A Musical Mnemonic for Logarithms of Small Integers

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When solving problems on the back of a napkin it is often useful to know approximate logarithms of small integers. Here we indicate a trick for working these out that requires only a little bit of musical knowledge.

The overtone series is familiar to anyone who plays a musical instrument. In particular, knowledge of this series is essential for brass players, because it gives the notes that can be played in open position, with no valves activated or slides pressed. For an (imaginary?) trombone pitched in the key of C, the open (first position) notes would be

Harmonic	Note	N
0th	C2	0
1st	C3	12
2nd	G3	19
3rd	C4	24
4th	E4	28
5th	G4	31
6th	Bb4	34-
7th	C5	36
8th	D5	38
9th	E5	40
10th	F5	41+
11th	G5	43

Here we use notation from Olson [1, page 38]. The third column, N, indicates the number of half steps above the 0th harmonic (fundamental). The sixth and tenth harmonics are audibly off from their notated values, in a direction indicated by the sign.

In the even tempered scale, two tones that differ by a half step have their frequencies in the ratio

$$r = \sqrt[12]{2} = 1.059463\ldots.$$

We can use this information to compute approximate logarithms. We illustrate for ln 3. To get started, we need

$$\ln 2 = 0.69315\ldots \approx 0.7$$

which everyone should remember anyway. The second harmonic has a frequency thrice the fundamental (note that indices have been shifted by 1 from what you would expect). So, to a very good approximation,

$$3 = r^{19} = 2^{19/12}.$$

This implies that

$$\ln 3 = \frac{19 \ln 2}{12} = \frac{19 \times 0.7}{12} = \frac{13.3}{12} = 1.1$$

approximately. Exactly, $\ln 3 = 1.09612\ldots$. Reasoning similarly we can find

$$\ln 5 = \frac{28 \ln 2}{12} = 1.6 \qquad \text{(exactly 1.609438...)}$$

$$\ln 7 = \frac{33.5 \ln 2}{12} = 1.95 \quad \text{(exactly 1.945910...)}$$

$$\ln 11 = \frac{41.5 \ln 2}{12} = 2.4 \quad \text{(exactly 2.397895...)}$$

Combining all of these additively we can get $\ln x$, for any integer $x \leq 12$. For example, $\ln 10$ is about $1.6 + 0.7 = 2.3$.

Here are a couple of continuation projects that would make interesting homework problems.

[For fans of Maynard Ferguson.] The harmonic series can be continued as far as desired, and exceptionally skilled brass players can play an octave above the standard symphony range, which stops at about the 6th harmonic. This can, of course, be used to to compute additional logarithms.

[For percussionists.] The harmonics for most objects are not given by neat integer ratios. Olson [1, pp. 75 ff.] gives then for a variety of clamped bars, streched skins, and so on. One could also convert these into musical notation and obtain logarithms for some other numbers.

References

[1] H. F. Olson, *Music, Physics, and Engineering*, 2nd Edn.,