimension n and let $\Gamma \subset M \times M$ denote the graph of f and $\Delta \subset M \times M$ denote the diagonal, Then a point $p \in M$ is a fixed point of f (i.e. $f(p) = p$) iff $(p, p) \in \Gamma \cap \Delta$ iff $p \in F^{-1}(\Delta)$. Then the submanifolds Γ and Δ of $M \times M$ intersect transversally at (p, p) if and only if the $I - df(p) : T_pM \to T_pM$ is invertible. In this case we say that the fixed point is **nondegenerate** and define the **fixed point index** of f at p by

$$i(f, p) := \operatorname{sgn} \det(1 - df((p)).$$

Since

$$\det(I - df(p)) = \det \begin{pmatrix} I - df(p) & 0 \\ df(p) & I \end{pmatrix} = \det \begin{pmatrix} I & I \\ df(p) & I \end{pmatrix}$$

we see that $i(f, p) = 1$ if the orientation of $T_{(p,p)}\Gamma$ induced by the graph map and the orientation of $T_{(p,p)}\Delta$ induced by the diagonal map add up to the orientation of $T_{(p,p)}M \times M$ and that $i(f, p) = -1$ otherwise.

Theorem 8 (Lefschetz Fixed Point Theorem)**.** *Assume that f has only nondegenerate fixed points. Then*

$$\sum_{f(p)=p} i(f, p) = \sum_{j=0}^{n} (-1)^j \operatorname{tr}\big(H_j(f) : H_j(M, \mathbb{R}) \to H_j(M, \mathbb{R})\big).$$

Proof. [1] page 129. □

Theorem 9. *Let f be the toral automorphism induced by $A \in \mathrm{GL}_n(\mathbb{Z})$ and assume A is hyperbolic (no eigenvalue on the unit circle) then there is an A invariant splitting $\mathbb{R}^n = E^s \oplus E^u$, an inner product on $\mathbb{R}^n$, and a number $\lambda \in (0, 1)$ such that $\|Av\| \leq \lambda\|v\|$ for $v \in E^s$ and $\|A^{-1}v\| \leq \lambda\|v\|$ for $v \in E^u$. Let $\zeta(t)$ be defined in Theorem 6 and $u = \dim E^u$. Then $\zeta_f(t) = \zeta(t)^{(-1)^u}$ if $A|E^u$ preserves orientation and $\zeta_f(t) = \zeta(t)^{(-1)^{u+1}}$ if $A|E^u$ reverses orientation.*

Proof. For $M = \mathbb{T}^n$ and f the toral automorphism induced by A we have $H_j(M, \mathbb{R}) = \Lambda^j \mathbb{R}^n$ and $H_j(f) = \Lambda^j A$. Since $df(p) = A$ the left hand side of the Lefschetz fixed point formula is $\pm\#\mathrm{Fix}(f)$ where $\pm = \operatorname{sgn}\det(I - A)$ and the right hand side is $\det(I - A) = \sum_j (-1)^j \mathrm{tr}(\Lambda^j A)$. Now $\pm = (-1)^u$ if $\det(A|E^u) > 0$ and $\pm = (-1)^{u+1}$ if $\det(A|E^u) < 0$. Now mimick the proof of Theorem 6 replacing t by $(\pm t)$. □

Definition 10. A diffeomorphism $f : M \to M$ is called **Anosov** iff there is an f invariant splitting $TM = E^s \oplus E^u$, a Riemannian metric on M, and a number $\lambda \in (0, 1)$ such that $\|df(p)v\| \leq \lambda\|v\|$ for $v \in E^s_p$ and $\|df^{-1}(p)v\| \leq \lambda\|v\|$ for $v \in E^u_p$.

Theorem 11. *The zeta function of an orientation preserving Anosov diffeomorphism on a compact orientable manifold is rational.*

Proof. If E^u is orientable and $df|E^u$ preserves orientation the proof is essentially the same as the proof of Theorem 6: read H_j for Λ_j and use that H_j is functorial. The tricky part is to see that at a point p of period k the index $i(f^k, p)$ is depends only on f and the parity of k.[2] (See [5] and the other references.) □

References

[1] R. Bott & L. W. Tu: *Differential Forms in Algebraic Topology*

[2] C. Robinson: *Dynamical Systems*, CRC Press, 1995.

[3] M. Shub: *Global Stability of Dynamical Systems*, Springer 1987.

[4] S. Smale: Differentiable dynamical systems, Bull. Amer. Math. Soc. **73** (1967) 747–817. repreinted in *The Mathematics of Time* , Springer-Verlag 1980, 1–82.

[5] R. Williams: The zeta function of an attractor, in *Conference on the topology of manifolds* (J.C. Hocking Ed), pp 155-161. Prendle Weber Schmidt

[2] Is this true?