The Probability Distribution of the Angle of Rotation

JWR

October 2006

1. Each element $g \in \mathrm{SO}(3)$ is a rotation through an angle $\theta = \Theta(g)$ about some fixed axis. Our objective is to calculate the probability

$$P(\alpha \leq \Theta \leq \beta) := \mu\big(\Theta^{-1}(\alpha, \beta)\big)$$

for $0 \leq \alpha \leq \beta \leq \pi$ where the probability measure μ is the normalized Haar measure on $\mathrm{SO}(3)$, i.e. the unique volume form on G of total mass one which is invariant under the left translations (and therefore also right translations). Now

$$P(\alpha \leq \Theta \leq \beta) = P(\cos\alpha \geq \cos\Theta \geq \cos\beta)$$

so it is enough to calculate the latter.

Lemma 2. *Let*

$$\mathrm{SU}(2) \to \mathrm{SO}(3) : q \mapsto g$$

denote the double cover, i.e. for $q \in \mathrm{SU}(2)$ and $\hat{q} \in T_e\mathrm{SU}(2) = \mathbb{R}^3$ we have

$$g(\hat{q}) = q\hat{q}q^{-1}.$$

Then

$$\cos\Theta(g) = \frac{\mathrm{Tr}(q)^2}{2} - 1$$

where $\mathrm{Tr}(q)$ denotes the trace of q.

Proof. The elements of $\mathrm{SU}(2)$ are the unit quaternions

$$q = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}, \qquad u\bar{u} + v\bar{v} = 1.$$

We can write $q \in \mathrm{SU}(2)$ as

$$q = x_0\sigma_0 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 \qquad (\#)$$

where $u = x_0 + ix_1$, $v = x_2 + ix_3$ and the σ_i are the Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The Lie algebra $T_e\mathrm{SU}(2)$ of SU(2 is the span of $\sigma_1,\, \sigma_2,\, \sigma_3$. For the diagonal matrix

$$q_\phi = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix},$$

the corresponding rotation g_ϕ has axis σ_1 and rotates the span of σ_2 and σ_3 through an angle of $\theta = 2\phi$. Hence

$$\cos\Theta(g_\phi) = \cos(2\phi) = 2\cos^2\phi - 1 = \frac{\mathrm{Tr}(q_\phi)^2}{2} - 1.$$

The lemma follows since every element $q \in \mathrm{SU}(2)$ is conjugate to a diagonal matrix. □

3. Because the Euclidean norm is multiplicative on the quaternions, the volume element on $\mathbb{S}^3 = \mathrm{SU}(2)$ inherited from $\mathbb{R}^4$ is invariant under left and right translations by a unit quaternion and is therefore (up to a normalization) the invariant volume form on SU(2). This volume element is obtained by interior multiplication with the radial vector field and restricting to $\mathbb{S}^3$ and is thus

$$\omega = x_0\, dx_1\, dx_2\, dx_3 - x_1\, dx_0\, dx_2\, dx_3 + x_2\, dx_0\, dx_1\, dx_3 - x_3\, dx_0\, dx_1\, dx_2.$$

The volume of $\mathbb{S}^3$ is $2\pi^2$. Hence for any measurable subset X of SO(3) we have that

$$\mu(X) = \frac{1}{2\pi^2}\int_{\tilde{X}} \omega$$

where $\tilde{X} \subset \mathrm{SU}(2)$ is the preimage of $X \subset \mathrm{SO}(3)$ under the double cover. In the notation of (#) the trace of q is $\mathrm{Tr}(q) = 2x_0$ so

$$\cos\Theta(g) = 2x_0^2 - 1$$

by the lemma. If $q \in \mathbb{S}^3$ is one of the preimage of $g \in \mathrm{SO}(3)$, then the other one is $-q$ so

$$P(b \le \cos\Theta \le a) = \frac{1}{\pi^2}\int_{X(b,a)} \omega \tag{†}$$

for $-1 \le b \le a \le 1$ where

$$X(b,a) := \left\{ (x_0, x_1, x_2, x_3) \in \mathbb{S}^3 : \sqrt{\frac{b+1}{2}} \le x_0 \le \sqrt{\frac{a+1}{2}} \right\}.$$

Note that $X(b,a)$ is a subset of the upper hemisphere $x_0 \ge 0$.

4. To evaluate the integral we parameterize the upper hemisphere by the unit ball $\mathbb{B}^3 \subset \mathbb{R}^3$ via the map $f : \mathbb{B}^3 \to \mathbb{S}^3$ defined by

$$f(x_1, x_2, x_3) = \left(\sqrt{1-\rho^2}, x_1, x_2, x_3\right), \qquad \rho := x_1^2 + x_2^2 + x_3^2.$$

Now

$$f^*(x_0\, dx_1\, dx_2\, dx_3) = \sqrt{1-\rho^2}\, dx_1\, dx_2\, dx_3$$

and

$$f^* dx_0 = -\frac{x_1\,dx_1 + x_2\,dx_2 + x_3\,dx_3}{x_0}$$

so

$$f^*(-x_1\,dx_0\,dx_2\,dx_3) = \frac{x_1^2}{x_0}\,dx_1\,dx_2\,dx_3$$

with similar formulas for $f^*(x_2\,dx_0\,dx_1\,dx_3)$ and $f^*(-x_3\,dx_0\,dx_1\,dx_2)$ so

$$f^*\omega = \left(\sqrt{1-\rho^2} + \frac{\rho^2}{\sqrt{1-\rho^2}}\right) dx_1\,dx_2\,dx_3 = \frac{dx_1\,dx_2\,dx_3}{\sqrt{1-\rho^2}}.$$

In spherical coordinates, $x_1 = \rho\sin\phi\cos\theta,\ x_2 = \rho\sin\phi\sin\theta,\ x_3 = \rho\cos\phi$, we get

$$f^*\omega = \frac{\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}{\sqrt{1-\rho^2}}.$$

Also

$$\begin{aligned} X(b,a) &= \left\{(x_0,x_1,x_2,x_3)\in\mathbb{S}^3 : b \le 1-2\rho^2 \le a\right\} \\ &= \left\{(x_0,x_1,x_2,x_3)\in\mathbb{S}^3 : \sqrt{\frac{1-a}{2}} \le \rho \le \sqrt{\frac{1-b}{2}}\right\}. \end{aligned}$$

Hence by (†) we have

$$P(b \le \cos\Theta \le a) = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_{\sqrt{(1-a)/2}}^{\sqrt{(1-b)/2}} \frac{\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}{\sqrt{1-\rho^2}}$$

This evaluates to

$$P(b \le \cos\Theta \le a) = F(a) - F(b)$$

where

$$F(x) = \frac{\sqrt{1-x^2}}{\pi} - \frac{2}{\pi}\sin^{-1}\left(\sqrt{\frac{1-x}{2}}\right).$$

Since

$$F(\cos\theta) = \frac{\theta - \sin\theta}{\pi}$$

the expression for the probability simplifies to

$$P(\alpha \le \Theta \le \beta) = \frac{\beta - \sin\beta}{\pi} - \frac{\alpha - \sin\alpha}{\pi}.$$