Cross Ratios and Conics *

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Monday January 21, 2002

1. A nonzero vector (x, y, z) determines a point $(x : y : z)$ in the projective plane $\mathbb{P}^2$. A nonzero vector (a, b, c) determines a line

$$[a : b : c] = \{(x : y : z) \in \mathbb{P}^2 : ax + by + cz = 0\}$$

in $\mathbb{P}^2$. We denote the projective space of lines in $\mathbb{P}^2$ by $\mathbb{P}_2$. Three points

$$P_i = (x_i : y_i : z_i)$$

$(i = 1, 2, 3)$ are collinear iff

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$

so the line P_1P_2 through points P_1 and P_2 has equation

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x & y & z \end{vmatrix} = 0,$$

i.e.

$$P_1P_2 = [y_1z_2 - z_1y_2 : z_1x_2 - x_1z_2 : x_1y_2 - y_1x_2].$$

*This document elaborates on some handwritten notes of R. H. Bruck dated May 4, 1978.

2. A **conic** is a set of form

$$[\Phi] = \{(x : y : z) : \Phi(x, y, z) = 0\}$$

where

$$\Phi(x, y, z) = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{12} & \phi_{22} & \phi_{23} \\ \phi_{13} & \phi_{23} & \phi_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

is a nondegenerate homogeneous quadratic form in three variables. We shall denote the quadratic form Φ and the matrix representing it by the same letter. The equation of the tangent line to the conic $[\Phi]$ at the point $(x_0 : y_0 : z_0)$ is $d\Phi(x_0, u_0, z_0)(x, y, z) = 0$, i.e.

$$0 = \begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{12} & \phi_{22} & \phi_{23} \\ \phi_{13} & \phi_{23} & \phi_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

More generally, if

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{12} & \phi_{22} & \phi_{23} \\ \phi_{13} & \phi_{23} & \phi_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

we say that the line $[a : b : c]$ is the **polar** of the point $(x : y : z)$ and the point $(x : y : z)$ is the **pole** of the line $[a : b : c]$ with respect to the conic $[\Phi]$. Thus the polar of a point on the conic is the tangent line at that point.

3. Theorem. *A line* $[a : b : c]$ *is tangent to* $[\Phi]$ *if and only if*

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{12} & \phi_{22} & \phi_{23} \\ \phi_{13} & \phi_{23} & \phi_{33} \end{bmatrix}^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0.$$

Hence the set of tangent lines to a conic in $\mathbb{P}^2$ *is a conic in* $\mathbb{P}_2$.

Proof. A line $[a : b : c]$ is tangent to $[\Phi]$ if and only if

$$\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix} \Phi$$

for some nonzero vector (x_0, y_0, z_0) satisfying $\Phi(x_0, y_0, z_0) = 0$. The theorem is a consequence of the matrix identity $\Phi = \Phi^{-1}\Phi\Phi^{-1}$. (Note that matrix Φ is symmetric; hence Φ^{-1} is also symmetric.) □

4. A conic $\Phi(x,y,z)=0$ passes through the points

$$A=(1:0:0),\qquad B=(0:1:0),\qquad C=(0:0:1),$$

if and only if $\Phi(1,0,0)=\Phi(0,1,0)=\Phi(0,0,1)=0$, i.e. if and only if V has form

$$\Phi(x,y,z)=\alpha yz+\beta xz+\gamma xy.$$

This means that the matrix Φ has vanishing diagonal. A conic is tangent to the lines BC, CA, AB if and only if the inverse of the matrix representing the quadratic form vanishes on the diagonal. The formula for the inverse of a matrix with vanishing diagonal is

$$\begin{bmatrix} 0 & \gamma & \beta \\ \gamma & 0 & \alpha \\ \beta & \alpha & 0 \end{bmatrix}^{-1} = \frac{1}{2\alpha\beta\gamma}\begin{bmatrix} -\alpha^2 & \alpha\beta & \alpha\gamma \\ \alpha\beta & -\beta^2 & \beta\gamma \\ \alpha\gamma & \beta\gamma & -\gamma^2 \end{bmatrix}.$$

5. The **cross ratio** of four points on the projective line is (well) defined by

$$(P_1,P_2;P_3,P_4) := \frac{\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}}{\begin{vmatrix} x_1 & x_4 \\ y_1 & y_4 \end{vmatrix}} \cdot \frac{\begin{vmatrix} x_2 & x_4 \\ y_2 & y_4 \end{vmatrix}}{\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}}$$

where $P_i=(x_i:y_i)$ and the vertical bars denote the determinant function. Note that

$$\begin{aligned}
(P_3,P_4;P_1,P_2) &= (P_1,P_2;P_3,P_4) \\
(P_1,P_2;P_4,P_3) &= 1/(P_1,P_2;P_3,P_4) \\
(P_1,P_3;P_2,P_4) &= 1-(P_1,P_2;P_3,P_4)
\end{aligned}$$

The pairs $\{P_1,P_2\}$ and $\{P_3,P_4\}$ are called **harmonically conjugate** iff $(P_1,P_2;P_3,P_4)=-1$. When $P_1=(1:0)$, $P_2=(0:1)$, $P_3=(x:1)$, $P_4=(x':1)$ we have that $\{P_1,P_2\}$ and $\{P_3,P_4\}$ are harmonically conjugate $\iff x'=-x$.

6. Each pair A, B of distinct points on the line determines an involution T of the line via the formula

$$T(W)=W' \iff (A,B;W,W')=-1;$$

the points A and B are fixed points of T. When $A = (1:0)$ and $B = (0:1)$ the involution is $T(x:y) = (-x:y)$. When the ground field is algebraically closed this defines a bijection between involutions of the line and (unordered) pairs of points on the line.

7. Definition. A **tri-harmonic configuration** is a system

$$\langle A, U, U'; B, V, V'; C, W, W' \rangle$$

of nine points in the projective plane where A, B, C are the vertices of a triangle (i.e. they are not collinear) and

- U and U' lie on the line BC and $(B, C; U, U') = -1$;
- V and V' lie on the line CA and $(C, A; V, V') = -1$;
- W and W' lie on the line AB and $(A, B; W, W') = -1$.

8. Theorem. *For a tri-harmonic configuration, the following are equivalent:*

(8.1) *The lines AU, BV, CW are concurrent;*

(8.2) *The points U', V', W' are collinear;*

(8.3) *There is a conic $[\Phi]$ tangent to the lines AU', BV', CW' at the points A, B, C respectively.*

(8.4) *There is a conic $[\Psi]$ tangent to BC, CA, AB at the points U, V, W respectively.*

We call a tri-harmonic configuration satisfying these conditions **coherent**.

Proof. Assume w.l.o.g. that

$$\begin{aligned}
A &= (1:0:0), & B &= (0:1:0), & C &= (0:0:1), \\
BC &= [1:0:0], & CA &= [0:1:0], & AB &= [0:0:1], \\
U &= (0:u:1), & V &= (1:0:v), & W &= (w:1:0), \\
U' &= (0:-u:1), & V' &= (1:0:-v), & W' &= (-w:1:0).
\end{aligned}$$

We show that each of the conditions (8.1)-(8.4) holds if and only if $uvw = 1$. The equations for the lines AU, BV, CW are

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & u & 1 \\ x & y & z \end{vmatrix} = 0, \qquad \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & v \\ x & y & z \end{vmatrix} = 0, \qquad \begin{vmatrix} 0 & 0 & 1 \\ w & 1 & 0 \\ x & y & z \end{vmatrix} = 0;$$

i.e.

$$AU = [0 : -1 : u], \qquad BV = [v : 0 : -1], \qquad CW = [-1 : w : 0].$$

These three lines are concurrent if and only if

$$uvw - 1 = \begin{vmatrix} 0 & -1 & u \\ v & 0 & -1 \\ -1 & w & 0 \end{vmatrix} = 0.$$

The points U', V', W' are collinear if and only if

$$1 - uvw = \begin{vmatrix} 0 & -u & 1 \\ 1 & 0 & -v \\ -w & 1 & 0 \end{vmatrix} = 0.$$

Let the conic $[\Phi]$ pass through the points A, B, C so that Φ has form

$$\Phi(x, y, z) = \alpha yz + \beta xz + \gamma xy.$$

The lines AU', BV', CW' are tangent to the conic $F = 0$ at the points A, B, C respectively if and only if all three functions

$$\begin{aligned} \Phi(1-t, -tu, t) &= -\alpha ut^2 + \beta(1-t)t - \gamma(1-t)tu, \\ \Phi(t, 1-t, -tv) &= -\alpha v(1-t)t - \beta vt^2 + \gamma t(1-t), \\ \Phi(-tw, t, 1-t) &= \alpha t(1-t) - \beta wt(1-t) - \gamma wt^2, \end{aligned}$$

have a double root at $t = 0$, i.e. if and only if

$$\beta = \gamma u, \qquad \gamma = \alpha v, \qquad \alpha = \beta w.$$

These equations have a nonzero solution (α, β, γ) if and only if $uvw = 1$. Finally, consider a quadratic form

$$\Psi(x, y, z) = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_{12} & \psi_{22} & \psi_{23} \\ \psi_{13} & \psi_{23} & \psi_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

A line is tangent to a conic $[\Psi]$ at one of its points if and only if the restriction of quadratic form Ψ to the line has a double root at that point. The restriction of quadratic form Ψ the lines BC, CA, AB have double roots at U, V, W if an only if there exist numbers λ, μ, ν with

$$\begin{array}{rcl} \psi_{22}y^2 + 2\psi_{23}yz + \psi_{33}z^2 & = & \lambda(y - uz)^2 \\ \psi_{11}x^2 + 2\psi_{13}xz + \psi_{33}z^2 & = & \mu(z - vx)^2 \\ \psi_{11}x^2 + 2\psi_{12}xy + \psi_{22}y^2 & = & \nu(x - wy)^2 \end{array}$$

for all (x, y, z), i.e.

$$\begin{array}{lll} \psi_{22} = \lambda, & \psi_{23} = -\lambda u, & \psi_{33} = \lambda u^2, \\ \psi_{33} = \mu, & \psi_{13} = -\mu v, & \psi_{11} = \mu v^2, \\ \psi_{11} = \nu, & \psi_{12} = -\nu w, & \psi_{22} = \nu w^2. \end{array}$$

These equations have a nontrivial solution Ψ if and only if the equations

$$\lambda = \nu w^2, \qquad \mu = \lambda u^2, \qquad \nu = \mu v^2$$

have a nontrivial solution (λ, μ, ν), i.e. if and only if $(uvw)^2 = 1$. After the normalization $\nu = 1$, the matrix Ψ takes the form

$$\Psi = \begin{bmatrix} \nu & -\nu w & -\mu v \\ -\nu w & \lambda & -\lambda u \\ -\mu v & -\lambda u & \mu \end{bmatrix} = \begin{bmatrix} 1 & -w & -u^2vw^2 \\ -w & w^2 & -uw^2 \\ -u^2w^2v & -uw^2 & u^2w^2 \end{bmatrix}.$$

Using the relation $(uvw)^2 = 1$ we find that

$$\det \Psi = -2u^2w^4(1 + uvw).$$

Hence the quadratic form Ψ is degenerate if $uvw = -1$ and nondegenerate if $uvw = 1$. Thus each of the conditions (8.1)-(8.4) is equivalent to $uvw = 1$ as claimed. □

9. Remark. The conditions of Theorem 8 are all equivalent to the condition $uvw = 1$. Since this condition is unchanged by reversing the signs of two of the factors, it follows that the condition that U, V, and W' be collinear is also equivalent.

10. let ABC be a triangle in $\mathbb{P}^2$. Denote by $\mathbb{P}_2$ the projective space of lines in $\mathbb{P}^2$, The equivalence of (8.1) and (8.2) defines a map

$$\ell : \mathbb{P}^2 \setminus \{A, B, C\} \to \mathbb{P}_2$$

as follows. Given a point $P \in \mathbb{P}^2 \setminus \{A, B, C\}$ let the lines AP, BP, CP intersect BC, CA, AB, in U_P, V_P, W_P respectively. Let U'_P, V'_P, W'_P be the points such that $\langle A, U_P, U'_P; B, V_P, V'_P; C, W_P, W'_P \rangle$ is a tri-harmonic configuration. Define $\ell(P)$ to be the line through U'_P, V'_P, and W'_P.

11. We calculate ℓ under the assumption that $A = (1 : 0 : 0)$, $B = (0 : 1 : 0)$, $C = (0 : 0 : 1)$. Let $P = (x : y : z)$. Then

$$AP = [0 : -z : y], \qquad BP = [z : 0 : -x], \qquad CP = [-y : x : 0],$$

and hence

$$\begin{aligned} U_P &= (0 : y : z), & U'_P &= (0 : y : -z), \\ V_P &= (x : 0 : z), & V'_P &= (-x : 0 : z), \\ W_P &= (x : y : 0), & W'_P &= (x : -y : 0), \end{aligned}$$

so the line containing U'_P, V'_P, W'_P is

$$\ell(P) = [yz : xz : xy].$$

When $xyz \neq 0$ we have

$$\ell(P) = \left[\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right]$$

which shows that ℓ maps $\mathbb{P}^2 \setminus (BC \cup CA \cup AB)$ bijectively onto the set $\mathbb{P}_2 \setminus (A \cup B \cup C)$ of lines not containing the points A, B, C.

12. Theorem. *Let $\langle A, U, U'; B, V, V'; C, W, W' \rangle$ be a coherent tri-harmonic configuration. Denote the point of concurrency in (8.1) by*

$$E = AU \cap BV \cap CW$$

and the common line in (8.2) by

$$L = U'V' = U'W' = V'W'.$$

Then

(12.1) *ℓ maps the points of $[\Phi]$ other than A, B, C bijectively onto the lines through E other than AU, BV, CW;*

(12.2) *ℓ maps the points of L other than U', V', W' to the tangents to $[\Psi]$ other than BC, CA, AB;*

(12.3) *the point E and the line L are pole and polar with respect to both conics $[\Phi]$ and $[\Psi]$.*

Proof. In the notation of the proof of Theorem 8, the point of concurrency in (8.1) is

$$E = (uw : u : 1),$$

the common lines in (8.2) is

$$L = [1 : w : uw],$$

the quadratic form in (8.3) has matrix

$$\Phi = \frac{1}{2}\begin{bmatrix} 0 & \gamma & \beta \\ \gamma & 0 & \alpha \\ \beta & \alpha & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & u \\ 1 & 0 & uw \\ u & uw & 0 \end{bmatrix}$$

where we have normalized so $\gamma = 2$, and the inverse of the matrix Ψ in (8.4) is

$$\Psi^{-1} = -\frac{1}{2u^2w^4}\begin{bmatrix} 0 & uw & w \\ uw & 0 & 1 \\ w & 1 & 0 \end{bmatrix}.$$

The formula

$$\Phi(x, y, z) = \begin{bmatrix} yz & xz & xy \end{bmatrix}\begin{bmatrix} uw \\ u \\ 1 \end{bmatrix}$$

proves (12.1), the formula

$$\begin{bmatrix} 1 & w & uw \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = xyz\begin{bmatrix} x^{-1} & y^{-1} & z^{-1} \end{bmatrix}\begin{bmatrix} 0 & uw & w \\ uw & 0 & 1 \\ w & 1 & 0 \end{bmatrix}\begin{bmatrix} x^{-1} \\ y^{-1} \\ z^{-1} \end{bmatrix}$$

proves (12.2), and the formulas

$$\begin{bmatrix} 1 \\ w \\ uw \end{bmatrix} = \frac{1}{2u}\begin{bmatrix} 0 & 1 & u \\ 1 & 0 & uw \\ u & uw & 0 \end{bmatrix}\begin{bmatrix} uw \\ u \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} uw \\ u \\ 1 \end{bmatrix} = \frac{1}{2w} \begin{bmatrix} 0 & uw & w \\ uw & 0 & 1 \\ w & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ w \\ uw \end{bmatrix}$$

prove (12.3). □