An Integral

JWR

Thursday September 27, 2001

Kepler's third law is

$$T = \frac{2\pi}{\sqrt{GM}} \left(\frac{r_{\min} + r_{\max}}{2} \right)^{3/2}$$

where T is the period of the planet, i.e. the time it takes the planet to go around the sun one time, $r_{\min}$ is the closest it comes to the sun, $r_{\max}$ is the farthest it is from the sun, M is the mass of the sun, and G is the gravitational constant. This is often expressed in words as "the square of the period varies as the cube of the mean distance". Milnor pointed out that calling the quantity

$$a = \frac{r_{\min} + r_{\max}}{2}$$

the mean distance is a misnomer; actually

$$\frac{1}{T} \int_0^T r\, dt = a \left(1 + \frac{e^2}{2} \right)$$

where e is the eccentricity of the orbit. To prove this we use the fact that in suitable polar coordinates (with the sun at the origin) the orbit has equation

$$r = \frac{k}{1 + e \cos \theta}$$

where $k = h^2/GM$ and h is the constant in Kepler's second law $r^2 d\theta/dt = h$. Thus $dt/d\theta = r^2/h$ so the formula for the average value of r is

$$\frac{1}{T} \int_0^T r\, dt = \frac{1}{hT} \int_0^{2\pi} r^3\, d\theta = \frac{k^3}{hT} \int_0^{2\pi} \frac{d\theta}{(1 + e \cos \theta)^3}$$

The integral on the right is a rational function of the trigonometric functions and can be done (as in Math 222) via the substitution $u = \tan(\theta/2)$ but it's not pretty. It is easy to check that $a = k/(1-e^2)$ and then that the constant on the right simplifies to

$$\frac{k^3}{hT} = \frac{a(1-e^2)^{5/2}}{2\pi}.$$

1 Ernesto's Solution

Our aim is to evaluate the integral

$$I_n = \int_0^{2\pi} \frac{d\theta}{(1 + e\cos\theta)^n}$$

where $n = 1, 2, \ldots$ and $0 < e < 1$. This can be written as a line integral

$$I_n = -i \oint \frac{dz}{z\left(1 + \frac{e}{2}(z + \frac{1}{z})\right)^n} = -i \oint \frac{z^{n-1}\,dz}{\left(z + \frac{e}{2}(z^2+1)\right)^n}$$

where the integral is around the unit circle $|z| = 1$ in the counter clockwise direction. We evaluate the integral by residues. The denominator vanishes for $z = p, q$ where

$$p = \frac{-1 + \sqrt{1-e^2}}{e}, \qquad q = \frac{-1 - \sqrt{1-e^2}}{e}.$$

The point q is outside the unit circle so

$$I_n = 2\pi \operatorname*{Res}_{z=p} \frac{z^{n-1}}{\left(z + \frac{e}{2}(z^2+1)\right)^n}.$$

Now

$$\frac{z^{n-1}}{\left(z + \frac{e}{2}(z^2+1)\right)^n} = \frac{2^n z^{n-1}}{e^n (z-p)^n (z-q)^n}$$

so

$$\operatorname*{Res}_{z=p} \frac{z^{n-1}}{\left(z + \frac{e}{2}(z^2+1)\right)^n} = \frac{2^n}{e^n (n-1)!} \left(\frac{d}{dz}\right)^{n-1} \frac{z^{n-1}}{(z-q)^n}\bigg|_{z=p}.$$

Take $n = 3$. We get

$$I_3 = 2\pi \frac{8}{e^3 2} \left(\frac{d}{dz}\right)^2 \frac{z^2}{(z-q)^3}\bigg|_{z=p} = \frac{8\pi}{e^3} \left(\frac{d}{dz}\right)^2 \frac{z^2}{(z-q)^3}\bigg|_{z=p}.$$

By Maple this is

$$I_3 = \frac{\pi\,(2+e^2)}{(1-e^2)^{5/2}}.$$

2 Eric's Solution

I[1] thought some more about the Kepler problem integral

$$\int_0^{2\pi} \frac{d\theta}{(1+e\cos\theta)^3}$$

and realized that it could be computed in a way that gives

$$I_n = \int_0^{2\pi} \frac{d\theta}{(1+e\cos\theta)^n}$$

for $n = 1, 2, 3, \ldots$. Here's the deal.

Expand $(1+e\cos\theta)^{-n}$ using the binomial theorem, and interchange summation and integration. The resulting sum turns out to be a value of the Gaussian hypergeometric function

$${}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; z\right] = 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{c(c+1)}z^2 + \cdots$$

When the smoke clears we get

$$\int_0^{2\pi} \frac{d\theta}{(1+e\cos\theta)^n} = 2\pi\ {}_2F_1\left[\begin{matrix} \frac{n+1}{2} & \frac{n}{2} \\ & 1 \end{matrix}; e^2\right]$$

Now use a transformation law for ${}_2F_1$, namely,

$${}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; z\right] = (1-z)^{-a}{}_2F_1\left[\begin{matrix} a & c-b \\ & c \end{matrix}; \frac{z}{z-1}\right],$$

[1]This section is the email from Eric.

and take b to be whichever of $(n+1)/2$ and $n/2$ is even. The resulting series is finite since $c-b$ is a negative integer, and so I_n is a power of $\sqrt{1-e^2}$ times a polynomial.

For the first four values of n I got

$$\begin{array}{ll} (n=1) & 2\pi/(1-e^2)^{1/2} \\ (n=2) & 2\pi/(1-e^2)^{3/2} \\ (n=3) & 2\pi/(1-e^2)^{5/2}(1+e^2/2) \\ (n=4) & 2\pi/(1-e^2)^{7/2}(1+3e^2/2) \end{array}$$

A good reference fot ${}_2F_1$ is Graham/Knuth/Patashnik's *Concrete Math.*

3 An elementary solution

For $0 < e < 1$ the equation

$$r = \frac{1}{1 + e\cos\theta}$$

is the polar equation for an ellipse with eccentricity e, major axis along the x axis, and a focus at the origin. The semimajor axis $a = (r_{\min} + r_{\max})/2$ of the ellipse, the semiminor axis b, and the distance $2c$ between the foci are given by

$$a = \frac{1}{1-e^2}, \qquad b = \sqrt{a^2 - c^2}, \qquad c = ea.$$

In rectangular coordinates $x = r\cos\theta$ and $y = r\sin\theta$ the equation of the ellipse is

$$\frac{(x+c)^2}{a^2} + \frac{y^2}{b^2} = 1$$

so the ellipse has parametric equations

$$x = -c + a\cos\phi, \qquad y = b\sin\phi, \qquad 0 \le \phi \le 2\pi.$$

We calculate $d\theta/d\phi$. Differentiate the equation

$$-c + a\cos\phi = x = r\cos\theta = \frac{\cos\theta}{1 + e\cos\theta}$$

to get

$$-a\sin\phi \frac{d\phi}{d\theta} = -\frac{\sin\theta}{(1+e\cos\theta)^2} = -yr.$$

substitute $y = b\sin\phi$ and divide by $-a\sin\phi$ to get

$$\frac{d\phi}{d\theta} = \frac{br}{a}$$

and hence

$$\int_0^{2\pi} r^3\,d\theta = \int_0^{2\pi} r^3\,\frac{d\theta}{d\phi}\,d\phi = \frac{a}{b}\int_0^{2\pi} r^2\,d\phi.$$

Using $r^2 = x^2 + y^2$, $a^2 = b^2 + c^2$, and $c = ea$ we have

$$\begin{aligned} r^2 &= c^2 - 2ac\cos\phi + a^2\cos^2\phi + b^2\sin^2\phi \\ &= a^2(1 - 2e\cos\phi + e^2\cos^2\phi) \\ &= a^2\left(1 - 2e\cos\phi + \frac{e^2(1-\cos 2\phi)}{2}\right) \end{aligned}$$

so

$$\int_0^{2\pi} r^3\,d\theta = \frac{a}{b}\int_0^{2\pi} r^2\,d\phi = \frac{a^3 2\pi}{b}\left(1 + \frac{e^2}{2}\right) = \frac{\pi(2+e^2)}{(1-e^2)^{5/2}}.$$