Real Solutions to Real Recurrences

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28 February 2002

Uche Okpara asked me how explicitly one can solve the recurrence relation

$$G(n) = G(n-1) + G(n-2) + G(n-3)$$

with initial condition

$$G(1) = 2, G(2) = 4, G(4) = 6$$

The characteristic polynomial is

$$f(z) = z^3 - z^2 - z - 1,$$

which has one real root $r = 1.8392...$ and two complex roots. Using Cardan's explicit solution to the cubic [[2], p. 189] one could write down $G(n)$ in terms of these roots.

The complexity of the result leaves something to be desired. However, there is another way to proceed, indicated by Euler in his famous book *Introductio in Analysin Infinitorum* [[1], §217]. The basic idea is that powers of complex numbers in the formula can be replaced by trigonometric functions. Here's what Euler's method leads to in this particular case.

First, let us work with $H(n) = G(n)/2$ and observe that $G(0) = 0$. From the expansion

$$\frac{1}{1 - x - x^2 - x^3} = 1 + x + 2x^2 + 4x^3 + \cdots$$

we deduce that

$$\sum_{n \geq 0} H(n)x^n = \frac{x + x^2}{1 - x - x^2 - x^3}.$$

(Note that the denominator is the reverse of f.) Dividing f by $z - r$ leads to the partial fraction

$$\frac{x + x^2}{1 - x - x^2 - x^3} = \frac{A}{1 - rx} + \frac{Bx + C}{1 + (r-1)x + r^{-1}x^2}.$$

By letting $x \to r^{-1}$ we get

$$A = \frac{r^{-1} + r^{-2}}{1 + (r-1)r^{-1} + r^{-3}} = \frac{r^2}{22} + \frac{3r}{11} - \frac{3}{22}.$$

From the coefficients of 1 and x^2 in the numerator of the generating function we find easily

$$C = -A$$

and

$$B = \frac{-13r^2}{22} + \frac{5r}{11} + \frac{17}{22}.$$

With

$$\rho = \sqrt{r^{-1}}$$

and

$$\phi = \cos^{-1}\left(-\frac{\sqrt{r}(r-1)}{2}\right),$$

we have

$$1 + (r-1)x + r^{-1}x^2 = 1 - 2\rho x \cos\phi + \rho^2 x^2.$$

Euler observes that

$$\frac{a + b\rho x}{1 - 2\rho x \cos\phi + \rho^2 x^2} = \sum_{n \geq 0} \frac{a \sin((n+1)\phi) + b \sin(n\phi)}{\sin\phi} \rho^n x^n.$$

This leads to the desired formula. We have

$$G(n) = 2Ar^n + 2\frac{C \sin((n+1)\phi) + B\rho^{-1} \sin(n\phi)}{\sin\phi} \rho^n.$$

As a check, Maple's floating point arithmetic gives 11.99999999 for $G(4)$.

Let's now look at this from the point of view of algebraic number theory. The standard (powers of roots) formula for $G(n)$ uses values from the splitting field K of f. This is a degree 6 extension of $\mathbf{Q}$, imaginary quadratic over

$\mathbf{Q}(r)$. Euler's solution method requires not only the dominant root r, but the additional algebraic number ρ. One can see that $\rho \notin \mathbf{Q}(r)$, as follows. Modulo $p = 7$, we have

$$f(x) = (x + 4)(x^2 + 2x + 5),$$

so $r \equiv 3 \bmod P$, for a degree 1 prime ideal P lying above 7 in the extension $\mathbf{Q}(r)/\mathbf{Q}$. (Note that 7 doesn't divide the discriminant of f, which is 44.) However, 3 is a quadratic nonresidue mod 7, so the extension $\mathbf{Q}(\rho)/\mathbf{Q}(r)$ must be proper. Since ρ is real, it cannot belong to K as well.

Thus, Euler's method has a price. One does get a "real only" solution, but at the cost of introducing more complicated algebraic numbers than one might at first expect.

References

[1] L. Euler, Introduction to Analysis of the Infinite, Book 1, English trans. by J. D. Blanton, Springer-Verlag 1988.

[2] B. L. van der Waerden, Algebra, 7th edn., vol. 1, New York: Ungar, 1970.