Surfaces

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These notes summarize the key points in the second chapter of *Differential Geometry of Curves and Surfaces* by Manfredo P. do Carmo. I wrote them to assure that the terminology and notation in my lecture agrees with that text.

1. Notation. Throughout $\mathbf{x} : U \to \mathbb{R}^3$ is a smooth[1] map defined of an open set $U \subseteq \mathbb{R}^2$ in the plane. Usually a typical point of U denoted by $q = (u, v)$ and the components of the map $\mathbf{x}$ are denoted

$$\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)).$$

The **differential** of this map at $q \in \mathbb{R}^2$ is the linear map $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ represented by the matrix of partial derivatives

$$d\mathbf{x}_q = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{pmatrix}$$

evaluated at the point $q = (u, v)$. See do Carmo page 54. On page 84 he introduces the notations

$$\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u} = \begin{pmatrix} \dfrac{\partial x}{\partial u} \\ \dfrac{\partial y}{\partial u} \\ \dfrac{\partial z}{\partial u} \end{pmatrix}, \qquad \mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v} = \begin{pmatrix} \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial v} \\ \dfrac{\partial z}{\partial v} \end{pmatrix}.$$

for the columns of $d\mathbf{x}_q$. Note the inconsistency of notation: in the expression $d\mathbf{x}_q$ the subscript q indicates where the partial derivatives are to be evaluated while in the expressions $\mathbf{x}_u$ and $\mathbf{x}_v$ the subscript indicates which partial derivative is being computed.

[1] For do Carmo the terms *smooth*, *differentiable* and *infinitely differentiable* are synonymous. I prefer the term *smooth*.

2. Definition. A **parameterized surface** is a map $\mathbf{x} : U \to \mathbb{R}^3$ as above. The image $\mathbf{x}(U) \subseteq \mathbb{R}^3$ is called the **trace** and the surface is called **regular** iff the differential $d\mathbf{x}_q$ is one-to-one for all $q \in U$. (See do Carmo page 78.)

3. Remarks. The definition is analogous to the definition of *regular parameterized curve* $\alpha : I \to \mathbb{R}^3$ given on pages 2 and 6 of do Carmo. The condition that $d\mathbf{x}_q$ be one-to-one holds if and only if $\mathbf{x}_u \wedge \mathbf{x}_v \neq 0$ and this is the analog of the regularity condition that $\alpha' \neq 0$. As for curves the real object of study is the trace. The following definitions restrict the trace and also enable us to define surfaces independently from any particular parameterization.

4. Definition. A subset $S \subseteq \mathbb{R}^3$ of $\mathbb{R}^3$ is called a **regular surface** iff for every point $p_0 \in S$ there is an open subset $V \subseteq \mathbb{R}^3$ and a regular parameterized surface $\mathbf{x} : U \to \mathbb{R}^3$ such that $p_0 \in S \cap V$, $\mathbf{x}(U) = S \cap V$, and the map $\mathbf{x}$ is a homeomorphism onto its trace $S \cap V$. The last condition means that the inverse map $\mathbf{x}^{-1} : S \cap V \to U$ is continuous. The map $\mathbf{x} : U \to S \cap V \subseteq \mathbb{R}^3$ is called a **local parameterization** of S and the functions $u, v : S \cap V \to \mathbb{R}$ defined by

$$\mathbf{x}^{-1}(p) = (u(p), v(p)), \qquad p \in S \cap V$$

are called **local coordinates** on S.

5. Change of Parameters Theorem. *Let $\mathbf{x} : U_1 \to S \cap V_1 \subseteq \mathbb{R}^3$ and $\mathbf{y} : U_2 \to S \cap V_2 \subseteq \mathbb{R}^3$ be two local parameterizations and define open subsets U_{12} and U_{21} of $\mathbb{R}^2$ by*

$$U_{12} := \mathbf{x}^{-1}(S \cap V_1 \cap V_2), \qquad U_{21} := \mathbf{y}^{-1}(S \cap V_1 \cap V_2).$$

Then the map $h : U_{12} \to U_{21}$ defined by

$$h(q) = \mathbf{y}^{-1}(\mathbf{x}(q))$$

is a diffeomorphism, i.e. both h and h^{-1} are smooth.

Proof: See do Carmo pages 70-71. □

6. Definition. A subset $C \subseteq \mathbb{R}^3$ of $\mathbb{R}^3$ is called a **regular curve** iff for every point $p_0 \in C$ there is an open subset $V \subseteq \mathbb{R}^3$ and a regular parameterized curve $\alpha : I \to \mathbb{R}^3$ such that $p_0 \in C \cap V$, $\mathbf{x}(I) = C \cap V$, and the map α is a homeomorphism onto its trace $C \cap V$. The map $\alpha \to C \cap V \subseteq \mathbb{R}^3$ is called a **local parameterization** of C. (Recall from Chapter 1 that the condition that α be a *regular parameterized curve* is that $\alpha'(t) \neq 0$ for $t \in I$.)

7. Change of Parameters Theorem for Curves. *Let $\alpha : I_1 \to S \cap V_1 \subseteq \mathbb{R}^3$ and $\beta : I_2 \to S \cap V_2 \subseteq \mathbb{R}^3$ be two local parameterizations of a regular curve C and define open intervals I_{12} and I_{21} of $\mathbb{R}^3$ by*

$$I_{12} := \alpha^{-1}(C \cap V_1 \cap V_2), \qquad I_{21} := \mathbf{y}^{-1}(C \cap V_1 \cap V_2).$$

Then the map $h : I_{12} \to I_{21}$ defined by

$$h(t) = \beta^{-1}(\alpha(t)))$$

is a diffeomorphism, i.e. both h and h^{-1} are smooth.

Proof: This is Exercise 2.3-15 on page 82 of Do Carmo. □

8. Example. Consider the curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\gamma(t) = (\cos t, \sin 2t).$$

The derivative γ' never vanishes and the trace $C = \gamma(\mathbb{R})$ is a figure eight crossing itself at the origin. Let $I_1 = (\pi/2, 5\pi/2)$, $I_2 = (-\pi/2, 3\pi/2)$ and let $\alpha : I_1 \to \mathbb{R}^2$ and $\beta : I_2 \to \mathbb{R}^2$ be the restrictions of γ to the indicated intervals. Then $C = \alpha(I_1) = \beta(I_2)$ and the maps α and β are one-to-one. However there do not exist open intervals I_{12} about about $3\pi/2$ and I_{21} about $\pi/2$ such that $\alpha^{-1} \circ \beta$ is a diffeomorphism. The hypothesis of the previous theorem fails. The inverse map $\alpha^{-1} : C \cap V \to I_1$ is not a homeomorphism onto its image no matter small is the neighborhood V of the origin in $\mathbb{R}^2$

9. Theorem. Let $S \subset \mathbb{R}^3$ be a regular surface and $f : S \to \mathbb{R}$. Then the following are equivalent.

(i) *For every* local parameterization $\mathbf{x} : U \to S \cap V$ the composition $f \circ \mathbf{x} : U \to \mathbb{R}$ is a smooth function.

(ii) For every $p_0 \in S$ *there is* a local parameterization $\mathbf{x} : U \to S \cap V$ with $p_0 \in S \cap V$ such that the composition $f \circ \mathbf{x}$ smooth.

(iii) For every $p \in S$ there is an open set $V \subseteq \mathbb{R}^3$ containing p_0 and a smooth function $F : V \to \mathbb{R}$ such that $F(p) = f(p)$ for $p \in S \cap V$.

(See do Carmo page 72.) A function satisfying these equivalent properties is called **smooth**. A map $f : S \to \mathbb{R}^n$ is called **smooth** iff each of its n components is a smooth function.

10. Regular Values. Let $V \subseteq \mathbb{R}^3$ be open subset and $F : V \to \mathbb{R}$ be a smooth function. A point $p \in V$ is called a **regular point** of F iff the differential

$$dF_p := \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)_p$$

is non zero. (Here the subscript p on the right indicates that the partial derivatives are to be evaluated at p.) A real number $a \in \mathbb{R}$ is called a **regular value** of F iff every point $p \in F^{-1}(a)$ is a regular point of F.

11. Regular Value Theorem. *A subset $S \subseteq \mathbb{R}^3$ is a smooth surface if and only if for every point $p \in S$ there is an open set $V \subseteq \mathbb{R}^3$ and a smooth function $F : V \to \mathbb{R}$ such that $p \in V$, 0 is a regular value of F, and $S \cap V = F^{-1}(0)$.*

Proof. (See do Carmo page 59.) If p is a regular point of F then at least one of the three partial derivatives is non zero at p. The Implicit Function Theorem[2]

[2] Click if reading online.

states that the corresponding variable is a function of the other two in a neighborhood of p. This means that there is a regular parameterization of one of the three forms

$$\mathbf{x}(u,v) = (x(u,v), u, v), \quad \mathbf{y}(u,v) = (u, y(u,v), v), \quad \mathbf{z}(u,v) = (u, v, z(u,v)).$$

Coordinates formed this way are called Monge coordinates. □

12. Remark. It is a theorem (page 114 of do Carmo) that a surface $S \subseteq \mathbb{R}^3$ is of form $S = F^{-1}(0)$ for some smooth $F : V \to \mathbb{R}$ having 0 is a regular value if and only if S is orientable. (See Definition 26 below for the definition of *orientable*.) This theorem requires that $S \subseteq V$ whereas Theorem 11 above is local; it only requires $S \cap V = F^{-1}(0)$. The point is that every surface is "locally orientable", but orientability is a "global condition".

13. Example. The ellipsoid is the set $F^{-1}(0)$ where

$$F(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1.$$

The only point of $\mathbb{R}^3$ which is not a regular point of F is the origin and F does not vanish at the origin. The ellipsoid can be be covered by six graphs, namely

$$\begin{aligned} \mathbf{x}_\pm(u,v) &= (\pm x(u,v), u, v), & x(u,v) &:= |a|\sqrt{1 - b^{-2}u^2 - c^{-2}v^2}, \\ \mathbf{y}_\pm(u,v) &= (u, \pm y(u,v), v), & y(u,v) &:= |b|\sqrt{1 - a^{-2}u^2 - c^{-2}v^2}, \\ \mathbf{z}_\pm(u,v) &= (u, v, \pm z(u,v)), & z(u,v) &:= |c|\sqrt{1 - a^{-2}u^2 - b^{-2}v^2}. \end{aligned}$$

In each case the open set $U \subseteq \mathbb{R}^2$ is defined by the condition that the quantity under the square root sign is positive (this the interior of an ellipse) and the open set $V \subseteq \mathbb{R}^3$ is the half space where the corresponding coordinate is either positive or negative as appropriate.

14. Definition. Let $S \subseteq \mathbb{R}^3$ be a regular surface and $p \in S$. The **tangent vector** to S at p is a vector $\alpha'(0)$ where $\alpha : I \to \mathbb{R}^3$ is a smooth curve such that $\alpha(I) \subseteq S$, $0 \in I$, and $\alpha(0) = p$. The space of all tangent vectors to S at p is denoted by T_pS and called the **tangent space** to S at p. (See do Carmo page 83.)

15. Theorem. *Let $\mathbf{x} : U \to S \cap V \subseteq \mathbb{R}^3$ be a local parameterization of a smooth surface S, $q \in U$, and $p = \mathbf{x}(q) \in S$. Then*

$$T_pS = d\mathbf{x}_q(\mathbb{R}^2),$$

i.e. the tangent space is the image of the differential $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$.

16. Remark. I prefer to call T_pS the **tangent space** and the translate $p + T_pS$ the **tangent plane**. The tangent space is a vector space; the tangent plane is not. On page 83 do Carmo writes "the plane $d\mathbf{x}_q(\mathbb{R}^2)$ which passes through $p = \mathbf{x}(q) \ldots$". This is incorrect as usually $p \notin d\mathbf{x}_q(\mathbb{R}^2)$. Of course, the point $p = p + 0$ lies in the tangent plane $p + T_pS$.

17. Maps between surfaces. Let $S_1, S_2 \subseteq \mathbb{R}^3$ be regular surfaces, and

$$\varphi : S_1 \to S_2$$

be a smooth map, i.e. each of its three components is a smooth function as in Theorem 9 above. An equivalent condition is that the map φ is smooth in local coordinates, i.e. for every point $p \in S$ and every local parameterization $\mathbf{y} : U_2 \to S_2 \cap V_2$ with $\varphi(p) \in S_2 \cap V_2$ there is a local parameterization $\mathbf{x} : U_1 \to S_1 \cap V_1$ such that $\varphi(U_1) \subseteq U_2$ and the map $\mathbf{y}^{-1} \circ \varphi \circ \mathbf{x} : U_1 \to U_2$ is a smooth map from the open set $U_1 \subseteq \mathbb{R}^2$ to the open set $U_2 \subseteq \mathbb{R}^2$. When $\varphi : S_1 \to S_2$ is smooth and $\alpha : I \to S_1$ is a curve in S_1 with $\alpha(0) = p$, then $\varphi \circ \alpha : I \to S_2$ is a curve with $(\varphi \circ \alpha)(0) = \varphi(p)$ so the differential

$$d\varphi_p : T_pS_1 \to T_{\varphi(p)}S_2$$

is a linear map from the tangent space to S_1 at p to the tangent space to S_2 at $\varphi(p)$. A map $\varphi : S_1 \to S_2$ is called a **diffeomorphism** iff φ is one-to-one and onto and both maps φ and φ^{-1} are smooth.

18. Inverse Function Theorem. *The differential $d\varphi_p : T_pS_1 \to T_{\varphi(p)}S_2$ is an invertible linear map if and only if f is a local diffeomorphism at p, i.e. if and only if there are open sets $S_1 \cap V_1$ and $S_2 \cap V_2$ such that $p \in S_1 \cap V_1$, $\varphi(p) \in S_2 \cap V_2$, $\varphi(S_1 \cap V_1) = S_2 \cap V_2$, and the map $\varphi : S_1 \cap V_1 \to S_2 \cap V_2$ is a diffeomorphism.*

Proof: In other words, for all $w_2 \in T_pS_2$ the equation $d\varphi_p(w_1) = w_2$ has a unique solution $w_1 \in T_pS_1$ if and only if for all $p_2 \in S_2$ near $\varphi(p)$ the equation $p_2 = \varphi(p_1)$ has a unique solution $p_1 \in S_1$ near p. A special case is where $S_1 = U_1$ and $S_2 = U_2$ are open subsets in $\mathbb{R}^2 = \mathbb{R}^2 \times \{0\} \subseteq \mathbb{R}^3$. The general case follows easily from the special case. For careful proofs of this and the other theorems (such as the Implicit Function Theorem and the Existence and Uniqueness Theorem for ODE) which Do Carmo leaves unproved see the little book *Calculus On Manifolds* by Michael Spivak. □

19. Definition. Let $S \subseteq \mathbb{R}^3$ be a regular surface and $p \in S$. The function $I_p : T_pS \to \mathbb{R}$ defined by

$$I_p(w) := \langle w, w \rangle = |w|^2, \qquad w \in T_pS \subseteq \mathbb{R}^3$$

is called the **first fundamental form** of S at p. (See do Carmo page 92.)

20. Remark. Here do Carmo uses the notation $\langle w_1, w_2 \rangle$ for what was denoted by $w_1 \cdot w_2$ in Chapter I and calls $\langle w_1, w_2 \rangle$ the **inner product** (rather than the *dot product*) of the vectors $w_1, w_2 \in \mathbb{R}^3$. When $w_1, w_2 \in T_pS$ he sometimes writes $\langle w_1, w_2 \rangle_p$ for $\langle w_1, w_2 \rangle$. Following do Carmo I will no longer write vectors in boldface. Note that do Carmo denotes local parameterizations in bold face, but $\mathbf{x}(u, v)$ should be viewed as a *point* of $\mathbb{R}^3$ *not* a vector.

21. The First Fundamental Form in Local Coordinates. Let $S \subseteq \mathbb{R}^3$ be a regular surface and $\mathbf{x} : U \to S \cap W$ be a local parameterization. Define functions $F, E, G : U \to \mathbb{R}$ by

$$E(q) = \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p, \qquad F(q) = \langle \mathbf{x}_u, \mathbf{x}_v \rangle_p, \qquad G(q) = \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p$$

for $q \in U$ and $p = \mathbf{x}(q) \in S$. Then

$$\langle \hat{p}_1, \hat{p}_2 \rangle_p = E(q)\hat{u}_1\hat{u}_2 + F(q)(\hat{u}_1\hat{v}_2 + \hat{v}_1\hat{u}_2) + G(q)\hat{v}_1\hat{v}_2$$

for $\hat{p}_i = (\hat{u}_i.\hat{v}_i) \in \mathbb{R}^2$. In particular, the first fundamental form is given by

$$I_p(\hat{p}) = E(q)\hat{u}^2 + 2F(q)\hat{u}\hat{v} + G(q)\hat{v}^2.$$

In matrix notation this formula is

$$I_p(\hat{p}) = \begin{pmatrix} \hat{u} & \hat{v} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}_q \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}.$$

22. Example (Stereographic Projection). (See do Carmo Exercise 16 page 67.) Let $S^2 \subseteq \mathbb{R}^3$ denote the unit sphere, i.e.

$$S^2 = \{(x, y, z) \in \mathbb{R}^3,\ x^2 + y^2 + z^2 = 1\}.$$

The point $n = (0, 0, 1) \in S^2$ is called the *north pole*. The map $\pi : S^2 \setminus \{n\} \to \mathbb{R}^2$ defined by the condition

$$\pi(p) = q \iff \text{the three points } n,\ p,\ (q, 0) \text{ are collinear}$$

is called **stereographic projection**. By similar triangles (see Figure 1) we see that

$$\pi(x, y, z) = \left(\frac{x}{1 - z}, \frac{y}{1 - z} \right)$$

and the inverse map is given by $\mathbf{x}(u, v) := \pi^{-1}(u, v) = (x, y, z)$ where

$$x = \frac{2u}{u^2 + v^2 + 1}, \qquad y = \frac{2v}{u^2 + v^2 + 1}, \qquad z = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}.$$

The partial derivatives are

$$\frac{\partial x}{\partial u} = \frac{-2u^2 + 2v^2 + 2}{(u^2 + v^2 + 1)^2}, \quad \frac{\partial y}{\partial u} = \frac{-4uv}{(u^2 + v^2 + 1)^2}, \quad \frac{\partial z}{\partial u} = \frac{-4u^2}{(u^2 + v^2 + 1)^2},$$

$$\frac{\partial x}{\partial v} = \frac{-4uv}{(u^2 + v^2 + 1)^2}, \quad \frac{\partial y}{\partial v} = \frac{2u^2 - 2v^2 + 2}{(u^2 + v^2 + 1)^2}, \quad \frac{\partial z}{\partial v} = \frac{-4v^2}{(u^2 + v^2 + 1)^2}.$$

Hence

$$\langle \mathbf{x}_u, \mathbf{x}_u \rangle = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = \frac{4}{u^2 + v^2 + 1}, \qquad \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0.$$

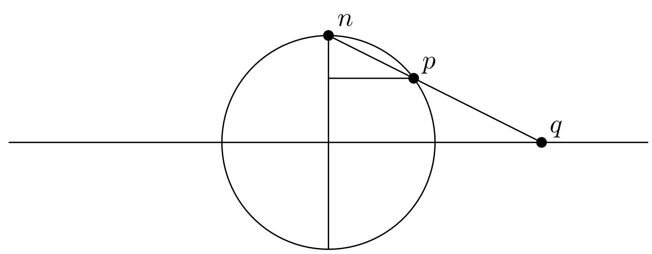

Figure 1: Stereographic Projection

Therefore for $p = (x, y, z) \in S^2 \setminus \{n\}$ and $q = (u, v) = \pi(p) \in \mathbb{R}^2$ we have

$$\langle \hat{p}_1, \hat{p}_2 \rangle = \mu(q) \langle \hat{q}_1, \hat{q}_2 \rangle$$

where

$$\hat{q}_i \in \mathbb{R}^2, \qquad \hat{p}_i = d\mathbf{x}_q(\hat{q}_i) \in T_pS^2, \qquad \mu(q) := \frac{4}{u^2 + v^2 + 1}.$$

In other words the first fundamental form satisfies $E = G$ and $F = 0$. This implies that the linear map $d\mathbf{x}_q : \mathbb{R}^2 \to T_pS^2$ preserves (cosines of) angles. A linear map which preserves angles is called **conformal**.

23. Remark. The book *Geometry and the Imagination* by David Hilbert and Stephan Cohn-Vossen (Chelsea Publishing Company, 1952) contains an elementary proof that stereographic projection is conformal on page 248. (The proof is elementary in that it doesn't use calculus.) An elementary proof can also be found online at `http://people.reed.edu/~jerry/311/stereo.pdf`. (I put a copy at `http://www.math.wisc.edu/~robbin/Do_Carmo/stereo.pdf`.)

24. Area Theorem. *Let $S \subseteq \mathbb{R}^3$ be a compact[3] regular surface. Then there is a unique function A which assigns a real number $A(S \cap V)$ to every open subset $S \cap V$ of S and satisfies the following two properties.*

(i) *For every local parameterization $\mathbf{x} : U \to S \cap V$ we have*

$$A(S \cap V) := \iint_U |\mathbf{x}_u \wedge \mathbf{x}_v| \, du \, dv$$

(ii) *If $V = V_1 \cup V_2$ and the sets $S \cap V_1$ and $S \cap V_2$ intersect only in their boundaries, then*

$$A(S \cap V) = A(S \cap V_1) + A(S \cap V_2).$$

[3] The term **compact** means *closed and bounded*.

The number $A(S)$ is called the **area** *of S.*

Proof: A careful proof of this theorem is best left for another course, but the geometric idea isn't so difficult. The key point is the change of variables formula for a double integral. (See do Carmo at the bottom of page 97.) This formula says that

$$\iint_{U_1} |\mathbf{x}_u \wedge \mathbf{x}_v|\, du\, dv = \iint_{U_2} |\mathbf{y}_u \wedge \mathbf{y}_v|\, du\, dv$$

if $\mathbf{x} : U_1 \to S \cap V$ and $\mathbf{y} : U_2 \to S \cap V$ are two local parameterizations with the same trace. i.e. $\mathbf{x}(U_1) = \mathbf{y}(U_2)$. Then we must show that S can be covered by open sets which overlap only in their boundaries. (A precise definition of *boundary* must be given.) Finally we must prove the addition formula in part (ii).

The formula in part (i) is plausible. Imagine that the set U is broken up into a large number of very small rectangles. Each rectangle has area $du\, dv$. The image of this rectangle under the map $\mathbf{x}$ will be approximately a parallelogram with edge vectors $\mathbf{x}_u\, du$ and, $\mathbf{x}_v\, dv$ and the area of this parallelogram is roughly

$$dA = |\mathbf{x}_u \wedge \mathbf{x}_v|\, du\, dv.$$

Now $|\mathbf{x}_u \wedge \mathbf{x}_v| = |\sin\theta|\, |\mathbf{x}_u|\, |\mathbf{x}_v|$ where θ is the angle from $\mathbf{x}_u$ to $\mathbf{x}_v$. But this is the area of the tiny parallelogram. Adding up all these tiny areas gives the total area as an integral. In terms of the first fundamental form the area element in local coordinates is

$$dA = \sqrt{EG - F^2}\, du\, dv.$$

This is a consequence of the formulas

$$\langle w_1, w_2\rangle = |w_1|\, |w_2| \cos\theta, \qquad |w_1 \wedge w_2| = |w_1|\, |w_2|\, |\sin\theta|$$

for the inner product and wedge product of two vectors $w_1, w_2 \in \mathbb{R}^3$. □

25. Example. As an example we will prove the formula

$$A(S^2) = 4\pi$$

in two different ways. A parameterization of the upper hemisphere is

$$\mathbf{x}(u, v) = (u, v, z(u, v)), \qquad z(u, v) := \sqrt{1 - u^2 - v^2}.$$

The coordinate vectors are

$$\mathbf{x}_u = \begin{pmatrix} 1 \\ 0 \\ \dfrac{-u}{\sqrt{1 - u^2 - v^2}} \end{pmatrix}, \qquad \mathbf{x}_v = \begin{pmatrix} 0 \\ 1 \\ \dfrac{-v}{\sqrt{1 - u^2 - v^2}} \end{pmatrix},$$

so

$$\mathbf{x}_u \wedge \mathbf{x}_v = \begin{pmatrix} \dfrac{v}{\sqrt{1 - u^2 - v^2}} \\ \dfrac{-u}{\sqrt{1 - u^2 - v^2}} \\ 1 \end{pmatrix}, \qquad |\mathbf{x}_u \wedge \mathbf{x}_v| = \frac{1}{\sqrt{1 - u^2 - v^2}}.$$

To evaluate the integral we use the change of variables

$$(0,1)\times(0,2\pi)\to\{(u,v),\ u^2+v^2<1\}:(r,\theta)\mapsto(u,v)=(r\cos\theta,r\sin\theta)$$

so $du\,dv=\dfrac{\partial(u,v)}{\partial(r,\theta)}\,dr\,d\theta$ where

$$\frac{\partial(u,v)}{\partial(r,\theta)}=\det\begin{pmatrix}\dfrac{\partial u}{\partial r} & \dfrac{\partial u}{\partial\theta}\\ \dfrac{\partial v}{\partial r} & \dfrac{\partial v}{\partial\theta}\end{pmatrix}=r$$

so

$$\int_{u^2+v^2<1}|\mathbf{x}_u\wedge\mathbf{x}_v|\,du\,dv=\int_0^{2\pi}\int_0^1\frac{r\,dr\,d\theta}{\sqrt{1-r^2}}=2\pi\int_0^1\frac{ds}{2\sqrt{s}}=2\pi.$$

The parameterization $(u,v)\mapsto(u,v,-z(u,v))$ of the lower hemisphere gives the same answer and the two hemispheres intersect only in their common boundary (the unit circle in the (x,y)-plane) so the area of S^2 is 4π.

A second way to prove $A(S^2)=4\pi$ is to use spherical coordinates

$$\mathbf{x}(\theta,\varphi)=(\cos\theta\sin\varphi,\sin\theta\sin\varphi,\cos\varphi).$$

Here $\mathbf{x}:(0,2\pi)\times(0,\pi)\to S^2\cap V$ where $V=\{(x,y,z)\in\mathbb{R}^3,\ x\neq 1,\ z\neq\pm 1\}$. Then

$$\mathbf{x}_\theta=\begin{pmatrix}-\sin\theta\cos\varphi\\ \cos\theta\cos\varphi\\ 0\end{pmatrix},\qquad \mathbf{x}_\varphi=\begin{pmatrix}-\cos\theta\sin\varphi\\ -\sin\theta\sin\varphi\\ \cos\varphi\end{pmatrix},$$

so

$$\mathbf{x}_\theta\wedge\mathbf{x}_\varphi=\begin{pmatrix}\cos\theta\cos^2\varphi\\ \sin\theta\cos^2\varphi\\ \cos\varphi\sin\varphi\end{pmatrix},\qquad |\mathbf{x}_\theta\wedge\mathbf{x}_\varphi|=|\cos\varphi|.$$

Now $S^2\cap V$ intersects itself only in its boundary (which is a semicircle) so

$$A(S^2)=\int_0^\pi\int_0^{2\pi}|\cos\varphi|\,d\theta\,d\varphi=4\pi.$$

26. The Unit Normals. For a two dimension vector subspace $W\subseteq\mathbb{R}^3$ there are exactly two unit vectors $n\in\mathbb{R}^3$ which are perpendicular to every vector in W, i.e. such that $|n|=1$ and $\langle n,w\rangle=0$ for $w\in W$. If n is one of these two vectors then $-n$ is the other one. In particular, when $W=T_pS$ is the tangent space at a point p to a regular surface $S\subseteq\mathbb{R}^3$ there are exactly two vectors $N(p)$ such that $|N(p)|=1$ and $\langle N(p),w\rangle=0$ for all $w\in T_pS$. If $\mathbf{x}:U\to S\cap V$ is a local parameterization of S and $p=\mathbf{x}(q)\in S$ where $q\in U$, then these two unit normal vectors are

$$N(p)=\pm\left(\frac{\mathbf{x}_u\wedge\mathbf{x}_v}{|\mathbf{x}_u\wedge\mathbf{x}_v|}\right)_q.$$

27. Definition. A regular surface is said to be **orientable** iff there is a smooth map $N : S \to S^2$ such that

$$\langle N(p), w \rangle = 0, \qquad \forall w \in T_pS.$$

Such a map determines an **orientation** on each tangent space: an ordered basis $w_1, w_2 \in T_pS$ is positively oriented iff $\langle N(p), w_1 \wedge w_2 \rangle > 0$. The vector field N is called the **unit normal** to the oriented surface S and the map $N : S \to S^2$ is called the **Gauss map**.

28. Remark. It is a difficult theorem that a compact regular surface S is orientable and the open set $\mathbb{R}^3 \setminus S$ has two connected components, one bounded and the other unbounded. In this case one chooses the orientation so that the normal vector N points into the unbounded component. This N is called the **outward unit normal** vector. For example, when $S = S^2$ the bounded component is the open ball $\{(x, y, z) \in R^3, x^2 + y^2 + z^2 < 1\}$ and the unbounded component is the open set $\{(x, y, z) \in R^3, x^2 + y^2 + z^2 > 1\}$. The outward unit normal for S^2 is $N(p) = p$ so the Gauss map is the identity map.

29. The Möbius Strip. (See do Carmo page 106.) This is the image S of the map $\mathbf{x} : \mathbb{R} \times (-1, 1) \to \mathbb{R}^3$ defined by

$$\mathbf{x}(\theta, r) = \mathbf{z}(\theta) + r\mathbf{n}(\theta), \quad \mathbf{z}(\theta) = (2\cos 2\theta, 2\sin 2\theta, 0), \quad \mathbf{n}(\theta) = (\sin\theta, \sin\theta, \cos\theta).$$

The curve $\mathbf{z}$ has period π and the curve $\mathbf{n}$ has period 2π. Note that the line segment $\ell(\theta)$ connecting the two points $\mathbf{x}(\theta, \pm 1)$ lies in S and if $0 < |\theta_1 - \theta_2| < \pi$ the two line segments $\ell(\theta_1)$ and $\ell(\theta_2)$ do not intersect. The two line segments $\ell(\theta)$ and $\ell(\theta + \pi)$ are equal as sets but they have opposite orientations. Since $\mathbf{x}_\theta = \mathbf{z}'(\theta) + r\mathbf{n}'(\theta)$ and $\mathbf{x}_r = \mathbf{n}(\theta)$ we get

$$(\mathbf{x}_\theta \wedge \mathbf{x}_r)(\theta, 0) = \mathbf{z}'(\theta) \wedge \mathbf{n}(\theta)$$

so $\mathbf{x}(\theta + \pi, 0) = \mathbf{x}(\theta, 0)$ but $\mathbf{x}_\theta \wedge \mathbf{x}_r(\theta + \pi, 0) = -\mathbf{x}_\theta \wedge \mathbf{x}_r(\theta, 0)$. Hence there is no continuous unit normal so the Möbius strip is not orientable.