Homwework

JWR

Jan 30 2014

Problem numbers refer to the do Carmo text.

1. 1.2-1 The curve $\alpha(s) = (\cos(-s), \sin(-s)) = (\cos(s), -\sin(s))$ parameterizes the circle $x^2 + y^2 = 1$ in the clockwise orientation.

2. 1.2-2 The distance form the point $\alpha(t) \in \mathbb{R}^n$ to the origin is $f(t) = |\alpha'(t)|$. At a point where this distance assumes its minimum, the derivative of the function must vanish. But

$$f'(t) = \frac{\alpha(t) \cdot \alpha'(t)}{|\alpha(t)|}$$

and this vanishes when the vectors $\alpha(t) - 0$ and $\alpha'(t)$ are orthogonal.

3. 1.2-3 Let $\alpha(t) = (x(t), y(t), z(t))$. Then $\alpha''(t) = (x''(t), y''(t), z''(t))$. If $\alpha''(t) = 0$ for all t, then $\alpha'(t)$ is a constant vector, say $\alpha'(t) = \mathbf{v}_0$ and hence (integrating once more) $\alpha(t) = t\mathbf{v}_0 + p_0$ for some $p_0 = (x_0, y_0, z_0)$. (This argument works in n dimensions.)

4. 1.3-1 The curve $\alpha(t) = (3t, 3t^2, 2t^3)$ has velocity vector $\mathbf{v}(t) = (3, 6t, 6t^2)$. The vector $\mathbf{u} = (1, 0, 1)$ point along the line $y = 0$, $x = z$. The cosine of the angle θ between these two vectors is defined by

$$\cos\theta = \frac{3 + 6t^2}{\sqrt{9 + 36t^2 + 36t^4}\sqrt{2}} = \cos(\pi/4)$$

where we used the formulas $(3 + 6t^2)^2 = 9 + 36t^2 + 36t^4$ and $\cos(\pi/4) = 1/\sqrt{2}$. *Note: The wording of the problem seems to suggest that all the tangent lines to the curve α intersect the line $y = 0$, $x = z$ but this is not correct. The tangent line to the curve α at the point $\alpha(t)$ has the parametric equations*

$$(x, y, z) = \alpha(t) + r\alpha'(t) = (3t + 3r, 3t^2 + 6rt, 2t^3 + 6rt^2).$$

(Here the variable r parameterizes the line.) The tangent line intersects the plane $y = 0$ when $r = -t/2$, but $3t + 3r \neq 2t^3 + 6rt^2$ for this value of r. Evidently, when do Carmo talks about the angle between two lines he means the angle between two vectors along the lines.

5. 1.3-5 Let $\alpha : (-1, \infty) \to \mathbb{R}^2$ be defined by

$$\alpha(t) = \big(x(t), y(t)\big) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}\right) = \big(g(t), g(t)t\big), \qquad g(t) := \frac{3t}{1+t^3}.$$

The derivative $\alpha'(t)$ is given by

$$\alpha'(t) = \big(g'(t), g'(t)t + g(t)\big), \qquad g'(t) = \frac{3(1+t^3) - 9t^3}{(1+t^3)^2} = \frac{3(1-2t^3)}{(1+t^3)^2}.$$

Now $g(0) = 0$ and $g'(0) = 3$ so $\alpha'(0) = (3, 0)$ which shows that the trace C of α is tangent to the x-axis at the point $\alpha(0)$. Now it is easy to see that

$$\lim_{t\to\infty} \alpha(t) = \lim_{t\to\infty} \alpha'(t) = (0, 0)$$

because in every fraction that appears, the degree of the numerator is smaller than the degree of the denominator.

Before discussing part **(c)** I'll make a preliminary remark. *Every line in the plane has an equation of form*

$$a_0x + b_0y + c_0 = 0$$

where either $a_0 \neq 0$ or $b_0 \neq 0$ (or both). Multiplying the vector (a_0, b_0, c_0) by a nonzero constant does not change the line, so the line does not determine the vector (a_0, b_0, c_0) uniquely. The vector $(-b_0, a_0)$ is perpendicular to the line at each point so there are only two equations for the line with a unit normal vector, namely $\pm(a_0x + b_0y + c_0) = 0$ where $a_0^2 + b_0^2 = 1$. The definition for what it means for a parameterized family

$$L(t) = \{(x, y) : a(t)x + b(t)y + c(t) = 0\}$$

of lines to "approach" a line

$$L_0 = \{(x, y,) : a_0x + b_0y + c_0 = 0\}$$

as $t \to t_0$ is that there is a choice of signs $\mu(t) = \pm 1$ such that

$$\lim_{t\to t_0} \mu(t)\frac{(a(t), b(t), c(t))}{\sqrt{a(t)^2 + b(t)^2}} = (a_0, b_0, c_0).$$

(This implies $a_0^2 + b_0^2 = 1$.) If we want to use language carefully, we should make this explicit before saying something like do Carmo says in part **(c)** *of the problem.*

Now to do part **(c)** we should compute the limit as $t \to -1$ of the unit normal vector

$$\mathbf{n}(t) = (a(t), b(t)) := \frac{(y'(t), -x'(t))}{\sqrt{y'(t)^2 + x'(t)^2}}$$

to the curve $\alpha(t) = (x(t), y(t))$. Now $x'(t) = g'(t)$ and $y'(t) = g'(t)t + g(t)$ so an explicit formula for $\mathbf{n}(t)$ is

$$\mathbf{n}(t) = \frac{\big(g(t) + g'(t)t, -g'(t)\big)}{\sqrt{(g(t) + g'(t)t)^2 + g'(t)^2}} = \frac{(h(t) + t, -1)}{\sqrt{(h(t) + t)^2 + 1}}, \qquad h(t) := \frac{g(t)}{g'(t)}.$$

(Here we used that $g'(t) > 0$ for $t \in (-1, 0)$.) Now

$$\begin{aligned} h(t) = \frac{g(t)}{g'(t)} &= \left(\frac{3t}{1+t^3}\right) \Big/ \left(\frac{3(1-2t^3)}{(1+t^3)^2}\right) \\ &= \left(\frac{3t}{1+t^3}\right)\left(\frac{(1+t^3)^2}{3(1-2t^3)}\right) \\ &= \frac{t(1+t^3)}{1-2t^3} \end{aligned}$$

so $\lim_{t\to-1} h(t) = 0$ and hence $\lim_{t\to-1} \mathbf{n}(t) = (-\frac{1}{2}, -\frac{1}{2})$. The tangent line to C at $\alpha(t)$ has equation $\mathbf{n}(t) \cdot (p - \alpha(t)) = 0$ where $p = (x, y)$. This expands to $a(t)x + b(t)y + c(t) = 0$ where

$$a(t) = \frac{h(t) + t}{\sqrt{(h(t) + t)^2 + 1}}, \qquad b(t) = \frac{t}{\sqrt{(h(t) + t)^2 + 1}},$$

and

$$\begin{aligned} c(t) = -a(t)x(t) - b(t)y(t) &= -\frac{3t(h(t) + t) - 3t^2}{\sqrt{(h(t) + t)^2 + 1}\,(1 + t^3)} \\ &= -\frac{3t^2}{\sqrt{(h(t) + t)^2 + 1}\,(1 - 2t^3)} \end{aligned}$$

so the limiting tangent line as $t \to -1$ has equation $-\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y - \frac{1}{\sqrt{2}} = 0$. This simplifies to $x + y + 1 = 0$ as in Figure 1-10 in do Carmo. The dashed curve in the figure is defined by the same formula with $t \in (-\infty, -1)$.

6. 1.3-6 The curve $\alpha(t) = (e^{-t}\cos t, e^{-t}\sin t)$ has velocity vector

$$\alpha'(t) = e^{-t}(-\cos t - \sin t, -\sin t + \cos t)$$

and speed $|\alpha'(t)| = 2e^{-t}$.

7. 1.4-5 A equation for the line through the three points $p_i = (x_i, y_i, z_i)$ is

$$\big((p_3 - p_1) \wedge (p_3 - p_2)\big) \cdot \big(p - p_3\big) = 0, \qquad p = (x, y, z).$$

This equation is linear, i.e. it can be written as $Ax + By + Cz + D = 0$ where $A = (y_3z_1 - z_1y_3)$, $B = (y_3z_2 - z_3y_2)$, $C = (x_3y_2 - y_3x_2)$, and

$$D = -\det\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}.$$

The three points p_i obviously lie on the surface

$$\big((p-p_1)\wedge(p-p_2)\big)\cdot\big(p-p_3\big)=0,$$

but it appears that this surface is cubic, not linear. However the surface *is* a plane. To see this write the triple product as a determinant

$$\big((p-p_1)\wedge(p-p_2)\big)\cdot\big(p-p_3\big)=\det\begin{pmatrix} x-x_1 & x-x_2 & x-x_3 \\ y-y_1 & y-y_2 & y-y_3 \\ z-z_1 & z-z_2 & z-z_3 \end{pmatrix}.$$

Subtracting the third column from the first and second gives

$$\big((p-p_1)\wedge(p-p_2)\big)\cdot\big(p-p_3\big)=\det\begin{pmatrix} x_3-x_1 & x_3-x_2 & x-x_3 \\ y_3-y_1 & y_3-y_2 & y-y_3 \\ z_3-z_1 & z_3-z_2 & z-z_3 \end{pmatrix}.$$

This is the same as the linear equation above.

8. 1.4-13 Assume that two maps $\mathbf{u},\mathbf{v}:I\to\mathbb{R}^3$ satisfy a differential equation

$$\mathbf{u}'=a\mathbf{u}+b\mathbf{v},\qquad \mathbf{v}'=c\mathbf{u}-a\mathbf{v}$$

where a,b,c are constants. Then

$$(\mathbf{u}\wedge\mathbf{v})'=\mathbf{u}'\wedge\mathbf{v}+\mathbf{u}\wedge\mathbf{v}'=(a\mathbf{u}+b\mathbf{v})\wedge\mathbf{v}+\mathbf{u}\wedge(c\mathbf{u}-a\mathbf{v})=0$$

as $\mathbf{v}\wedge\mathbf{v}=\mathbf{u}\wedge\mathbf{u}=0$ and the wedge product is distributive.

9. 1.5-1 The helix

$$\alpha(s)=(a\cos\theta,a\sin\theta,b\theta),\qquad \theta=\frac{s}{c},\qquad a^2+b^2=c^2$$

has derivative $\alpha'(s)=c^{-1}(-a\sin\theta,a\cos\theta,b)$ with length $|\alpha'(s)|=\sqrt{a^2+b^2}/|c|$ so α is parameterized by arc length. Since $\alpha''(s)=-ac^{-2}(\cos\theta,\sin\theta,0)$ the curvature is $\kappa=ac^{-2}$, and the Frenet trihedron is

$$\begin{array}{rcl} \mathbf{t} & = & c^{-1}(\;-a\sin\theta,\; a\cos\theta,\; b), \\ \mathbf{n} & = & (\;\cos\theta,\; \sin\theta,\; 0), \\ \mathbf{b} & = & c^{-1}(\;-b\sin\theta,\; b\cos\theta,\; -a). \end{array}$$

Since $\mathbf{b}'=-bc^{-2}(\cos\theta,\sin\theta,0)$ the torsion is $\tau=bc^{-2}$ by the third Frenet formula. A point p lies on the osculating plane at $\alpha(s)$ if and only if .

$$\mathbf{b}\cdot(p-\alpha(s))=0.$$

When $p=(x,y,z)$ this expands to

$$(-b\sin\theta)(x-a\sin\theta)+(b\cos\theta)(y-a\cos\theta)-a(z-b\theta)=0$$

and simplifies to

$$(-b\sin\theta)x + (b\cos\theta)y - az + ab(1+\theta) = 0.$$

The line through $\alpha(s)$ along the normal $\mathbf{n}(s)$ has parametric equation

$$\gamma(r) = \alpha(s) + r\mathbf{n}(s) = \big((a+r)\cos\theta, (a+r)a\sin\theta, b\theta\big)$$

and meets the z-axis when $r = -a$. The cosine of the angle between unit tangent $\mathbf{t}$ and the vector $(0,0,1)$ is the constant

$$\mathbf{t}\cdot(0,0,1) = c^{-1}(-a\sin\theta, a\cos\theta, b)\cdot(0,0,1) = a/c.$$

10. 1.5-4 Suppose that the curve $\alpha = \alpha(s)$ is parameterized by arc length and all its normal lines pass through the point $o \in \mathbb{R}^3$. Then there is a function $r = r(s)$ such that

$$\alpha(s) + r(s)\mathbf{n}(s) = o.$$

Differentiating and applying the Frenet formulas we get

$$\mathbf{t} + r'\mathbf{n} + r(-\kappa\mathbf{t} + \tau\mathbf{b}) = 0.$$

But the vectors $\mathbf{t}, \mathbf{n}, \mathbf{b}$ are linearly independent so $r' = 0$, $1 = r\kappa$, and $\tau = 0$. Hence r' is constant, $r = 1/\kappa$, $\tau = 0$, $\mathbf{b}$ is constant, and the curve lies in the plane through o perpendicular to $\mathbf{b}$. (The penultimate assertion uses the third Frenet equation $\mathbf{b}' = -\tau\mathbf{n}$. The last statement follows from the fact that the osculating plane has equation $\mathbf{b}\cdot(p-o) = 0$ and the fact that $\mathbf{b}$ is constant.)

The following problem is important (but was not assigned).

11. 1.5-12 Let the position of a particle at time t be given by

$$\alpha(t) = \beta(\sigma(t))$$

where β is parameterized by arclength and $\dot\sigma(t) = |\dot\alpha(t)|$ is the speed of the particle. Denote differentiation with respect to t by an overdot and differentiation with respect to s by a prime so if $g(t) = f(\sigma(t))$ we have $\dot g(t) = f'(\sigma(t))\dot\sigma(t)$ by the chain rule. By the Frenet equations

$$\begin{aligned}
\dot\alpha &= \dot\sigma\mathbf{t} \\
\ddot\alpha &= \ddot\sigma\mathbf{t} + \dot\sigma^2\mathbf{t}' = \ddot\sigma\mathbf{t} + \dot\sigma^2\kappa\mathbf{n} \\
\dddot\alpha &= \dddot\sigma\mathbf{t} + \ddot\sigma\dot\sigma\mathbf{t}' + (2\ddot\sigma\dot\sigma\kappa + \dot\sigma^3\kappa')\mathbf{n} + \dot\sigma^3\kappa\,\mathbf{n}' \\
&= (\dddot\sigma - \dot\sigma^3\kappa\tau)\mathbf{t} + (\ddot\sigma\dot\sigma\kappa + 2\ddot\sigma\dot\sigma\kappa + \dot\sigma^3\kappa')\mathbf{n} + \dot\sigma^3\kappa\tau\,\mathbf{b}
\end{aligned}$$

From the first two equations we get $\dot\alpha \wedge \ddot\alpha = \dot\sigma^3\kappa\,\mathbf{b}$ and hence $|\dot\alpha \wedge \ddot\alpha| = \dot\sigma^2\kappa$. Represent all three equations in the matrix form

$$\begin{pmatrix} \dot\alpha \\ \ddot\alpha \\ \dddot\alpha \end{pmatrix} = \begin{pmatrix} \dot\sigma & 0 & 0 \\ * & \dot\sigma^2\kappa & 0 \\ * & * & \dot\sigma^3\kappa\tau \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}$$

and we get $(\dot\alpha \wedge \ddot\alpha) \cdot \dddot\alpha = \dot\sigma^6\kappa^2\tau$. Combining gives

$$\kappa = \frac{|\dot\alpha \wedge \ddot\alpha|}{\dot\sigma^3}, \qquad \tau = \frac{(\dot\alpha \wedge \ddot\alpha)\cdot\dddot\alpha}{|\dot\alpha \wedge \ddot\alpha|^2}.$$

A plane curve is a special case: take $\alpha(t) = (x(t), y(t), 0)$. But in the case of a plane curve we can define the unit normal vector by rotation the unit tangent vector clockwise by a right angle. To maintain the second Frenet equation $\mathbf{t}' = \kappa\mathbf{n}$ we must allow κ to take negative values. The above equation for κ becomes

$$\kappa = \frac{\dot\alpha \wedge \ddot\alpha}{\dot\sigma^3} = \frac{\dot x\,\ddot y - \dot y\,\ddot x}{(\dot x^2 + \dot y^2)^{3/2}}$$

In terms of the velocity vector $\mathbf{v} = \dot\alpha$ and the acceleration vector $\mathbf{a} = \ddot\alpha$ the above equations for $\dot\alpha$ and $\ddot\alpha$ take the form

$$\mathbf{b} = \dot\sigma\mathbf{t}, \qquad \mathbf{a} = \ddot\sigma\mathbf{t} + \dot\sigma^2\kappa\mathbf{n}.$$

The second equation resolves the acceleration into its tangential and normal componenets and explains what happens when a car goes around a sharp curve in the road.