The Gauss map

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1. Let $C \subset \mathbb{R}^3$ be a curve and $p \in C$. Let $\alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ be a parameterization of C by arc length centered at p, i.e.

$$\|\alpha'(s)\|^2 = 1, \qquad \alpha(0) = p.$$

The vector $\alpha''(0)$ is called the **curvature vector** at p. Differentiating shows that $\langle \alpha'', \alpha' \rangle = 0$ so the curvature vector is orthogonal to the tangent vector $\alpha'(0)$ to the curve at p. Reversing the orientation of the curve (i.e. replacing s by $-s$) reverses the direction of the tangent vector but leaves the curvature vector unchanged.

2. Let $S \subset \mathbb{R}^3$ be an oriented surface. The **Gauss map** is the map $N : S \to S^2$ which assigns to $p \in S$ the unit normal. There are two unit normals ($-N$ is the other one); the meaning of the word *oriented* is that we have chosen one. Thus[1]

$$\|N(p)\| = 1, \qquad \langle N(p), \mathbf{v} \rangle = 0 \text{ for } \mathbf{v} \in T_pS. \qquad \text{page 136}$$

The **first fundamental form** assigns to each $p \in S$ the quadratic form $I_p : T_pS \to \mathbb{R}$ defined by

$$I_p(\mathbf{v}) = \langle \mathbf{v}, \mathbf{v} \rangle = \|\mathbf{v}\|^2 \qquad \text{page 92}$$

It assigns to each tangent vector $\mathbf{v} \in T_pS \subset \mathbb{R}^3$ the square of its length. The **second fundamental form** is defined by

$$II_p(\mathbf{v}) = \langle N(p), \alpha''(0) \rangle, \qquad \mathbf{v} = \alpha'(0)$$

where $\alpha : (\varepsilon, \varepsilon) \to S$ is a curve whose tangent vector at p is $\mathbf{v}$. Equation (†) below says that $II_\alpha(\alpha')$ is the normal component of the curvature vector α''.

3. Lemma. *The second fundamental form is independent of the choice of curve α used to define it.*

Proof. Since $\alpha(s) \in S$ we have $\alpha'(s) \in T_{\alpha(s)}S$ and hence $\langle N(\alpha(s), \alpha'(s) \rangle = 0$. Differentiating gives

$$\langle dN_p(\alpha'(0), \alpha'(0) \rangle + \langle N(p), \alpha''(0) \rangle$$

This shows that

$$II_p(\mathbf{v}) = -\langle dN_p(\mathbf{v}), \mathbf{v} \rangle \text{ for } \mathbf{v} \in T_pS \qquad \text{page 141}$$

is independent of the second derivative. □

[1] All page references are to the Do Carmo text.

4. Lemma. *The derivative $dN_p : T_pS \to T_{N(p)}S^2$ of the Gauss map is a map from a vector space to itself, i.e.*

$$T_pS = T_{N(p)}S^2$$

for $p \in S$.

Proof. $T_pS = N(p)^\perp$ and $T_wS^2 = w^\perp$ for $w \in S^2$. □

5. Lemma. *The derivative $dN_p : T_pS \to T_pS$ is self adjoint, i.e.*

$$\langle dN_p(\mathbf{u}), \mathbf{v} \rangle = \langle \mathbf{u}, dN_p(\mathbf{v}) \rangle$$

for $\mathbf{u}, \mathbf{v} \in T_pS$.

Proof. See Proposition 1 page 140. Choose a parameterization $\mathbf{x} : U \to S$ with

$$\mathbf{x}(0,0) = p, \qquad \mathbf{x}_u(0,0) = \mathbf{u}, \qquad \mathbf{x}_v(0,0) = \mathbf{v}.$$

Here (u, v) are the standard coordinates on the open set $U \subset \mathbb{R}^2$ and the subscripts u and v indicate partial differentiation.[2] Since $N(\mathbf{x}) \perp T_{\mathbf{x}}S$ and $\mathbf{x}_u, \mathbf{x}_v \in T_{\mathbf{x}}S$ we have

$$\langle N, \mathbf{x}_u \rangle = \langle N, \mathbf{x}_v \rangle = 0$$

so

$$\langle N_v, \mathbf{x}_u \rangle + \langle N, \mathbf{x}_{uv} \rangle = \langle N_u, \mathbf{x}_v \rangle + \langle N, \mathbf{x}_{vu} \rangle = 0. \qquad (*)$$

The lemma follows from $\mathbf{x}_{uv} = \mathbf{x}_{vu}$. □

6. Remark. Let $\alpha : (-\varepsilon, \varepsilon) \to S$ be a curve in S parameterized by arclength. By the geometric definition of the cross product, the vectors $N, \alpha', N \wedge \alpha'$ are orthonormal at each point $\alpha(s)$. The vector α' is a unit vector tangent to S (at α) and $N(\alpha)$ is a unit vector normal to S so $N \wedge \alpha'$ is a unit vector tangent to S and is orthogonal to both N and α'. Since $\|\alpha'\| = 1$ we also have $\langle \alpha', \alpha'' \rangle = 0$. Hence the curvature vector can be written as

$$\alpha'' = k_n N + k_g (N \wedge \alpha'), \qquad k_n := \langle \alpha'', N \rangle, \qquad k_g := \langle \alpha'', N \wedge \alpha' \rangle \qquad (\dagger)$$

The coefficient k_n is called the **normal curvature** and coefficient k_g is called the **geodesic curvature**. By definition

$$II_\alpha(\alpha') = -\langle \alpha'', N(\alpha) \rangle = -k_n.$$

By the Pythagorean theorem

$$k^2 = k_n^2 + k_g^2.$$

See page 249. A **geodesic** in S is a curve whose geodesic curvature is zero, i.e. whose curvature vector is normal to S.

[2] However the subscript p in the expression $dN_p(\mathbf{u})$ indicates that that the derivative is to be evaluated at p.

7. Remark. The curvature vector is the acceleration from classical mechanics so a particle moving in S and acted on by a force which is perpendicular to to S (and no other forces) moves along a geodesic.

8. Definition. The eigenvalues k_1, k_2 of dN_p are called the **principal curvatures** and the determinant

$$K := \det(dK_p) = k_1 k_2$$

is called the **Gauss curvature**. The average value

$$H := \frac{k_1 + k_2}{2}$$

of the principal curvatures is the called the **mean curvature**. Thus $\lambda = k_1$ and $\lambda = k_2$ are the two solutions of the characteristic equation

$$\lambda^2 + 2H\lambda + K = 0.$$

9. Remark. If dA denotes the area of an infinitesimal region on S containing the point p, then $K(p)\,dA$ is the area of the image of that infinitesimal region under the Gauss map. Thus $K(p)$ is the analog for surfaces of the curvature $k = d\theta/ds$ of a plane curve.

10. Let $U \subset \mathbb{R}^2$ be open and $\mathbf{x} : U \to S$ be a parameterization. The unit normal is

$$N = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{\|\mathbf{x}_u \wedge \mathbf{x}_v\|}.$$ page 135

A curve $\alpha : (-\varepsilon, \varepsilon) \to S$ can be written

$$\alpha(t) = \mathbf{x}(u(t), v(t))$$

where $(u(t), v(t)) \in U$. In these coordinates the fundamental forms are given by

$$I_\alpha(\alpha') = E(u')^2 + 2Fu'v' + G(v')^2,$$ page 92

$$II_\alpha(\alpha') = e(u')^2 + 2fu'v' + g(v')^2$$ page 154

where

$$\begin{array}{lll} E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle, & F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle, & G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle, \\ e = -\langle N_u, \mathbf{x}_u \rangle, & f = -\langle N_u, \mathbf{x}_v \rangle, & g = -\langle N_v, \mathbf{x}_v \rangle. \end{array}$$

are functions on U. The subscript on N means partial differentiation so

$$N_u = dN_{\mathbf{x}}(\mathbf{x}_u), \qquad N_v = dN_{\mathbf{x}}(\mathbf{x}_v).$$

By $(*)$ f can be written four ways.

11. Weingarten Equations.

$$N_u = a_{11}\mathbf{x}_u + a_{12}\mathbf{x}_v. \qquad N_v = a_{21}\mathbf{x}_u + a_{22}\mathbf{x}_v \qquad \text{page 154}$$

where

$$a_{11} = \frac{fF - eG}{EG - F^2}, \qquad a_{12} = \frac{gF - fG}{EG - F^2},$$
$$a_{21} = \frac{eF - fE}{EG - F^2}, \qquad a_{22} = \frac{fF - gE}{EG - F^2}. \qquad \text{page 155}$$

12. Corollary. *The Gauss curvature is given by*

$$K = \frac{eg - f^2}{EG - F^2}$$

and the mean curvature is given by

$$H = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}.$$

13. Suppose that the surface S is a graph, i.e. it is defined by an equation

$$z = h(x, y).$$

The tangent space at $p = (x, y, z) \in S$ is the graph of dh i.e. the set of all vectors (x', y', z') such that

$$z' = h_x(x, y)x' + h_y(x, y)y'.$$

The vector

$$N = \frac{(-h_x, -h_y, 1)}{\|N\|}, \qquad \|N\| = \sqrt{h_x^2 + h_y^2 + 1}$$

is one of the two unit normal vectors to S. There is an obvious parameterization $\mathbf{x}(u, v) = (x, y, z)$ where

$$x = u, \qquad y = v, \qquad z = h(u, v). \qquad (\#)$$

For this parameterization

$$\mathbf{x}_u = (1, 0, h_x), \qquad \mathbf{x}_v = (0, 1, h_y)$$

so

$$E = 1 + h_x^2, \qquad F = h_x h_y, \qquad G = 1 + h_y^2,$$
$$e = \frac{h_{xx}}{\|N\|}, \qquad f = \frac{h_{xy}}{\|N\|}, \qquad g = \frac{h_{yy}}{\|N\|},$$
$$K = \frac{h_{xx}h_{yy} - h_{xy}^2}{\|N\|^2},$$
$$2H = \frac{(1 + h_x^2)h_{yy} - 2h_x h_y h_{xy} + (1 + h_y^2)h_{xx}}{\|N\|^{3/2}}.$$

14. Theorem. *If $K(p) > 0$, then S lies to one side of $p + T_pS$ near p. If $K(p) < 0$, then S intersects $p + T_pS$.*

Proof. Choose coordinates on $\mathbb{R}^3$ so that $p = (0,0,0)$, T_pS = the xy-plane. Then S is a graph near p with equation $z = h(x,y)$ and $h(0,0) = h_x(0,0) = h_y(0,0) = 0$ and $d^2h(0,0)$ is the second fundamental form. Rotate the (x,y) plane so that $(1,0)$ and $(0,1)$ are eigenvectors of Hessian matrix

$$d^2h(0,0) = \begin{pmatrix} h_{xx}(0,0) & h_{xy}(0,0) \\ h_{yx}(0,0) & h_{yy}(0,0) \end{pmatrix}.$$

Then $x_{xy}(0,0) = h_{yx}(0,0) = 0$ so the principle curvatures are $k_1 = h_{xx}(0,0)$ and $k_2 = h_{yy}(0,0)$. The second fundamental form is $k_1x^2 + k_2y^2$. Then

$$h(x,y) = k_1x^2 + k_2y^2 + \text{higher order terms}.$$

See Proposition 3 in section 2-2 on page 63 and problem 26 on page 91. □

15. Remark. The *Implicit Function Theorem* says that if $N(p)$ does not lie in the xy-plane then p lies in the image of a local parameterization as in equation (#). This is Proposition 3 in section 2-2 on page 63. Since $N(p)$ cannot lie in all three coordinate planes it is always possible to choose two of the three coordinates x, y, z to parameterize the surface (near p) as a graph. For example, the unit sphere is covered by six parameterizations

$$\begin{array}{ccc} z = \sqrt{1-x^2-y^2}, & y = \sqrt{1-x^2-z^2}, & x = \sqrt{1-y^2-z^2}, \\ z = -\sqrt{1-x^2-y^2}, & y = -\sqrt{1-x^2-z^2}, & x = -\sqrt{1-y^2-z^2}. \end{array}$$

Other local parameterizations of the unit sphere are by cylindrical coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = \sqrt{1-r^2}$$

(this parameterizes the northern hemisphere), by spherical coordinates

$$x = \cos\theta\sin\varphi, \qquad y = \sin\theta\sin\varphi. \qquad z = \cos\varphi$$

(this parameterizes everything but the north and south poles) and stereographic projection

$$x = \frac{2u}{1+u^2+v^2}, \qquad y = \frac{2v}{1+u^2+v^2}, \qquad z = \frac{1-u^2-v^2}{1+u^2+v^2}$$

(this parameterizes everything but the south pole $(0,0,-1)$. See exercise 16 page 67.) The Gauss curvature of the unit sphere is (obviously) identically equal to one as the Gauss map is the identity map.

16. The point $(\cos(u \pm \nu), \sin(u \pm \nu), \pm 1)$ lies in the plane $z = \pm 1$. When $\nu = 0$ these points lie on the same vertical line but for $\nu > 0$ the upper one has been rotated clockwise and the lower one has been rotate counter clockwise. The line connecting these two points has parametric equations

$$x = x_0 + v\xi, \qquad y = y_0 + v\eta, \qquad z = v$$

where $(x_0, y_0, 0)$ is the midpoint of the line segment connecting them and $(2\xi, 2\eta, 2)$ is the vector from the lower point to the upper, i.e.

$$x_0 = \tfrac{1}{2}(\cos(u+\nu) + \cos(u-\nu)) = \cos u \cos \nu,$$
$$y_0 = \tfrac{1}{2}(\sin(u+\nu) + \sin(u-\nu)) = \sin u \cos \nu$$

and

$$\xi = \tfrac{1}{2}(\cos(u+\nu) - \cos(u-\nu)) = -\sin u \sin \nu,$$
$$\eta = \tfrac{1}{2}(\sin(u+\nu) - \sin(u-b)) = \cos u \sin \nu.$$

Since $x_0\xi + y_0\eta = 0$ we get

$$x^2 + y^2 = \cos^2 \nu + v^2 \sin^2 \nu = a^2 + b^2 z^2$$

where $a = \cos \nu$ and $b = \sin \nu$. This is the equation of a hyperboloid of one sheet. Replacing ν by $-\nu$ gives the same equation so the hyperboloid of one sheet contains two lines though every point. The tangent plane at any point intersects the hyperboloid in these two lines so the hyperboloid has negative Gauss curvature.

17. The equation $x^2 + y^2 = z^2 + 1$ defines a hyperboloid of one sheet, and the equation $x^2 + y^2 = z^2 - 1$ defines a hyperboloid of two sheets. The latter has positive Gauss curvature and therefore contains no lines.

18. Stereographic projection $\mathbb{R}^2 \to S^2$ is defined by the condition that the three points

$$s = (0, 0, -1), \qquad p = (x, y, z), \qquad w = (u, v, 0), \qquad x^2 + y^2 + z^2 = 1$$

are collinear. It covers the entire sphere *except for the south pole* $s = (0, 0, -1)$ in a one-one way. The analogous condition that the three points

$$s = (0, 0, -1), \qquad p = (x, y, z), \qquad w = (u, v, 0), \qquad x^2 + y^2 - z^2 = -1$$

be collinear be used to parameterize the upper sheet of the hyperboloid of one sheet by the unit disk $u^2 + v^2 < 1$. In this example the parameterization covers the whole upper sheet in a one-one way. (The south pole $s = (0, 0, -1)$ is on the lower sheet.)