Homework

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Feb 6, 2014

1. Exercise 1.5-12. Let the position of a particle at time t be given by

$$\alpha(t) = \beta(\sigma(t))$$

where β is parameterized by arclength and $\dot\sigma(t) = |\dot\alpha(t)|$ is the speed of the particle. Denote differentiation with respect to t by an overdot and differentiation with respect to s by a prime so if $g(t) = f(\sigma(t))$ we have $\dot g(t) = f'(\sigma(t))\dot\sigma(t)$ by the chain rule. By the Frenet equations

$$\begin{array}{rcl} \dot\alpha & = & \dot\sigma\mathbf{t} \\ \ddot\alpha & = & \ddot\sigma\mathbf{t} + \dot\sigma^2\mathbf{t}' = \ddot\sigma\mathbf{t} + \dot\sigma^2\kappa\mathbf{n} \\ \dddot\alpha & = & \dddot\sigma\mathbf{t} + \ddot\sigma\dot\sigma\mathbf{t}' + (2\ddot\sigma\dot\sigma\kappa + \dot\sigma^3\kappa')\mathbf{n} + \dot\sigma^3\kappa\,\mathbf{n}' \\ & = & (\dddot\sigma - \dot\sigma^3\kappa\tau)\mathbf{t} + (\ddot\sigma\dot\sigma\kappa + 2\ddot\sigma\dot\sigma\kappa + \dot\sigma^3\kappa')\mathbf{n} + \dot\sigma^3\kappa\tau\,\mathbf{b} \end{array}$$

From the first two equations we get $\dot\alpha \wedge \ddot\alpha = \dot\sigma^3\kappa\,\mathbf{b}$ and hence $|\dot\alpha\wedge\ddot\alpha| = \dot\sigma^2\kappa$. Represent all three equations in the matrix form

$$\begin{pmatrix} \dot\alpha \\ \ddot\alpha \\ \dddot\alpha \end{pmatrix} = \begin{pmatrix} \dot\sigma & 0 & 0 \\ * & \dot\sigma^2\kappa & 0 \\ * & * & \dot\sigma^3\kappa\tau \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}$$

and we get $(\dot\alpha\wedge\ddot\alpha)\cdot\dddot\alpha = \dot\sigma^6\kappa^2\tau$. Combining gives

$$\kappa = \frac{|\dot\alpha\wedge\ddot\alpha|}{\dot\sigma^3}, \qquad \tau = \frac{(\dot\alpha\wedge\ddot\alpha)\cdot\dddot\alpha}{|\dot\alpha\wedge\ddot\alpha|^2}.$$

A plane curve is a special case: take $\alpha(t) = (x(t), y(t), 0)$. But in the case of a plane curve we can define the unit normal vector by rotation the unit tangent vector clockwise by a right angle. To maintain the second Frenet equation $\mathbf{t}' = \kappa\mathbf{n}$ we must allow κ to take negative values. The above equation for κ becomes

$$\kappa = \frac{\dot\alpha\wedge\ddot\alpha}{\dot\sigma^3} = \frac{\dot x\,\ddot y - \dot y\,\ddot x}{(\dot x^2 + \dot y^2)^{3/2}} \qquad (\heartsuit)$$

2. Exercise 1.5-12. *Let $\alpha : I \to \mathbb{R}^3$ be a curve parameterized by arclength whose curvature κ and torsion τ do not vanish. If the trace of α lies in a sphere, then*

$$\frac{1}{\kappa^2} + \left(\frac{\kappa'}{\kappa^2\tau}\right)^2 = \textit{constant} \qquad (\dagger)$$

The converse holds if the derivative of κ' of the curvature does not vanish.[1]

Assume that $\alpha(I)$ lies in the sphere of radius $a > 0$ centered at $p_0 \in \mathbb{R}$, i.e. that

$$|\alpha(s) - p_0|^2 = a.$$

The curve $\alpha - p_0$ has the same curvature and torsion as α so we may assume that $p_0 = 0$. Then $\alpha \cdot \alpha' = 0$. But $\alpha' = \mathbf{t}$ so

$$\alpha \cdot \mathbf{t} = 0.$$

Differentiating again gives $\alpha' \cdot \mathbf{t} + \alpha \cdot \mathbf{t}' = 0$. As $\alpha' = \mathbf{t}$, $|\mathbf{t}| = 1$, $\mathbf{t}' = \kappa\mathbf{n}$, and $\mathbf{t} \cdot \mathbf{n} = 0$ this gives

$$\alpha \cdot \mathbf{n} = -\frac{1}{\kappa}$$

Differentiating a third time gives $\alpha' \cdot \mathbf{n} + \alpha \cdot \mathbf{n}' = \kappa'\kappa^{-2}$. As $\alpha' = \mathbf{t}$ and $\mathbf{t} \cdot \mathbf{n} = 0$ this simplifies to $\alpha \cdot \mathbf{n}' = \kappa'\kappa^{-2}$. Using the second Frenet equation this becomes $\alpha \cdot (-\kappa\mathbf{t} + \tau\mathbf{b}) = \kappa'\kappa^{-2}$ and since $\alpha \cdot \mathbf{t} = 0$ we get

$$\alpha \cdot \mathbf{b} = \frac{\kappa'}{\kappa^2\tau}.$$

Now since $\mathbf{t}, \mathbf{n}, \mathbf{b}$ is an orthonormal basis we get

$$\alpha = (\alpha \cdot \mathbf{t})\mathbf{t} + (\alpha \cdot \mathbf{n})\mathbf{n} + (\alpha \cdot \mathbf{b})\mathbf{b} = -\frac{1}{\kappa}\mathbf{n} + \frac{\kappa'}{\kappa^2\tau}\mathbf{b}$$

and hence

$$a^2 = |\alpha|^2 = \frac{1}{\kappa^2} + \left(\frac{\kappa'}{\kappa^2\tau}\right)^2. \qquad (*)$$

To prove the converse let

$$\gamma := -\frac{1}{\kappa}\mathbf{n} + \frac{\kappa'}{\kappa^2\tau}\mathbf{b}.$$

denote the right hand side of $(*)$ and assume that $|\gamma|$ is constant, i.e. that γ lies on a sphere centered at the origin. It is enough to show that $\alpha' = \gamma'$ for then $\alpha = \gamma + p_0$ for some $p_0 \in \mathbb{R}^3$ so α lies on a translate of the sphere containing γ. Differentiate γ to get

$$\gamma' = \frac{\kappa'}{\kappa^2}\mathbf{n} - \frac{1}{\kappa}\mathbf{n}' + \left(\frac{\kappa'}{\kappa^2\tau}\right)'\mathbf{b} + \frac{\kappa'}{\kappa^2\tau}\mathbf{b}'.$$

By Frenet we have $\mathbf{n}' = -\kappa\mathbf{t} + \tau\mathbf{b}$ and $\mathbf{b}' = -\tau\mathbf{n}$ so

$$\begin{aligned}\gamma' &= \frac{\kappa'}{\kappa^2}\mathbf{n} - \frac{1}{\kappa}(-\kappa\mathbf{t} + \tau\mathbf{b}) + \left(\frac{\kappa'}{\kappa^2\tau}\right)'\mathbf{b} - \frac{\kappa'}{\kappa^2\tau}\tau\mathbf{n} \\ &= \mathbf{t} - \frac{\tau}{\kappa}\mathbf{b} + \left(\frac{\kappa'}{\kappa^2\tau}\right)'\mathbf{b} \\ &= \alpha' - \left(\frac{\tau}{\kappa} - \left(\frac{\kappa'}{\kappa^2\tau}\right)'\right)\mathbf{b}.\end{aligned}$$

[1] However, the converse does not hold in general. A helix has constant curvature and torsion so it also satisfies (†).

so we must prove that the coefficient of $\mathbf{b}$ is zero. The assumption that $|\gamma|$ is constant is the same as the assumption that the derivative of the right hand side of $(*)$ is zero so

$$0 = \left(\frac{1}{\kappa^2} + \left(\frac{\kappa'}{\kappa^2\tau} \right)^2 \right)' = \frac{-2\kappa'}{\kappa^3} + 2\left(\frac{\kappa'}{\kappa^2\tau} \right)\left(\frac{\kappa'}{\kappa^2\tau} \right)' = -\frac{2\kappa'}{\kappa^2\tau}\left(\frac{\tau}{\kappa} - \left(\frac{\kappa'}{\kappa^2\tau} \right)' \right)$$

As $\kappa' \neq 0$, the other factor vanishes as required. □

3. Exercise 1.6-3. Let $\alpha : I \to \mathbb{R}^3$ be a regular curve parameterized by arclength and $p_0 = \alpha(s_0)$ be a point on its trace. Assume the curvature $\kappa_0 = \kappa(s_0)$ of C at p_0 is nonzero. Let $P_0 \subseteq \mathbb{R}^3$ be the osculating plane to C at p_0 and $\pi : \mathbb{R}^3 \to \mathbb{P}_0$ be the orthogonal projection. We will show that the space curve α and the plane curve $\pi \circ \alpha$ have the same curvature at s_0.

On page 17 do Carmo defines the osculating plane as the "plane determined by the unit tangent and normal vectors $\alpha'(s)$ and $\mathbf{n}(s)$" but in Figure 1-15 on the same page he draws the osculating plane so that it touches the curve. What he intends is that the osculating plane is the image of the two dimensional vector subspace

$$V_0 := \{t_1\mathbf{t}_0 + t_2\mathbf{n}_0,\ t_1, t_2 \in \mathbb{R}\}, \qquad \mathbf{t}_0 := \alpha'(s_0), \quad \mathbf{n}_0 := \mathbf{n}(s_0)$$

under the translation $\mathbf{v} \mapsto p_0 + \mathbf{v}$, i.e.

$$P_0 = p_0 + V_0.$$

The point is that V_0 is a vector subspace of $\mathbb{R}^3$ and thus contains the origin whereas P_0 is a plane which probably does not pass through the origin. The orthogonal projection $\pi_0 : \mathbb{R}^3 \to V_0$ is defined by

$$\pi_0(\mathbf{v}) = (\mathbf{v} \cdot \mathbf{t}_0)\mathbf{t}_0 + (\mathbf{v} \cdot \mathbf{n}_0)\mathbf{n}_0$$

whereas by the projection on the osculating plane do Carmo means the map $\mathbf{v} \mapsto \pi_0(\mathbf{v} - p_0) + p_0$. We can use either interpretation in this problem since the curvature is invariant under translations.

The problem is easy if we use the local canonical form on page 27 of do Carmo. Here $s_0 = 0$, $p_0 = 0$, and $\alpha(s) = (x(s), y(s), z(s))$ where

$$x = s - \frac{\kappa_0 s^3}{6} + \cdots, \qquad y = \frac{\kappa_0 s^2}{2} + \cdots, \qquad z = \frac{\kappa_0\tau_0}{6}s^3 + \cdots,$$

where κ_0 and τ_0 are the curvature and torsion at $s_0 = 0$ and the dots represent terms which vanish to order three. Thus $\mathbf{t}_0 = (1, 0, 0)$, $\mathbf{n}_0 = (0, 1, 0)$, and the osculating plane is the (x, y)-plane. The plane curve $\pi_0 \circ \alpha(t) = (x(t), y(t))$ has a Taylor expansion

$$x = t - \frac{\kappa_0 t^3}{6} + \cdots, \qquad y = \frac{\kappa_0 t^2}{2} + \cdots,$$

and the curvature at $s = 0$ of this plane curve $\pi \circ \alpha$ is κ_0 by Equation ($\heartsuit$) in Exercise 1.5-12 above.

4. Theorem. *Let $\alpha : I \to \mathbb{R}^2$ be a plane curve parameterized by arclength, $s_0 \in I$, $p_0 = \alpha(s_0)$, and $\mathbf{t}_0 = \alpha'(s_0)$ be the unit tangent vector at s_0. Assume that the curvature κ_0 of α at s_0 is not zero. Then there is a unique circle*

$$\beta(s) = (x_0 + r_0 \cos\theta, y_0 + r_0 \sin\theta), \qquad \theta = \frac{s}{r_0}$$

which has second order contact with α at s_0, i.e.

$$\beta(s_0) = \alpha(s_0), \qquad \beta'(s_0) = \alpha'(s_0), \qquad \beta''(s_0) = \alpha''(s_0).$$

The curve α and the circle β have the same curvature $\kappa_0 = 1/r_0$ at s_0 and the center $q_0 = (x_0, y_0)$ of the circle β is given by

$$q_0 = p_0 + r_0 \mathbf{n}_0$$

where $\mathbf{n}_0 = \kappa_0^{-1}\alpha''(s_0)$ is the unit normal vector.

5. Definition. In the notation of the theorem the circle β is called the **osculating circle** to the curve α at s_0, its center $q_0 = (x_0, y_0)$ is called the **center of curvature**, and its radius $r_0 = 1/\kappa_0$ is called **radius of curvature**. The fact that the curvature of a circle is the reciprocal of its radius is an immediate consequence of Equation (♡) in Exercise 1.5-12 above.

6. Remark. In Exercise 1.6-2 on page 30 do Carmo defines the **osculating circle** at a point $p_0 = \alpha(s_0)$ of a space curve $\alpha : I \to \mathbb{R}^3$. The curves α and $\pi_0 \circ \alpha$ (see Problem 1.6-3 above) have the same osculating circle. Do Carmo doesn't give a precise definition for what it means for a parameterized family of circles to converge. An adequate definition is that the centers, the radii, and the unit normal vectors of the ambient planes all converge.

7. Exercise 1.7-3. By Equation (♡) in Exercise 1.5-12 above the curvature of the ellipse $\alpha(\theta) = (a\cos\theta, b\cos\theta)$ is

$$\kappa = \frac{\dot\alpha \wedge \ddot\alpha}{\dot\sigma^3} = \frac{\dot x\,\ddot y - \dot y\,\ddot x}{(\dot x^2 + \dot y^2)^{3/2}} = \frac{2ab\sin\theta\cos\theta}{(a^2\cos^2\theta + b^2\sin^2\theta)^{3/2}}$$

which vanishes at the four points $\theta = 0, \pi/2, \pi, 3\pi/2$ so the vertices are $(\pm a, 0)$, $(0, \pm b)$. (The Four Vertex Theorem (see do Carmo pages 37 and 40) says that every simple closed plane curve has at least four vertices.)

8. Remark. The locus of centers of curvature of a plane curve is called its <u>evolute</u>[2] By Equation (♡) in Exercise 1.5-12 above the evolute of a parameterized curve $\alpha(t) = (x(t), y(t))$ has parameterized equations $\beta(t) = (\xi(t), \eta(t))$ where

$$\beta = (-m\dot y, m\dot x), \qquad m = \frac{\dot x^2 + \dot y^2}{\dot x\ddot y - \dot y\ddot x}$$

The evolute of an ellipse is called an *asteroid*. Figure **??** shows this curve together with the normal lines to the ellipse. It is easy to prove that the normal

[2] Click if reading online.

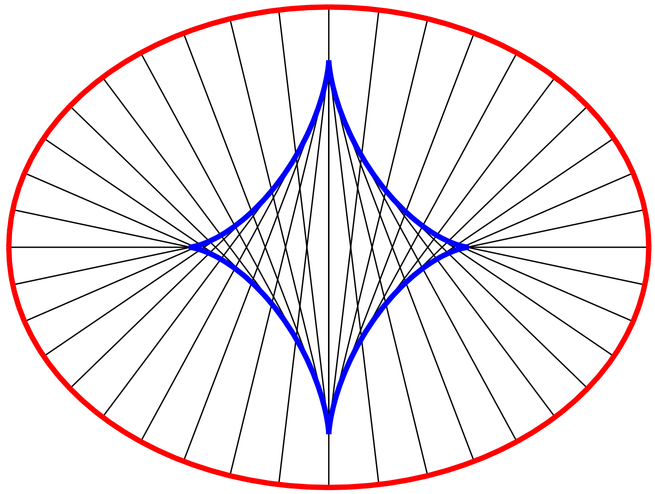

Figure 1: The evolute of an ellipse

lines to a plane curve are tangent to the evolute of that curve and that the curve is traced out by unwinding a string along its evolute. A curve which is obtained by unwinding a taut string along a given curve is called an involute of the given curve.

Here is a sage program which will draw Figure 1.

```
var('x','y','t','X','Y')
a=2.0; b=1.5
x(t)=a*cos(t); y(t)=b*sin(t)
dx(t)=x.derivative(t); dy(t)=y.derivative(t);
ddx(t)=dx.derivative(t); ddy(t)=dy.derivative(t)
m(t)=(dx(t)*dx(t)+dy(t)*dy(t))/(dx(t)*ddy(t)-dy(t)*ddx(t))
X(t)=x(t)-m(t)*dy(t); Y(t)=y(t)+m(t)*dx(t)
G=Graphics()
G+=parametric_plot((x(t),y(t)), (t,0.0,2*pi),color='red')
G+=parametric_plot((X(t),Y(t)), (t,0.0,2*pi),color='blue')
for i in range(0,40):
    t=2*i*pi/40
    G+=line([(x(t),y(t)),(X(t),Y(t))],color='black')
G.show(axes=False)
```