Homework

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Feb 13, 2014

1. Exercise 2.2-1. The cylinder $\{(x, y, z),\ x^2 + y^2 = 1\}$ is the image of the map $\mathbf{z}(\theta, z) = (\cos\theta, \sin\theta, z)$. This map is regular as $\mathbf{z}_\theta = (-\sin\theta, \cos\theta, 0)$ and $\mathbf{z}_z = (0, 0, 1)$ and these two vectors are obviously linearly independent. Restricting the map $\mathbf{z}$ to $I \times \mathbb{R}$ where I is any open interval of length less than 2π gives a local parameterization. The cylinder can also be covered by the four local parameterizations

$$\mathbf{x}_\pm(x, z) = (x, \pm\sqrt{1 - x^2}, z), \qquad \mathbf{y}_\pm(y, z) = (\pm\sqrt{1 - y^2}, y, z).$$

(Note that a graph map is always regular.)

2. Exercise 2.2-7ab. All three partial derivatives of the function $f(x, y, z) = (x + y + z - 1)^2$ vanish at (and only at) the points of the plane $x + y + z - 1 = 0$. The set $f^{-1}(c)$ is the plane $x + y + z - 1{=}0$ if $c = 0$ and is empty if $c < 0$. It is *always* a regular surface even when c is the only critical value 0. (The empty set is a regular surface since the definition of regular surface begins with the phrase *For all* $p \in S$ and every point in the empty set satisfies every property.)

3. Exercise 2.2-8 Two vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ are linearly dependent exactly when one is a multiple of the other. Since $|a \wedge b| = |a|\,|b|\,|\sin\theta|$ where θ is the angle between the two vectors, this occurs exactly when $a \wedge b = 0$. One can also see this using the formula

$$a \wedge b = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).$$

The differential $d\mathbf{x}_q : R^2 \to \mathbb{R}^3$ (at a point $q \in U \subseteq \mathbb{R}^2$) is one-to-one if and only if the vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent (i.e. not linearly dependent), i.e. if and only if $\mathbf{x}_u \wedge \mathbf{x}_v \neq 0$.

4. Exercise 2.2-11 A graph $S = \{(x, y, z),\ z = f(x, y)\}$ is *always* a regular surface since it has a parameterization

$$\mathbf{z}(x, y) = (x, y, f(x, y))$$

and the vectors

$$\mathbf{z}_x = \left(1, 0, \frac{\partial f}{\partial x}\right), \qquad \mathbf{z}_y = \left(0, 1, \frac{\partial f}{\partial y}\right)$$

are linearly independent. In the example $f(x, y) = x^2 - y^2$ the map

$$\mathbf{x}(u, v) = (u + v, u - v, 4uv)$$

is a parameterization of the same graph since it is the composition $\mathbf{x} = \mathbf{z} \circ \varphi$ where $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is the diffeomorphism

$$\varphi(u, v) = (\tfrac{1}{2}(u + v), \tfrac{1}{2}(u - v)).$$

The image of the map

$$\mathbf{y}(u, v) = (u \cosh v, u \sinh v, u^2)$$

lies in the graph (since $\cosh^2 v - \sinh^2 v = 1$) and the vectors

$$\mathbf{y}_u(u, v) = (\cosh v, \sinh v, 2u), \qquad \mathbf{y}_v(u, v) = (u \sinh v, u \cosh v, 0)$$

are linearly independent when $u \neq 0$. The image of $\mathbf{y}$ is the intersection of the graph with the upper half space $z > 0$.

5. Exercise 2.2-17ac The definition that do Carmo expects here is given later on page 75.

> *A set $C \subseteq \mathbb{R}^n$ is a* regular curve *iff for every point $p \in C$ there is an open set $V \subseteq \mathbb{R}^n$ containing p and regular parameterized curve $\alpha : I \to V$ such that α is one-to-one, $\alpha(I) = C \cap V$, and the inverse map $\alpha^{-1} : C \cap V \to I$ is a homeomorphism.*

This is just like the definition of *regular surface* on page 52 of do Carmo. The following equivalent definition will be used in the answer to Exercise 2.2-15a below.

> *A set $C \subseteq \mathbb{R}^n$ is a* regular curve *iff for every point $p \in C$ there is a open set $V \subseteq \mathbb{R}^n$ containing p an open interval I about $0 \in R$, an open neighborhood W of $0 \in \mathbb{R}^{n-1}$, and a diffeomorphism $\Phi : I \times W \to V$ such that $C \cap V = \Phi(I \times \{0\})$.*

The proof that the two definitions are the same uses the inverse function theorem and is just like the argument in do Carmo on page 71. We'll take $n = 3$ to simplify the notation. It is obvious that the second definition implies the first. To prove the converse choose $p \in C$ and let $\alpha : I \to S \cap V$ as in the first definition. Assume that $\alpha(0) = p$ and choose $e_1, e_2 \in \mathbb{R}^3$ so that the three vectors $\alpha(t_1), e_1, e_2$ are linearly independent. Define $\Phi : I \times \mathbb{R}^2 \to \mathbb{R}^3$ by $\Phi(t, s_1, s_2) = \alpha(t) + s_1 e_1 + s_2 e_2$. Then $\Phi(t, 0, 0) = \alpha(t)$. The differential $d\Phi_q$ is invertible at $q = (0, 0, 0)$ so (shrinking I if necessary) the inverse function theorem gives us neighborhood W of $(0, 0) \in \mathbb{R}^2$ such that the restriction of Φ to $I \times W$ is a diffeomorphism onto its image. By continuity we can shrink $I \times W$ to get $\Phi(I \times W) \subseteq V$. By the continuity of α^{-1} can shrink V so $\alpha^{-1}(C \cap V) \subseteq I$. Now replace I by the interval $\alpha^{-1}(C \cap V)$ to get $C \cap V = \Phi(I \times \{0\})$. $\square$

(a) When $n = 2$ the preimage of a regular value is a regular curve by the implicit function theorem as follows. If $p_0 = (x_0, y_0) \in F^{-1}(0)$ and $dF_{p_0} \neq 0$ then either $\partial F/\partial x \neq 0$ at p_0 or $\partial F/\partial y \neq 0$ at p_0 (or both). In the former case the implicit function theorem says that there is an open set $V \subseteq \mathbb{R}^2$, an interval $I = (x_0 - \varepsilon, x_0 + \varepsilon)$ and a function $f : I \to \mathbb{R}$ such that

$$F^{-1}(0) \cap V = \{(x, y) \in \mathbb{R}^2,\ x \in I,\ y = f(x)\}$$

In the latter case the same thing happens with the roles of x and y reversed. (A graph is always a regular curve.) Every point on the hyperbola $y^2 - x^2 = 1$ is a regular point but the hyperbola is not connected; it is the union of two disjoint curves $y = \pm\sqrt{1 + x^2}$.

(c) The semicubical parabola $\{(x, y),\ y^3 = x^2\}$ is not a regular curve. If it were regular, then by the implicit function theorem there would either be a smooth function $y = f_1(x)$ defined on an interval $-\varepsilon < x < \varepsilon$ such that $f_1(x)^3 = x^2$ or there would be a smooth function $x = f_2(y)$ defined on an interval $-\varepsilon < y < \varepsilon$ satisfying $x^3 = f_2(y)^2$. The semicubical parabola contains points where $x < 0$ so there can be no such function f_2. The only possible function f_1 is $f_1(x) = x^{2/3}$ and this function is not differentiable at $x = 0$.

6. Exercise 2.3-2. For a regular surface $S \subseteq \mathbb{R}^3$ a map $f : S \to \mathbb{R}^2$ is (by definition) smooth if and only if the composition $f \circ \mathbf{x} : U \to \mathbb{R}^2$ is smooth for every local parameterization $\mathbf{x} : U \to S$. But each component of a local parameterization is smooth by definition. Applying this principle to the projection $f = \pi : S \to \mathbb{R}^2$ and a local parameterization $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$ we see that the composition $\pi \circ \mathbf{x}(u, v) = (x(u, v), y(u, v))$ is smooth *a fortiori.*

7. Exercise 2.3-6. If $\mathbf{x} : U_1 \to S \cap V$ and $\mathbf{y} : U_2 \to S \cap V$ are two local parameterizations with the same image, then the map $\varphi : U_2 \to U_1$ defined by $\mathbf{y} = \mathbf{x} \circ \varphi$ is a diffeomorphism by change of parameters theorem. Hence if $f \circ \mathbf{x}$ is smooth so is $f \circ \mathbf{y} = (f \circ \mathbf{x}) \circ \varphi$ by the chain rule. Since $\varphi^{-1} : U_1 \to U_2$ is also a diffeomorphism, the converse hold as well.

8. Exercise 2.2-15a. If $\alpha : I \to C \subseteq \mathbb{R}^n$ and $\beta : J \to C \subseteq \mathbb{R}^n$ be two regular parameterizations as in the first definition in Exercise 2.2-17 and let $h = \alpha^{-1} \circ \beta$ so $\beta = \alpha \circ h$. We must show that $h : I \to J$ is a diffeomorphism. For this it is enough to show that h is smooth (the same argument will show that h^{-1} is smooth) and for this we need only show that h is smooth in a small interval about each point.

The following argument is tempting. By the chain rule

$$\beta'(t) = h'(t)\alpha'(h(t)).$$

Now $\beta'(t) \neq 0$ so $h'(t) \neq 0$. Hence $h : I \to J$ is a diffeomorphism by the inverse function theorem. The problem with this argument is that it assumes what we are trying to prove, namely that h is differentiable. The following argument addresses this objection. It uses the second definition of *regular curve* that I gave above in my answer to Exercise 2.2-17. With this definition we may assume that $C = I \times \{(0,0)\} \subseteq I \times W$. Then $\alpha(t) = (a(t), 0, 0)$ and $\beta(t) = (b(t), 0, 0)$ so $\alpha'(t) = (a'(t), 0, 0)$ and $\beta'(t) = (b'(t), 0, 0)$ and hence $b = a \circ h$. Now h is smooth by the (one dimensional) inverse function theorem.