

GIT and μ -GIT

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Many important problems in geometry can be reduced to a partial differential equation of the form

$$\mu(x) = 0,$$

where x ranges over a complexified group orbit in an infinite dimensional symplectic manifold X and $\mu : X \rightarrow \mathfrak{g}$ is an associated moment map. Here we study the finite dimensional version.

Because we want to gain intuition for the infinite dimensional problems, our treatment avoids the structure theory of compact groups. We also generalize from projective manifolds (GIT) to Kähler manifolds (μ -GIT).

- In GIT you start with (X, J, G) and try to find Y with $R(Y) \simeq R(X)^G$.
- In μ -GIT you start with (X, ω, G) and try to solve $\mu(x) = 0$.
- GIT = μ -GIT + rationality.

Kähler manifolds.

A Kähler manifold (X, ω, J) satisfies

$$\omega_x(J_x \hat{x}_1, J_x \hat{x}_2) = \omega_x(\hat{x}_1, \hat{x}_2), \quad |\hat{x}|_x^2 := \omega_x(\hat{x}, J_x \hat{x}) > 0$$

for $x \in X$, $\hat{x}, \hat{x}_1, \hat{x}_2 \in T_x X$. Hence

$$\langle \hat{x}_1, \hat{x}_2 \rangle_x := \omega_x(\hat{x}_1, J_x \hat{x}_2)$$

is a Riemannian metric. A function $H : X \rightarrow \mathbb{R}$ has a **symplectic gradient** $\mathbf{X}_H \in \text{Vect}(X)$ and a **Riemannian gradient** $\nabla H \in \text{Vect}(X)$ characterized by

$$dH(x)\hat{x} = \omega_x(\mathbf{X}_H, \hat{x}) = \langle \nabla H, \hat{x} \rangle_x.$$

They are related by the formula

$$\nabla H = J\mathbf{X}_H.$$

The symplectic gradient is also called the **Hamiltonian vector field**.

Projective manifolds.

Ingredients:

- The complex vector space $V = \mathbb{C}^N$ and the projective space $P(V)$.
- The (frame bundle of the) tautological line bundle $\pi : V \setminus 0 \rightarrow P(V)$.

Any complex submanifold $X \subseteq P(V)$, $V = \mathbb{C}^N$ of projective space is an example of a Kähler manifold. The symplectic form is the restriction to X of the Fubini–Study form

$$\pi^* \omega_{FS} = \partial \bar{\partial} K, \quad K(v) = i\hbar \log |v|.$$

By Chow, a complex submanifold of $P(V)$ is the same as a **projective manifold**, i.e. a smooth projective variety.

In GIT the ingredients are

- A compact subgroup $G \subseteq U_N$ and its complexification $G^c \subseteq GL_N(\mathbb{C})$.
- A G^c invariant closed complex submanifold $X \subseteq P(V)$.
- The homogenous coordinate ring $R(X)$.
- The ring $R(X)^G$ of invariants.

Hilbert showed how to construct a projective variety Y with

$$R(Y) = R(X)^G.$$

A sequence of generators $\varphi_0, \dots, \varphi_n$ of $R(X)^G$ gives an embedding

$$\varphi : P(U) \rightarrow P(\mathbb{C}^n).$$

The set $U \subseteq V$ is the complement of the **null cone**, the set of points in V where all the invariants vanish. (The embedding φ is not defined on this set. If necessary raise the components to suitable powers so that the map is homogeneous.) The closure Y of the image of φ is an algebraic variety. A **syzygy** for φ gives equations for Y .

In μ -GIT the ingredients are

- A compact symplectic manifold (X, ω) .
- A Lie algebra $\mathfrak{g} \subseteq \mathfrak{u}_N$ of G .
- The $\text{ad}(G)$ -invariant inner product $\langle \xi, \eta \rangle = \text{trace}(\xi^* \eta)$ on \mathfrak{g} .
- A Hamiltonian action $\mathfrak{g} \rightarrow \text{Vect}(X) : \xi \mapsto \mathbf{X}_\xi$.
- An equivariant moment map $\mu : X \rightarrow \mathfrak{g}$ for this action.
- The norm squared function $f : X \rightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{2} |\mu(x)|^2$.

That μ is a **moment map** means that the Hamiltonian for \mathbf{X}_ξ is

$$H_\xi := \langle \mu, \xi \rangle, \quad dH_\xi = \omega(\mathbf{X}_\xi, \cdot).$$

That μ is **equivariant** means that $\mu(ux) = u\mu(x)u^{-1}$ for $x \in X$, $u \in G$. This implies

$$\langle \mu(x), [\xi, \eta] \rangle = \omega(\mathbf{X}_\xi(x), \mathbf{X}_\eta(x)) =: \{H_\xi, H_\eta\}$$

for $\xi, \eta \in \mathfrak{g}$.

Kähler Actions.

A Hamiltonian action $\mathfrak{g} \rightarrow \text{Vect}(X) : \xi \mapsto \mathbf{X}_\xi$ on a closed Kähler manifold extends to an action

$$\mathfrak{g}^c \rightarrow \text{Vect}(X) : \xi + \mathbf{i}\eta \mapsto \mathbf{X}_\xi + J\mathbf{X}_\eta$$

by holomorphic vector fields, so the action of G on X extends to a holomorphic action of G^c on X .

Any subgroup of $GL(V) = GL_N$ induces an action on $P(V)$. When $G \subseteq U_N$ and $X \subseteq P(V)$ is a complex G -invariant submanifold,¹ the moment map is

$$\langle \mu(\pi(v)), \xi \rangle_{\mathfrak{g}} = \frac{1}{2} \langle \mathbf{i}\xi v, v \rangle_V, \quad v \in S(V) := S^{2N-1}.$$

¹By Weyl's unitarian trick G -invariance implies G^c -invariance. 

Rationality Theorem.

A Kähler G -manifold is isomorphic to a projective G -manifold if and only if

(A) Integrality of ω . The cohomology class of ω lies in $H^2(X; 2\pi\hbar\mathbb{Z})$.

(B) Integrality of μ . The action integral

$$\mathcal{A}_\mu(x_0, u, \bar{x}) := - \int_{\mathbb{D}} \bar{x}^* \omega + \int_0^1 \langle \mu(u(t)^{-1}x_0), u(t)^{-1}\dot{u}(t) \rangle dt$$

is integral in the sense that

$$\mathcal{A}_\mu(x_0, u, \bar{x}) \in 2\pi\hbar\mathbb{Z}$$

whenever $x_0 \in X$, $u : \mathbb{R}/\mathbb{Z} \rightarrow G$, and $\bar{x} : \mathbb{D} \rightarrow X$ satisfy $\bar{x}(e^{2\pi it}) = u(t)^{-1}x_0$.

(The inner product on \mathfrak{g} satisfies

(C) Integrality of \mathfrak{g} . If $\xi, \eta \in \mathfrak{g}$, $\exp(\xi) = \exp(\eta) = 1$, and $[\xi, \eta] = 0$, then $\langle \xi, \eta \rangle \in \mathbb{Z}$.)

Moreover

Where $\Lambda := \{\xi \in \mathfrak{g} \setminus \{0\} \mid \exp(\xi) = 1\}$ (see §24)

- (i) The action integral $\mathcal{A}_\mu(x, u, v)$ is invariant under homotopy.
- (ii) There is an $N \in \mathbb{N}$ such that $N\alpha = 0$ for every torsion class $\alpha \in H_1(G; \mathbb{Z})$.
- (iii) If $\langle \omega, \pi_2(X) \rangle \subseteq 2\pi\hbar N\mathbb{Z}$ then there is a central element $\tau \in Z(\mathfrak{g})$ such that the moment map $\mu - \tau$ satisfies condition (B).
- (iv) Assume (A), (B), (C), and let $\tau \in Z(\mathfrak{g})$ be a central element, so $\mu + \tau$ is an equivariant moment map. Then $\mu + \tau$ satisfies (B) if and only if $\tau \in 2\pi\hbar\Lambda$.

Note that if ξ in (iii) is replaced by $\xi + \tau$ where $\tau \in Z(\mathfrak{g})$ then the moment map μ is replaced by $\mu + \tau$ and

$$\mathcal{A}_{\mu+\tau}(x_0, u, \bar{x}) = \mathcal{A}_\mu(x_0, u, \bar{x}) + \langle \tau, \xi \rangle.$$

This is because $\tau \in \Lambda$ and $\xi \in \Lambda$ so $\langle \tau, \xi \rangle \in 2\pi\hbar\mathbb{Z}$ by taking $\eta = \tau/2\pi\hbar$ in (C).

The simplest Kähler manifold.

The sphere S^2 is a Kähler manifold. The complex structure, symplectic form, and Riemannian metric are

$$J_q \hat{q} = q \times \hat{q}, \quad \omega_q(\hat{q}_1, \hat{q}_2) = q \cdot (\hat{q}_1 \times \hat{q}_2), \quad \langle \hat{q}_1, \hat{q}_2 \rangle_q = \hat{q}_1 \cdot \hat{q}_2$$

for $q \in S^2$ and $\hat{q}, \hat{q}_1, \hat{q}_2 \in T_q S^2$. With the identification $\mathfrak{so}_3 \simeq (\mathbb{R}^3, \times)$ the action is Hamiltonian with

$$\mathbf{X}_\xi(q) = \xi \times q, \quad \langle \xi, \eta \rangle = \xi \cdot \eta, \quad H_\xi(q) = \langle \mu(q), \xi \rangle = q \cdot \xi.$$

(rotation about ξ). The gradient

$$\nabla H_\xi(q) = -J_q \mathbf{X}_\xi(q) = (\xi \times q) \times q = \xi - (\xi \cdot q)q$$

of H_ξ generates a north pole – south pole flow with poles on the axis of rotation of ξ .

The action of \mathbb{C}^* on S^2

Let $\xi = (0, 0, 1) \in \mathbb{R}^3 \simeq \mathfrak{so}_3$ generate rotation about the z -axis. Then $H_\xi(q) = z$ for $q = (x, y, z) \in S^2$ and the ODE $\dot{q} = \mathbf{X}_\xi(q)$ is

$$\dot{x} = -y, \quad \dot{y} = x, \quad \dot{z} = 0.$$

The gradient vector field is

$$\nabla H_\xi(q) = \xi - (\xi \cdot q)q$$

so the equation $\dot{q} = -\nabla H_\xi(q)$ is

$$\dot{x} = zx, \quad \dot{y} = zy, \quad \dot{z} = z^2 - 1.$$

The solution with $q(0) = q_0$ is

$$x = \frac{2e^t x_0}{z_0 + 1 - (z_0 - 1)e^{2t}}, \quad y = \frac{2e^t y_0}{z_0 + 1 - (z_0 - 1)e^{2t}}, \quad z = \frac{z_0 + 1 + (z_0 - 1)e^{2t}}{z_0 + 1 - (z_0 - 1)e^{2t}}.$$

Another gradient flow on S^2 .

The moment map squared for the S^1 action on S^2 is

$$f(q) = \frac{1}{2}z^2$$

for $q = (x, y, z) \in S^2$. The gradient is

$$\nabla f(q) = \xi - (\xi \cdot q)q$$

where $\xi = (0, 0, z)$ so the ODE

$$\dot{q} = -\nabla f(q)$$

is²

$$\dot{x} = z^2x, \quad \dot{y} = z^2y, \quad \dot{z} = z^3 - z.$$

The equator consists of rest points and the orbits run away from the poles along the meridians towards the equator.

²Check: $x\dot{x} + y\dot{y} + z\dot{z} = z^2x^2 + z^2y^2 + z^4 - z^2 = z^2(x^2 + y^2 + z^2 - 1) = 0$.

The archetypal example.

Let SO_3 act diagonally on $X = \underbrace{S^2 \times \cdots \times S^2}_n$ with moment map

$$\mu(x) := \sum_{i=1}^n q_i, \quad x = (q_1, \dots, q_n) \in X.$$

The preimage $\mu^{-1}(0)$ consists of those x with center of mass at the origin. The moment map squared is

$$f(x) = \frac{1}{2} |\mu(x)|^2, \quad df(x)\hat{x} = \frac{1}{2} \sum_{i \neq j} q_i \cdot \hat{q}_j$$

and negative gradient flow of f is

$$\dot{q}_i = -\nabla f(x) = -\sum_{j \neq i} (q_j - (q_j \cdot q_i)q_i).$$

This example is closely related to the space of binary forms of degree n .

Critical points of f where $\mu \neq 0$.

In the archetypal example we can characterize the critical points using Lagrange multipliers. Since

$$f(x) = \frac{1}{2}|\mu(x)|^2 = \frac{n}{2} + \sum_{i < j} q_i \cdot q_j,$$

we have that $df(x) = 0$ if and only if there exist $\lambda_1, \dots, \lambda_n$ such that

$$\sum_{j \neq i} q_j = \lambda_i q_i$$

for $i = 1, \dots, n$ and when this holds, $\lambda_i = \left(\sum_{j \neq i} q_j \right) \cdot q_i$ so

$$df(x) = 0 \iff \mu(x) = (\lambda_i + 1)q_i.$$

Since $|q_i| = 1$, the critical points x of f where $\mu(x) \neq 0$ are the points x of form $x = (q_1, \dots, q_n)$ where $q_i = \pm p$ for $i = 1, \dots, n$ for some $p \in S^2$.

The moment map squared.

In the Kähler case define $f : X \rightarrow \mathbb{R}$ by

$$f(x) := \frac{1}{2} |\mu(x)|^2$$

satisfies $f^{-1}(0) = \mu^{-1}(0)$ so any $x \in \mu^{-1}(0)$ is a critical point of f (as it is an absolute minimum).

In the projective case for $x = \pi(v)$, $v \in V \setminus 0$ the following are equivalent:

- v minimizes the distance from the orbit $G^c v$ to the origin.
- $x \in \mu^{-1}(0)$.

Proof: Assume w.l.o.g. that $|v| = 1$. The tangent space to the orbit is

$$T_v(G^c v) = \{\zeta v, \zeta \in \mathfrak{g}^c\}, \quad T_v(Gv) = \{\xi v, \xi \in \mathfrak{g}\}.$$

The derivative of the distance is $\hat{v} \mapsto 2 \langle v, \hat{v} \rangle_v$. It vanishes on $T_g(G^c V)$ iff $\langle v, \hat{v} \rangle_v = \langle v, \zeta v \rangle_v = 0$ for all ζ . But for $\zeta = \xi + i\eta$, $\xi, \eta \in \mathfrak{g}$ we have

$$\frac{1}{2} \langle v, i\zeta v \rangle_v = \langle \mu(x), i\xi - \eta \rangle_{\mathfrak{g}^c} = -\langle \mu(x), \eta \rangle_{\mathfrak{g}} = 0$$

if $x \in \mu^{-1}(0)$. For the converse see §20 items (ii) and (iii).

Convergence Theorem.

Every solution of the negative gradient equation

$$\dot{x} = -\nabla f(x), \quad x(0) = x_0$$

converges, i.e. the limit

$$x_\infty := \lim_{t \rightarrow \infty} x(t)$$

exists in X . (Proof: Lojasiewicz.)

The (not quite right) idea is to view f as a G -invariant Morse–Bott function. The stable manifold of the G -invariant set $\mu^{-1}(0)$ is an open dense set in $U \subseteq X$ and the map

$$U \rightarrow \mu^{-1}(0) : x_0 \rightarrow x_\infty$$

gives an isomorphism $U/G^c \simeq \mu^{-1}(0)/G$. (This is like $R(Y) = R(X)^G$.)

The homogeneous space G^c/G .

Equip G^c with the unique left invariant Riemannian metric which agrees with the inner product

$$\langle \xi_1 + \mathbf{i}\eta_1, \xi_2 + \mathbf{i}\eta_2 \rangle_{\mathfrak{g}^c} = \langle \xi_1, \xi_2 \rangle_{\mathfrak{g}} + \langle \eta_1, \eta_2 \rangle_{\mathfrak{g}}$$

on the tangent space \mathfrak{g}^c to G^c at the identity. This Riemannian metric is invariant under the right G -action. Let

$$\pi : G^c \rightarrow G^c/G$$

be the projection onto the right cosets of G . This is a principal G -bundle. The (orthogonal) splitting

$$\mathfrak{g}^c = \mathfrak{g} \oplus \mathbf{i}\mathfrak{g}$$

extends to a left invariant principal connection on π . Projection from the horizontal bundle (i.e. the summand corresponding to $\mathbf{i}\mathfrak{g}$) defines a G^c -invariant Riemannian metric of nonpositive curvature on G^c/G .

The Moment Conjugacy Theorem.

Fix $x \in X$ and define a map $\psi_x : G^c \rightarrow G^c x \subseteq X$ by

$$\psi_x(g) = g^{-1}x.$$

Then there is a function $\Phi_x : G^c \rightarrow \mathbb{R}$ such that ψ_x intertwines the two gradient vector fields $\nabla\Phi_x \in \text{Vect}(G^c)$ and $\nabla f \in \text{Vect}(X)$, i.e.

$$d\psi_x(g)\nabla\Phi_x(g) = \nabla f(\psi_x(g)). \quad (\heartsuit)$$

In particular, ∇f is tangent to the G^c -orbits.

Definition. The function $\Phi_x : G^c \rightarrow \mathbb{R}$ will be called the **lifted Kempf–Ness function** based at x . (It is unique if normalized by the condition $\Phi_x(1) = 0$.) It is G -invariant and hence descends to a function

$$\Phi_x : G^c/G \rightarrow \mathbb{R}$$

denoted by the same symbol and called the **Kempf–Ness function**.

Proof of the Conjugacy Theorem.

Define a vector field $F_x \in \text{Vect}(G^c)$ and a one form α_x on H^c by

$$F_x(g) := -g\mathbf{i}\mu(g^{-1}x), \quad \alpha_x(g)\hat{g} = -\langle \mu(g^{-1}x), \mathfrak{S}(g^{-1}\hat{g}) \rangle$$

for $g \in G^c$, $\hat{g} \in T_g G^c$. Then

Step 1. There is a unique Φ_x such that $\Phi_x(1) = 0$ and

$$d\Phi_x = \alpha_x.$$

Step 2. The map ψ_x intertwines the vectorfields F_x and ∇f .

$$d\psi_x(g)F_x(g) = \nabla f(\psi_x(g)).$$

Step 3. The gradient $\nabla\Phi_x$ of Φ_x is F_x , i.e.

$$\alpha_x(g)\hat{g} = \langle F_x(g), \hat{g} \rangle_g$$

where the inner product on the right is the left-invariant inner Riemannian metric on G^c

The vector field F is horizontal since $\mu(x) \in \mathfrak{g}$ and is right G -equivariant, i.e.

$$F_x(gu) = F_x(g)u$$

for $g \in G^c$ and $u \in G$.

The Kempf–Ness function for projective manifolds.

The Kempf–Ness function Φ_x for a projective manifold $X \subseteq P(V)$ is

$$\Phi_x(g) = \frac{1}{2} (\log |g^{-1}v|_V - \log |v|_V) \quad (\#)$$

for $g \in G^c$ where $x = \pi(v) \in X \subseteq P(V)$.

Proof: Define Φ_x by (#). Then $\Phi_x(1) = 0$ and

$$d\Phi_x(g)\hat{g} = -\frac{\langle g^{-1}\hat{g}g^{-1}v, g^{-1}v \rangle_V}{2|g^{-1}v|_V^2}.$$

The moment map $\mu : X \rightarrow \mathfrak{g}$ is characterized by the formula

$$\langle \mu(y), \eta \rangle_{\mathfrak{g}} = H_{\eta}(y) = \frac{1}{2} \langle \mathbf{i}\eta w, w \rangle_V, \quad y = \pi(w), \quad |w|_V^2 = 1, \quad \eta \in \mathfrak{g}.$$

(See §3.) Let $\zeta = g^{-1}\hat{g} = \xi + \mathbf{i}\eta$ where $\xi, \eta \in \mathfrak{g}$. Then

$$d\Phi_x(g)\hat{g} = -\frac{\langle \zeta g^{-1}v, g^{-1}v \rangle_V}{2|g^{-1}v|_V^2} = -\langle \mathbf{i}\mu(g^{-1}x), \zeta \rangle_{\mathfrak{g}^c} = -\langle \mu(g^{-1}x), \eta \rangle_{\mathfrak{g}^c}$$

and $\eta = \mathfrak{S}(\zeta) = \mathfrak{S}(g^{-1}\hat{g})$ so $d\Phi_x = \alpha_x$.



Properties of the Kempf–Ness function.

- (i) The Kempf–Ness function $\Phi_x : G^c/G \rightarrow \mathbb{R}$ is Morse–Bott (usually Morse).
- (ii) The critical set of Φ_x is a (possibly empty) closed connected submanifold of G^c/G . It is given by

$$\text{Crit}(\Phi_x) = \{ \pi(g) \in G^c/G, \mu(g^{-1}x) = 0 \}.$$

- (iii) If the critical manifold of Φ_x is nonempty, then it consists of the absolute minima of Φ_x and every negative gradient flow line of Φ_x converges exponentially to a critical point.
- (iv) Even if the critical manifold of Φ_x is empty, every negative gradient flow line $\gamma : \mathbb{R} \rightarrow G^c/G$ of Φ_x satisfies

$$\lim_{t \rightarrow \infty} \Phi_x(\gamma(t)) = \inf_{G^c/G} \Phi_x.$$

(The infimum may be minus infinity.)

Stability in symplectic geometry.

A point $x \in X$ is called

- (i) μ -unstable ($x \in X^{\text{us}}$) iff $\overline{G^c x} \cap \mu^{-1}(0) = \emptyset$,
- (ii) μ -semistable ($x \in X^{\text{ss}}$) iff $\overline{G^c x} \cap \mu^{-1}(0) \neq \emptyset$,
- (iii) μ -polystable ($x \in X^{\text{ps}}$) iff $G^c x \cap \mu^{-1}(0) \neq \emptyset$,
- (iv) μ -stable ($x \in X^{\text{s}}$) iff x is μ -polystable and $\mathfrak{g}_x^c = 0$.³

In the archetypal example $x \in X^{\text{us}} \iff$ more than half the q_i coincide and $x \in X^{\text{ps}} \iff$ exactly half the points coincide.

The Moment Limit Theorem. With x_0 and x_∞ as in §15,

- (i) $x_0 \in X^{\text{us}}$ if and only if $\mu(x_\infty) \neq 0$.
- (ii) $x_0 \in X^{\text{ss}}$ if and only if $\mu(x_\infty) = 0$.
- (iii) $x_0 \in X^{\text{ps}}$ if and only if $\mu(x_\infty) = 0$ and $x_\infty \in G^c x_0$.
- (iv) $x_0 \in X^{\text{s}}$ if and only if $\mathfrak{g}_{x_\infty} = 0$.

Moreover, X^{ss} and X^{s} are open subsets of X .

³i.e. the isotropy subgroup $G_{x_\infty}^c$ is discrete.

Stability in algebraic geometry.

A vector $v \in V \setminus 0$ is called

- (i) **unstable** ($v \in V^{\text{us}}$) iff $0 \in \overline{G^c v}$,
- (ii) **semistable** ($v \in V^{\text{ss}}$) iff $0 \notin \overline{G^c v}$,
- (iii) **polystable** ($v \in V^{\text{ps}}$) iff $G^c v = \overline{G^c v}$,
- (iv) **stable** ($v \in V^{\text{s}}$) iff $G^c v = \overline{G^c v}$ and G_V^c is discrete.

In the archetypal example $x \in V^{\text{us}} \iff$ more than half the roots coincide, and $x \in V^{\text{ps}} \iff$ exactly half the roots coincide.

Kempf–Ness Theorem. The two notions of stability agree for projective space in the following sense. If $x \in X \subseteq P(V)$, then

- (i) $x \in X^{\text{us}}$ if and only if $\pi^{-1}(x) \subseteq V^{\text{us}}$.
- (ii) $x \in X^{\text{ss}}$ if and only if $\pi^{-1}(x) \subseteq V^{\text{ss}}$.
- (iii) $x \in X^{\text{ps}}$ if and only if $\pi^{-1}(x) \subseteq V^{\text{ps}}$.
- (iv) $x \in X^{\text{s}}$ if and only if $\pi^{-1}(x) \subseteq V^{\text{s}}$.

The Kempf–Ness Theorem generalized.

For any Kähler G -manifold (X, ω, J, μ) the Kempf–Ness function Φ_x characterizes μ -stability as follows.

- (i) x is μ -unstable $\iff \Phi_x$ is unbounded below.
- (ii) x is μ -semistable $\iff \Phi_x$ is bounded below.
- (iii) x is μ -polystable $\iff \Phi_x$ has a critical point.
- (iv) x is μ -stable $\iff \Phi_x$ is bounded below and proper.

In the archetypal example take $V = \mathbb{C}^{n+1}$. A point $v \in V$ is a binary form

$$v(x, y) = v_0 x^n + v_1 x^{n-1} y + \cdots + v_n y^n.$$

The north pole-south pole flow is

$$(\exp(t\xi)v)(x, y) = v(e^t x, e^{-t} y).$$

The Kempf–Ness function is

$$\Phi_x(e^t) = \log(|v_0 e^n t|^2 + |v_1 e^{(n-2)t}|^2 + \cdots + |v_n e^{-nt}|^2) - \log(|v|^2).$$

Toral elements.

A nonzero element $\zeta \in \mathfrak{g}^c$ is called **toral** iff it satisfies the following equivalent conditions.

- ζ is semi-simple and has purely imaginary eigenvalues.
- The subset $T_\zeta := \overline{\{\exp(t\zeta) \mid t \in \mathbb{R}\}}$ is a torus in G^c .
- The element ζ is conjugate to an element of \mathfrak{g} .

Denote the set of toral elements by

$$\mathcal{T}^c := \text{ad}(G^c)(\mathfrak{g} \setminus \{0\})$$

and also use the abbreviations

$$\Lambda := \{\xi \in \mathfrak{g} \setminus \{0\} \mid \exp(\xi) = 1\}, \quad \Lambda^c := \{\zeta \in \mathfrak{g}^c \setminus \{0\} \mid \exp(\zeta) = 1\}.$$

- The set $\Lambda \cup \{0\}$ intersects the Lie algebra $\mathfrak{t} \subset \mathfrak{g}$ of any maximal torus $T \subseteq G$ in a spanning lattice.
- The generator of any one parameter subgroup $\mathbb{C}^* \rightarrow G^c$ is conjugate to an element of $\Lambda \cap \mathfrak{t}$.

For $x \in X$ and $\xi \in \mathfrak{g} \setminus \{0\}$ the μ -**weight** of the pair (x, ξ) is the real number

$$w_\mu(x, \xi) := \lim_{t \rightarrow \infty} \langle \mu(\exp(it\xi)x), \xi \rangle.$$

In the projective case the μ -weight is

$$w_\mu(x, \xi) = \hbar \max_{v_i \neq 0} \lambda_i$$

for $x = \pi(v)$ where

- $\lambda_1 < \dots < \lambda_k$ are the eigenvalues of $\mathbf{i}\xi$,
- $V_i \subseteq V$ are the corresponding eigenspaces, and
- $v = \sum_{i=1}^k v_i$ with $v_i \in V_i$.

The Hilbert–Mumford criterion.

The μ -weight characterizes μ -stability as follows.

- (i) $x \in X^{\text{us}}$ \iff there exists a $\xi \in \Lambda$ such that $w_\mu(x, \xi) < 0$.
- (ii) $x \in X^{\text{ss}}$ \iff $w_\mu(x, \xi) \geq 0$ for all $\xi \in \Lambda$.
- (iii) $x \in X^{\text{ps}}$ \iff $x \in X^{\text{ps}}$ and $\lim_{t \rightarrow \infty} \exp(it\xi)x \in G^c x$ if $w_\mu(x, \xi) = 0$.
- (iv) $x \in X^s$ \iff $w_\mu(x, \xi) > 0$ for all $\xi \in \Lambda$.

The original Hilbert–Mumford criterion is that $v \in V^{\text{us}}$ \iff there exists an element $\xi \in \Lambda$ such that

$$\lim_{t \rightarrow \infty} \exp(it\xi)v = 0.$$

In the archetypal example take the north pole at the heavy cluster. The center of mass will lie on the line through the north pole and the center of mass of the remaining points.⁴ The center of mass lies on the polar axis, at the origin in the polystable case.

⁴Slogan: The center of mass of the centers of mass is the center of mass.