The Central Force Problem in One Dimension

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This is the second order differential equation

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}.$$

Conservation of energy leads to the first order equation

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \frac{GM}{r} + c.$$

We solve this equation by evaluating the integral

$$t = \int \sqrt{\frac{r}{1+r}}\, dr.$$

(We chose units so that $2GM = 2c = 1$.) This amounts to integrating the Abelian differential $s\,dr$ on the algebraic curve

$$s^2 = \frac{r}{1+r}$$

which is the zero set of $F = s^2 + s^2r - r$. This curve is smooth in the affine plane but after the change of variable $s = y/x$ and $r = 1/x$ the curve becomes

$$y^2 + y^2x - x^2 = 0$$

which has a double point at the origin. The curve is of degree three so by the genus formula it has genus zero and is thus rational. To find the parameterization by the Riemann sphere we blow up at $x = y = 0$, i.e. we substitute $y = wx$. We get

$$(w^2 + w^2x - 1)x^2 = 0$$

so the paremeterization is

$$x = \frac{1-w^2}{w^2}, \qquad y = \frac{1-w^2}{w}.$$

We get

$$\int s\,dr = -\int \frac{y\,dx}{x^3} = \int \frac{1-w^2}{w} \cdot \left(\frac{w^2}{1-w^2}\right)^3 \cdot \frac{2\,dw}{w^3} = \int \frac{2w^2\,dw}{(1-w^2)^2}$$

which can be done by partial fractions. The substitution $w = y/x = s = \sqrt{r/(1+r)}$ which we found is exactly what is taught in Math 222.