27 Lines

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This note shows how to use transversality theory to prove that a generic cubic hypersurface in $\mathbb{P}^3$ contains 27 lines.

1. Let $\mathbb{G}(k,n)$ denote the Grassmann manifold of k-dimensional vector subspaces of $\mathbb{C}^n$. It has dimension $k(n-k)$. This manifold has an atlas consisting of $\binom{n}{k}$ charts (α_I, U_I), one for every k element subset I of $N_n = \{1, 2 \ldots, n\}$. These are defined as follows. Let $e_1, e_2, \ldots, e_n$ be the standard basis for $\mathbb{C}^n$, $\mathbb{C}^I$ denote the k-dimensional subspace spanned by $\{e_i : i \in I\}$, $J = N_n \setminus I$, $\mathbb{C}^J$ denote the $(n-k)$-dimensional subspace spanned by $\{e_j : j \in J\}$, and $\mathbb{C}^{J\times I}$ denote the vector space of complex $(n-k)\times k$ matrices viewed as linear maps from $\mathbb{C}^I$ to $\mathbb{C}^J$. The graph

$$\mathrm{Gr}(u) = \{z \in \mathbb{C}^n : z_J = uz_I\}$$

of an element $u \in \mathbb{C}^{J\times I}$ is an element of $\mathbb{G}(k,n)$ and the chart $\alpha_I : U_I \to \mathbb{C}^{J\times I}$ is defined by

$$U_I = \big\{\lambda \in \mathbb{G}(k,n) : \lambda \cap \mathbb{C}^J = \{0\}\big\}, \qquad u = \alpha_I(\lambda) \iff \lambda = \mathrm{Gr}(u).$$

2. For any complex vector space V let $S^k(V)$ denote the vector space of homogeneous complex polynomials of degree k on V. The complex dimension is given by

$$\dim_{\mathbb{C}}\big(S^k(V)\big) = \binom{n+k-1}{n}, \qquad n = \dim(V).$$

3. Let $X = \mathbb{G}(2,4)$ be the Grassmann manifold of 2-dimensional subspaces of $\mathbb{C}^4$, i.e. the space of (projective) lines in CP^3. Let Y be the total space of the vector bundle $Y \to X$ whose fiber over $\lambda \in X = \mathbb{G}(2,4)$ is the 4-dimensional vector space $Y_\lambda = S^3(\lambda)$ and let $W \subset Y$ be the (image of the) zero section of this vector bundle. Each $g \in S^3(\mathbb{C}^4)$ determines a section $F_g : X \to Y$ of $Y \to X$ via restriction: $F_g(\lambda) = g|\lambda$. Let

$$\mathcal{A} = \left\{ g \in S^3(\mathbb{C}^4) : g^{-1}(0) \cap (Dg)^{-1}(0) = \{0\} \right\}.$$

By the Homogeneous Resultant Theorem, $\mathcal{A}$ is the complement of an algebraic variety and is therefore a connected open subset of the 15-dimensional vector space $S^3(\mathbb{C}^4)$. The image of $\mathcal{A}$ in CP^{14} may be identified with the space of nonsingular cubic hypersurfaces in CP^3. Evidently

$$\lambda \in F_g^{-1}(W) \iff \lambda \subseteq g^{-1}(0),$$

i.e. *the cardinality of the set* $F_g^{-1}(W)$ *is precisely the number of (projective) lines in the cubic hypersurface* $\{g = 0\} \subseteq CP^3$.

4. The evaluation map $F : \mathcal{A} \times X \to Y$ given by $F(g, \lambda) = F_g(\lambda)$ is a degree 4 polynomial (linear in g, cubic in λ) when expressed in the above local coordinates. In the local trivialization corresponding to $I = \{1,2\}$ and $J = \{3,4\}$ the formula for F is $F(g,u) = (u, f)$ where

$$f(x_1, x_2) = g(x_1, x_2, u_{31}x_2 + u_{32}x_2, u_{41}x_1 + u_{42}x_2)$$

so that in multi index notation

$$g(x_1, x_2, x_3, x_4) = \sum_{p+q+r+s=3} g_{ijrs} x_1^p x_2^q x_3^r x_4^s, \qquad f(x_1, x_2) = \sum_{i+j=3} f_{ij} x_1^i x_2^j,$$

$$\begin{aligned} f_{30} \quad = \quad & g_{3000} + g_{2010}u_{31} + g_{2001}u_{41} + g_{1020}u_{31}^2 + g_{1011}u_{31}u_{41} + g_{1002}u_{41}^2 \\ & + g_{0030}u_{31}^3 + g_{0021}u_{31}^2 u_{41} + g_{0012}u_{31}u_{41}^2 + g_{0003}u_{41}^3 \end{aligned}$$

etc. It is easy to see that $F \pitchfork W$: to solve $DF(g,u)(\hat{g}, \hat{u}) + (\hat{w}, 0) = (\hat{v}, \hat{f})$ we take $\hat{u} = \hat{v}$, $\hat{w} = 0$, $\hat{g}_{ij00} = \hat{f}_{ij}$, and $\hat{g}_{pqrs} = 0$ if $(r,s) \neq (0,0)$. By the Transversal Density Theorem conclude that $\mathcal{A}_W$ dense in $\mathcal{A}$. By the Transversal Openness Theorem $\mathcal{A}_W$ is an open set. Below we show that $\mathcal{A}_W$ is connected. Hence, by the Transversal Isotopy Theorem, the cardinality of $F_g^{-1}(W)$ is a constant function of $g \in \mathcal{A}_W$.

5. We can reach these same conclusions by arguing as follows. Each $g \in \mathcal{A} \subseteq S^3(\mathbb{C}^4)$ and each two element subset $I \subseteq N_4$ determines a polynomial map $f_I : \mathbb{C}^{2\times 2} \to S^3(\mathbb{C}^2)$ as above. Define the set

$$S_I(g) = \{u \in \mathbb{C}^{2\times 2} : f_I(u) = 0, \quad \det(Df_I(u)) = 0\}.$$

Then

$$\mathcal{A}_W = \{g \in \mathcal{A} : S_I(q) = \emptyset \; \forall I\}.$$

By the Resultant Theorem the set $\mathcal{A}_W$ is the complement of a closed algebraic set in $S^3(\mathbb{C}^4)$ and is hence open dense and connected provided that it is nonempty.

6. Theorem. *The homogeneous polynomial*

$$g(x_1, x_2, x_3, x_4) = x_1^2 + x_2^3 + x_3^3 + x_4^3$$

is an element of $\mathcal{A}_W$. The associated section F_g has exactly 27 zeros.

Proof. For each 2 element subset $I \subseteq N_4$ and each pair (μ, ν) of cube roots of -1 we define the 2 plane $\lambda = \lambda(I, \mu, \nu)$ by

$$\lambda = \{x \in \mathbb{C}^4 : x_r = \mu x_p, \; x_s = \nu x_q\}$$

where $I = \{p, q\}$, $N_4 \setminus I = \{r, s\}$. There are 27 such lines; each is a zero of F_g since $g|\lambda = 0$.

Now take $I = \{1, 2\}$. In the corresponding coordinate system on $\mathbb{G}(2, 4)$ the map $\lambda \mapsto (\lambda, f)$ is given by

$$f_{30} = 1 + u_{31}^3 + u_{41}^3, \qquad f_{21} = 3u_{31}^2 u_{32} + 3u_{41}^2 u_{42},$$

$$f_{12} = 3u_{31} u_{32}^2 + 3u_{41} u_{42}^2, \qquad f_{03} = 1 + u_{32}^3 + u_{42}^3.$$

We first show that $f = 0$ has no solution where all $u_{pq} \neq 0$. Suppose the contrary. From $f_{21} = 0$ and $f_{12} = 0$ we get that $u_{31}/u_{32} = u_{41}/u_{42}$ so that $u_{31} = mu_{41}$ and $u_{32} = mu_{42}$ for some m. From $f_{21} = 0$ we get $(m^3 + 1)u_{41}^2 u_{42} = 0$ so $m^3 + 1 = 0$ so $u_{31}^3 + u_{41}^3 = 0$ which contradicts $f_{30} = 0$. Hence some $u_{pq} = 0$. If (say) $u_{41} = 0$ then from $f_{30} = 0$ we get $u_{31} \neq 0$ so from $f_{21} = 0$ we get $u_{32} = 0$ and $\lambda = \lambda(I, u_{31}, u_{42})$ as required.

To finish the proof we must show that the 4×4 matrix $\partial f/\partial u$ is invertible. Evaluate at the point

$$u = \begin{pmatrix} u_{31} & u_{32} \\ u_{41} & u_{42} \end{pmatrix} = \begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix}.$$

The result is

$$\begin{pmatrix} 3u_{31}^2 & 0 & 3u_{41}^2 & 0 \\ 6u_{31}u_{32} & 3u_{31}^2 & 6u_{41}u_{42} & 3u_{41}^2 \\ 3u_{32}^2 & 6u_{31}u_{32} & 3u_{42}^2 & 6u_{41}u_{42} \\ 0 & 3u_{32}^2 & 0 & 3u_{42}^2 \end{pmatrix} = \begin{pmatrix} 3\mu^2 & 0 & 0 & 0 \\ 0 & 3\mu^2 & 0 & 0 \\ 0 & 0 & 3\nu^2 & 0 \\ 0 & 0 & 0 & 3\nu^2 \end{pmatrix}$$

which is clearly invertible. $\square$