Curves

JWR

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These notes summarize the key points in the first chapter of *Differential Geometry of Curves and Surfaces* by Manfredo P. do Carmo. I wrote them to assure that the terminology and notation in my lecture agrees with that text. *All page references in these notes are to the Do Carmo text.*

1. Definition. A **parameterized smooth curve** is a map $\alpha : I \to \mathbb{R}^n$ where $I \subseteq \mathbb{R}$ is an interval. The set theoretic image

$$C = \alpha(I) := \{\alpha(t) : t \in I\}$$

is called the **trace** of α and α is called a **parameterization** of C. See do Carmo page 2.

2. Remark. For do Carmo the words *differentiable* and *smooth* are synonymous. I prefer the word *smooth*. The adjective *differentiable* is often omitted by do Carmo.

3. Remark. On page 2 do Carmo says that the interval I should be open but on page 30 he extends the notion of smoothness to closed intervals. A function defined on a closed interval $[a, b]$ is said to be **smooth** iff it extends to an open interval containing $[a, b]$. This means that the derivatives of the function are defined at the end points a and b.

4. Definition. A **reparameterization** of $\alpha : I \to \mathbb{R}^n$ is a smooth map $\beta : J \to \mathbb{R}^n$ of form $\beta = \alpha \circ \sigma$ where $\sigma : J \to I$ is a diffeomorphism. That σ is a diffeomorphism means σ is one-to-one and onto (so there is an inverse map $\sigma^{-1} : J \to I$) and that $\sigma'(t) \neq 0$ for $t \in I$ (so that the map σ^{-1} is also smooth).

5. Remark. If β is a reparameterization of α, then the maps α and β have the same trace C. The idea of the definition is that we should think of α and β as different ways of describing the same curve C. However do Carmo avoids giving a precise definition of an (unparameterized) curve.

6. Example. The circle $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is the trace of the parameterized curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\alpha(\theta) = \big(\cos\theta, \sin\theta\big) = \big(\cos(\theta + 2\pi), \sin(\theta + 2\pi)\big).$$

Define a map $\beta : \mathbb{R} \to \mathbb{R}^2$ by

$$\beta(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right).$$

This map is a reparameterization of the restriction of α to the open interval $(-\pi, \pi)$ as follows:

$$\begin{aligned} \big(\cos(2\varphi), \sin(2\varphi)\big) &= \left(\frac{\cos^2\varphi - \sin^2\varphi}{\cos^2\varphi + \sin^2\varphi}, \frac{2\sin\varphi\cos\varphi}{\cos^2\varphi + \sin^2\varphi} \right) \\ &= \left(\frac{1-\tan^2\varphi}{1+\tan^2\varphi}, \frac{2\tan\varphi}{1+\tan^2\varphi} \right). \end{aligned}$$

Take $2\varphi = \theta$, $t = \tan\varphi = \tan(\theta/2)$, and we get $\alpha = \beta \circ \sigma$ where $\sigma : (-\pi, \pi) \to \mathbb{R}$ is defined by $\sigma(\theta) = \tan(\theta/2)$. The common trace of (the restriction of) α and the map β is the punctured circle $C \setminus (-1, 0)$. (This particular reparameterization is called the **Weierstrass substitution** or **half angle substitution**. It is one of the main techniques used to evaluate integrals in calculus.)

7. Definition. Let $\alpha : I \to \mathbb{R}^n$ be a smooth parameterized curve. The derivative $\alpha'(t)$ is called **velocity vector** at t. The map α is called **regular** iff its velocity vector never vanishes. The map α is said to be **parameterized by arc length** iff its tangent vector always has length one.

8. Theorem. *A smooth regular parameterized curve α has a reparameterization by arc length, i.e. there is a reparameterization $\beta : J \to \mathbb{R}^n$ of α such that $|\beta'(s)| = 1$ for $s \in J$.*

Proof: This is the content of Remark 2 in do Carmo page 21. The reparametrization is defined by $\beta = \alpha \circ \sigma$ where σ is a solution of the differential equation

$$\sigma'(s) = \frac{1}{|\alpha'(\sigma(s))|}.$$

By the chain rule $\beta'(s) = \alpha'(\sigma(s))\sigma'(s)$ so $|\beta'(s)| = 1$. □

9. Remark. The arc length

$$\ell(C) = \int_a^b |\alpha'(t)|\, dt$$

of the trace C of a regular parameterized curve $\alpha : [a, b] \to \mathbb{R}^n$ is independent of the parameterization α used to define it. This is an easy consequence of the formula for changing variables in a definite integral: if $\sigma : [a, b] \to [c, d]$ is a diffeomorphism, then

$$\int_a^b |\alpha'(t)|\, dt = \int_c^d |\alpha'(\sigma(r))|\, |\sigma'(r)|\, dr.$$

(The change of variables is $t = \sigma(r)$ so $dt = \sigma'(r)\, dr$.) When α is parameterized by arc length, $\ell(C) = |b - a|$.

10. The reparameterization in Theorem 8 is unique in the following sense: If $\beta_1 : J_1 \to \mathbb{R}^n$ and $\beta_2 : J_2 \to \mathbb{R}^n$ are two reparameterizations of the same map α then $\beta_2 = \beta_1 \circ \sigma$ where $\sigma : J_2 \to J_1$ has one of the two forms $\sigma(s) = s + c$ or $\sigma(s) = -s + c$. (This is because $|\sigma'(s)| = 1$.) On page 6 do Carmo says that when $\sigma(s) = -s + c$ the two curves β_1 and β_2 are said to **differ by a change of orientation**.

This use of the word *orientation* can be viewed as a special case of the definition of *orientation of a vector space* that do Carmo gives on pages 11 and 12. For a regular curve α the one dimensional vector space $\mathbb{R}\alpha'(t) \subseteq \mathbb{R}^n$ is called the **tangent space** to the curve at the point $\alpha(t)$. The velocity vector $\alpha'(t)$ is a basis for this space. Changing the orientation of the curve changes the sign of the velocity vector $\alpha'(t)$ and thus reverses the orientation of the tangent space.

11. Remark. Note the distinction between the *tangent space* and the *tangent line*. The **tangent line** is the line containing the points $\alpha(t)$ and $\alpha(t) + \alpha'(t)$. (See do Carmo page 5.) This line need not pass through the origin of $\mathbb{R}^n$ and thus is not a vector subspace of the vector space $\mathbb{R}^n$. This illustrates the difference between points and vectors.

12. Remark. The two orientations of $\mathbb{R}^3$ correspond to the thumb, forefinger, and middle finger of the right and left hands. (Recall the *right hand rule* from calculus.) The two orientations of $\mathbb{R}^2$ correspond to *clockwise* and *counter clockwise*. The two orientations of $\mathbb{R} = \mathbb{R}^1$ correspond to the two directions *increasing* and *decreasing*.

13. Definition. A map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is called an **isometry** iff it preserves distance i.e. iff it satisfies

$$|\Phi(p) - \Phi(q)| = |p - q|$$

for $p, q \in \mathbb{R}^n$. A map $\mathbb{R}^n \to \mathbb{R}^n$ is called a **translation** iff there is a vector $\mathbf{c} \in \mathbb{R}^n$ such that the map sends the point $p \in \mathbb{R}^n$ to the point $p + \mathbf{c}$. A linear transformation $\rho : \mathbb{R}^n \to \mathbb{R}^n$ is called **orthogonal** iff it satisfies $(\rho\mathbf{u})\cdot(\rho\mathbf{v}) = \mathbf{u}\cdot\mathbf{v}$ for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. A **rigid motion** of $\mathbb{R}^n$ is an isometry Φ such that the corresponding orthogonal linear transformation ρ preserves orientation, i.e. has positive determinant.

14. Theorem. *A map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry if and only if it is the composition of a translation and an orthogonal linear transformation.*

Proof: For *if* see do Carmo page 23 Exercise 6 and do Carmo page 228 Exercise 7. The converse is not very difficult but is not needed in the rest of these notes so the proof is omitted. □

15. Theorem. *Let $\alpha : [a, b] \to \mathbb{R}^n$ be smooth, $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry, and $\beta = \Phi \circ \alpha$:. Then the curves α and β have the same arc length. If α is parameterized by arc length so is β.*

Proof: This is because Φ preserves the length of vectors. The first part also follows from Exercise 8 on page 10 of do Carmo. □

16. Definition. Let $\alpha : I \to \mathbb{R}^n$ be parameterized by arc length. Then the **unit tangent vector** is the vector valued function $\mathbf{t} : I \to \mathbb{R}^n$ defined by

$$\mathbf{t}(s) = \alpha'(s) = \frac{d}{ds}\alpha(s),$$

the **curvature vector** is the vector valued function $I \to \mathbb{R}^n$

$$\alpha''(s) = \frac{d}{ds}\mathbf{t}(s) = \frac{d^2}{ds^2}\alpha(s),$$

and the **curvature** is the length κ of the curvature vector, i.e.

$$\kappa(s) = |\mathbf{t}'(s)| = |\alpha''(s)|.$$

The **unit normal vector** is the normalized curvature vector

$$\mathbf{n} = \frac{\mathbf{t}'}{|\mathbf{t}'|}.$$

(The vector $\mathbf{n}$ is defined only where the curvature κ is not zero.) The **binormal vector** is the vector product

$$\mathbf{b} = \mathbf{t} \wedge \mathbf{n}$$

of the unit tangent vector $\mathbf{t}$ and the unit normal vector $\mathbf{n}$. (The binormal vector is defined only when $n = 3$.)

17. Theorem. *Let $\alpha : I \to \mathbb{R}^n$ be parametrized by arc length, $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry, and $\beta = \Phi \circ \alpha : I \to \mathbb{R}^n$. Then β is also parametrized by arc length and α and β have the same curvature. If $n = 3$ and Φ is a rigid motion they have the same torsion.*

Proof: Exercise 6 page 23 of do Carmo. □

18. Standing Assumption. *Henceforth we assume that $\alpha : I \to \mathbb{R}^3$ is a regular curve parameterized by arc length.*

19. Theorem. *Then the vectors* $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ *are orthonormal, i.e.*

$$|\mathbf{t}| = |\mathbf{n}| = |\mathbf{b}| = 1, \qquad \mathbf{t}\cdot\mathbf{n} = \mathbf{t}\cdot\mathbf{b} = \mathbf{n}\cdot\mathbf{b} = 0.$$

The ordered orthonormal basis $\mathbf{t}, \mathbf{n}, \mathbf{b}$ *is called the* **Frenet trihedron**.

Proof: (See do Carmo pages 18-19.) The equations $|\mathbf{t}| = |\mathbf{n}| = 1$ hold by definition. Since $|\mathbf{t}|^2 = \mathbf{t}\cdot\mathbf{t}$ is constant we get

$$0 = \frac{d}{ds}|\mathbf{t}|^2 = \frac{d}{ds}\mathbf{t}\cdot\mathbf{t} = 2\,\mathbf{t}\cdot\mathbf{t}' = 2\kappa\,\mathbf{t}\cdot\mathbf{n}$$

so $\mathbf{t}\cdot\mathbf{n} = 0$. Now $\mathbf{b}$ is the vector product of two orthogonal unit vectors $\mathbf{t}$ and $\mathbf{n}$ so it is itself a unit vector and is orthogonal to both $\mathbf{t}$ and $\mathbf{n}$. □

20. Corollary. *The derivative* $\mathbf{b}'$ *of the binormal vector* $\mathbf{b}$ *is parallel to the unit normal vector* $\mathbf{n}$*, i.e. there is a real valued function* τ *such that*

$$\mathbf{b}' = \tau\mathbf{n}, \qquad \tau = \mathbf{b}'\cdot\mathbf{n}.$$

The function τ *is called the* **torsion**.

Proof: Since $\mathbf{t}'\wedge\mathbf{n} = \kappa\,\mathbf{n}\wedge\mathbf{n} = 0$ we have

$$\mathbf{b}' = (\mathbf{t}\wedge\mathbf{n})' = \mathbf{t}'\wedge\mathbf{n} + \mathbf{t}\wedge\mathbf{n}' = \mathbf{t}\wedge\mathbf{n}'$$

so $\mathbf{b}'\cdot\mathbf{t} = \mathbf{b}'\cdot\mathbf{n}' = 0$. □

21. Frenet Formulas. *The Frenet trihedron satisfies the differential equations*

$$\boxed{\mathbf{t}' = \kappa\mathbf{n}, \qquad \mathbf{n}' = -\kappa\mathbf{t} - \tau\mathbf{b}, \qquad \mathbf{b}' = \tau\mathbf{n}.}$$

Proof: The first and last formulas hold by definition. For the middle formula differentiate the identities $\mathbf{n}\cdot\mathbf{t} = \mathbf{n}\cdot\mathbf{b} = 0$ and $\mathbf{n}\cdot\mathbf{n} = 1$ to get

$$\begin{aligned} 0 &= \mathbf{n}'\cdot\mathbf{t} + \mathbf{n}\cdot\mathbf{t}' = \mathbf{n}'\cdot\mathbf{t} + \kappa\,\mathbf{n}\cdot\mathbf{n} = \mathbf{n}'\cdot\mathbf{t} + \kappa \\ 0 &= \mathbf{n}'\cdot\mathbf{b} + \mathbf{n}\cdot\mathbf{b}' = \mathbf{n}'\cdot\mathbf{t} + \tau\,\mathbf{n}\cdot\mathbf{n} = \mathbf{n}'\cdot\mathbf{b} + \tau \\ 0 &= 2\mathbf{n}'\cdot\mathbf{n} \end{aligned}$$

Since the Frenet trihedron is orthonormal

$$\mathbf{n}' = (\mathbf{n}'\cdot\mathbf{t})\mathbf{t} + (\mathbf{n}'\cdot\mathbf{n})\mathbf{n} + (\mathbf{n}'\cdot\mathbf{b})\mathbf{b}.$$

This proves the middle Frenet formula. □

22. Remark. The Frenet formulas may be written in matrix form as

$$\begin{pmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}$$

The coefficient matrix is skew symmetric. This is no coincidence. The two triples

$$\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s), \qquad \mathbf{t}(s_0), \mathbf{n}(s_0), \mathbf{b}(s_0)$$

are both bases for the vector space $\mathbb{R}^3$ so there is a unique change of basis matrix $U(s)$ satisfying

$$\begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix} = U(s) \begin{pmatrix} \mathbf{t}(s_0) \\ \mathbf{n}(s_0) \\ \mathbf{b}(s_0 \end{pmatrix}.$$

Since the two bases are both orthonormal, the matrix $U(s)$ is orthogonal. Differentiating with respect to s and evaluating at $s = s_0$ gives the Frenet formula (in matrix form) evaluated at $s = s_0$. But $U(s_0)$ is the identity matrix and $U(s)^* = U(s)^{-1}$ so $U^*(s)U(s)$ is the identity matrix so differentiating at s and evaluating at s_0 gives

$$U'(s_0)^* + U'(s_0) = 0.$$

□

23. Theorem. *Reversing the orientation of α leaves the curvature κ and the torsion τ unchanged, i.e. if $\beta(s) = \alpha(-s)$ the curves α and β have the same curvature and torsion at $s = 0$.*

Proof: By definition the curvature κ is nonnegative, the normal vector is only defined at points where the curvature κ is not zero, reversing the orientation of α reverses the sign of the unit tangent vector $\mathbf{t}$ and leaves the sign of the curvature vector unchanged. Reversing the orientation of α reverses the sign of $\mathbf{t}$, preserves the sign of $\mathbf{n}$, and therefore reverses the sign of $\mathbf{b} = \mathbf{t} \wedge \mathbf{n}$. But reversing the orientation of $\mathbf{b}$ reverses the sign of $\mathbf{b}'$ so reversing the orientation of α preserves the sign of $\mathbf{b}'$ and hence (by the Frenet formula $\mathbf{b}' = \tau\mathbf{n}$) preserves the sign of τ. □

24. Fundamental Theorem. *Let $\kappa, \tau : I \to \mathbb{R}$ be smooth functions defined on an interval I. Assume $\kappa > 0$. Then*

(Existence.) There is a curve $\alpha : I \to \mathbb{R}^3$ parameterized by arc length with curvature κ and torsion τ.

(Uniqueness.) If $\alpha, \beta : I \to \mathbb{R}^3$ are two curves paramaterized by arc length both having curvature κ and torsion τ, then there is a rigid motion $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\beta = \Phi \circ \alpha$.

Proof: See do Carmo page 309. □

25. Corollary. The curvature and torsion of the helix $\alpha(\theta) = (a\cos\theta, a\sin\theta, b\theta)$ are both constant so for any two points p and q on the helix there is a rigid motion carrying p to q and mapping the helix to itself.

26. Gauss curvature. In the case of a plane curve ($n = 2$) it is possible to choose a normal vector even when the curvature is zero. In this case since $\mathbf{t}$ and $\mathbf{n}$ are orthogonal unit vectors we can define $\mathbf{n}$ by rotating $\mathbf{t}$ clockwise through 90 degrees:

$$\mathbf{t} = (\xi, \eta), \qquad \mathbf{n} = (\eta, -\xi).$$

With this definition both $\mathbf{t}$ and $\mathbf{n}$ change sign when the orientation is reversed so to maintain the equation $\mathbf{t}' = \kappa\mathbf{n}$ it is necessary to allow κ to be negative. For a plane curve $\alpha : I \to \mathbb{R}^2$ parameterized by arc length we can view the unit normal vector as a map to the unit circle and define an angle $\theta = \theta(s)$ by the formula

$$\mathbf{n}(s) = (\cos\theta(s), \sin\theta(s)).$$

We then define the **signed curvature** by the formula

$$\kappa = \frac{d\theta}{ds}.$$

The signed curvature κ for a plane curve $C \subseteq \mathbb{R}^2$ is analogous to the Gauss curvature K of a surface $S \subseteq \mathbb{R}^3$. (See do Carmo pages 146, 155, 167.) Note that when $\alpha(s) = (\cos s, \sin s)$ is the counter clockwise parameterization of the unit circle in $\mathbb{R}^2$, the vector $\mathbf{n}$ defined by rotation of $\mathbf{t}$ as above is the outward normal (=radius vector) to the circle and the curvature κ is identically one. Thus the curvature compares the curve α to the unit circle.

27. Setup for local canonical form. Assume that $\alpha : I \to \mathbb{R}^3$ has positive curvature and $s_0 \in I$. The Taylor expansion

$$\alpha(s) = \alpha(s_0) + (s - s_0)\alpha'(s_0) + \frac{(s-s_0)^2}{2}\alpha''(s_0) + \frac{(s-s_0)^3}{6}\alpha'''(s_0) + \cdots$$

tells us what the trace C of α looks like near the point $\alpha(s_0) \in \mathbb{C}$. Because any reparameterization of C has the same trace we assume that α is parameterized by arc length. Because the reparameterization defined by $\sigma(s) = s - s_0$ is also a parameterization by arc length, we assume that $s_0 = 0$. Because the arc length, curvature, and torsion are invariant under rigid motions, we assume that

$$\alpha(0) = (0,0,0), \qquad \mathbf{t}(0) = (1,0,0), \qquad \mathbf{n}(0) = (0,1,0), \qquad \mathbf{b}(0) = (0,0,1).$$

28. Local Canonical Form. *In the notation of Setup 27 above, the Taylor expansion of* $\alpha(s) = (x(s), y(s), z(s))$ *is*

$$\begin{aligned} x(s) &= s - \frac{\kappa(0)s^3}{6} + R_x \\ y(s) &= \frac{\kappa(0)s^2}{2} - \frac{\kappa'(0)s^3}{6} + R_y \\ z(s) &= -\frac{\kappa(0)\tau(0)s^3}{6} + R_z \end{aligned}$$

where $R_x, R_y, R_z = o(s^3)$.

Proof: There is no constant term in these formulas because $\alpha(0) = 0$. By definition

$$\alpha' = \mathbf{t}, \qquad \alpha'' = \kappa\mathbf{n}.$$

Differentiating once more gives

$$\alpha''' = \kappa'\mathbf{n} + \kappa\mathbf{n}' = \kappa'\mathbf{n} + \kappa(-\kappa\mathbf{t} - \tau\mathbf{b})$$

by the second Frenet formula. Now evaluate at $s = s_0 = 0$. □

29. Application. Recall (Remark 11 above and do Carmo page 5) that the **tangent line** to the trace C of a regular curve α at a point $p_0 = \alpha(s_0) \in C$ is the line containing the two points p_0 and $p_0 + \mathbf{t}_0$ where $\mathbf{t}_0 = \mathbf{t}(s_0)$. The **osculating plane** to C at p_0 is the plane containing the three points p_0, $p_0 + \mathbf{t}_0$, $p_0 + \mathbf{n}_0$ where $\mathbf{n}_0 = \mathbf{n}(s_0)$. (See do Carmo pages 17, 29, 30. The definition assumes that the curvature $\kappa(s_0)$ at p_0 is positive.) Let $p_1 = \alpha(s_1)$ and $p_2 = \alpha(s_2)$ be two other points on C distinct from p_0 and each other. Then as $s_1 \to s_0$ the limit of the line through p_0 and p_1 is the tangent line at p_0 and the limit as $s_1, s_2 \to s_0$ of the plane through p_0, p_1, and p_2 is the osculating plane.