# The Uniformization Theorem

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The proof given here is a loose translation of [3]. There is another proof of the Uniformization Theorem in [2] where it is called the *Riemann Mapping Theorem.*

## 1 Harmonic functions

**1.1.** Throughout this section $X$ denotes a connected Riemann surface, possibly noncompact. The open unit disk in $\mathbb{C}$ is denoted by $\mathbb{D}$. A **conformal disk** in $X$ centered at $p \in X$ is a pair $(z, D)$ where $z$ is a holomorphic coordinate on $X$ whose image contains the closed disk of radius $r$ about the origin in $\mathbb{C}$, $z(p) = 0$, and

$$D = \{q \in X : |z(q)| < r\}.$$

For a conformal disk $(z, D)$ we abbreviate the average value of a function $u$ on the boundary of $D$ by by

$$M(u, z, \partial D) := \frac{1}{2\pi} \int_0^{2\pi} v(re^{i\theta})\, d\theta, \qquad v(z(q)) = u(q).$$

**1.2.** The **Poisson kernel** is the function $P : \mathbb{D} \times \partial\mathbb{D} \to \mathbb{R}$ defined by

$$P(z, \zeta) := \frac{1}{2\pi} \cdot \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2}.$$

The unique solution of Dirichlet's problem

$$\Delta u = 0, \qquad u|\partial\mathbb{D} = \phi$$

where $\phi \in C^0(\partial\mathbb{D})$ on the unit disk $\mathbb{D}$ is given by the **Poisson integral formula**

$$u(z) = \int_{\partial\mathbb{D}} P(z, e^{i\theta})\phi(e^{i\theta})\, d\theta.$$

(See [1] page 13 for the proof.) For a conformal disk $(z, D)$ in a Riemann surface $X$ and a continuous function $u : X \to \mathbb{R}$ we denote by $u_D$ the unique continuous function which agrees with $u$ on $X \setminus D$ and is harmonic in $D$. (It is given by reading $u|\partial D$ for $\phi$ in the Poisson integral formula.)

**1.3.** The **Hodge star operator** on Riemann surface $X$ is the operation which assigns to each 1-form $\omega$ the 1-form $*\omega$ defined by

$$(*\omega)(\xi) = -\omega(i\xi)$$

for each tangent vector $\xi$. If $z$ is a holomorphic coordinate on $X$ a real valued 1-form has the form

$$\omega = a\, dx + b\, dy$$

where $a$ and $b$ are real valued functions, $x = \Re(z)$, and $y = \Im(z)$; the form $*\omega$ is then given by

$$*\omega = -b\, dx + a\, dy.$$

A function $u$ is called **harmonic** iff it is $C^2$ and $*du$ is closed, i.e. $d * du = 0$. In terms of the coordinate $z$ we have

$$d{*}du = (\Delta u)\, dx \wedge dy, \qquad \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Note that the operator $\Delta$ is not intrinsic, i.e. $\Delta u$ depends on the choice of holomorphic coordinate. (However the operator $d{*}d$ is independent of the choice of coordinate and hence also the property of being harmonic.)

**Theorem 1.4.** *Let $X$ a Riemann surface and $u : X \to \mathbb{R}$ be continuous. Then the following are equivalent:*

**(i)** *$u$ is harmonic.*

**(ii)** *$u$ satisfies* **mean value property** *i.e. $u(p) = M(u, z, \partial D)$ for every conformal disk $(D, z)$ centered at $p$;*

**(iii)** *$u$ is locally the real part of a holomorphic function.*

*Proof.* It is enough to prove this for an open subset of $\mathbb{C}$. That (ii) $\implies$ (i) follows from the Poisson integral formula, namely

$$u(0) = \int_{\partial\mathbb{D}} P(0, e^{i\theta})\phi(e^{i\theta})\, d\theta = M(u, \mathrm{id}, \partial\mathbb{D}).$$

The general case follows by the change a change of variables $z \mapsto az+b$. Note that the mean value property implies the **maximum principle**: the function $u$ has no strict maximum (or minimum) on any open set. A function which is continuous on the closure of $D$ and satisfies the mean value property in $D$ must therefore assume its maximum and minimum on the boundary of $D$. Hence (ii) $\implies$ (i) because $u - u_D$ (see 1.2) satisfies the mean value property in $D$ and vanishes on $\partial D$ so $u - u_D = 0$ on $D$ so $u$ is harmonic on $X$ as $D$ is arbitrary. For (i) $\iff$ (iii) choose a conformal disk $(z, D)$. The function $u$ is harmonic if and only if the form $*du$ is closed. The equation $*du = dv$ encodes the Cauchy Riemann equations; it holds if and only if the function $u + iv$ is holomorphic. ☐

**Corollary 1.5.** *The form $*du$ is exact if and only there is a holomorphic function $f : X \to \mathbb{C}$ with $u = \Re(f)$.*

**Theorem 1.6 (Removable Singularity Theorem).** *If $u$ is harmonic and bounded on the punctured disk, it extends to a harmonic function on the disk.*

*Proof.* By shrinking the disk and subtracting the solution of the Dirichlet problem we may assume that $u$ vanishes on $\partial D$. For $\varepsilon > 0$ the harmonic function

$$v_\varepsilon(z) = u(z) + \varepsilon \log|z| - \varepsilon$$

is negative on $\partial\mathbb{D}$ and near 0 and thus negative on $\mathbb{D} \setminus \{0\}$. Fix $z$ and let $\varepsilon \to 0$; we conclude that $u \leq 0$ on $\mathbb{D} \setminus \{0\}$. Similarly $-u \leq 0$. ☐

**Remark 1.7.** Bôrcher's Theorem (see [1] page 50) says that a positive function which harmonic on $\mathbb{D} \setminus 0$ has the form

$$u(z) = -b\log|z| + h(z)$$

where $b \geq 0$ and $h$ is harmonic on $\mathbb{D}$. This implies the Removable Singularity Theorem (add a constant). To prove Bôrcher's Theorem choose $b$ so that $*du + b\log|z|$ is exact. Hence assume w.l.o.g. that $u$ is the real part of a holomorphic function with a possible singularity at the origin. However if the Laurent expansion for this function contains any negative powers of $z$ its real part $u$ will be unbounded in both directions.

**Theorem 1.8 (Harnack's Principle).** *A pointwise nondecreasing sequence of harmonic functions converges uniformly on compact sets either to $\infty$ or to a harmonic function.*

*Proof.* If we assume that the sequence converges uniformly, this follows from the characterization via the mean value property (Theorem 1.4). For the general argument see [1] page 49. □

**Theorem 1.9.** *If a sequence of harmonic functions converges uniformly on compact subsets then the limit is harmonic and for $k = 1, 2, \ldots$ the sequence converges in $C^k$ (uniformly on compact subsets).*

*Proof.* In each holomorphic disk $(z, D)$ we have

$$\partial^\kappa u = \int_0^{2\pi} \partial^\kappa P_\zeta \cdot u(\zeta) d\theta$$

for each multi-index $\kappa = (\kappa_1, \kappa_2)$ where $\zeta = re^{i\theta} = z(q)$, $q \in \partial D$ (i.e. $|z(q)| = r$), and $P_\zeta(z) = P(z, \zeta)$ is the Poisson kernel. ( See [1] page 15.) □

**Theorem 1.10 (Compactness Theorem).** *A uniformly bounded sequence of harmonic functions contains a subsequence which converges uniformly on compact sets.*

*Proof.* By the estimate in the proof of 1.9 the the first derivatives of the sequence are uniformly bounded on any compact subset of any open disk on which the functions $u_n$ are uniformly bounded. Use Arzela Ascoli and diagonalize over compact sets. (See [1] page 35.) □

# 2 The Dirichlet Problem

Compare the following lemma and definition with Theorem 1.4.

**Lemma 2.1.** *Let $u : X \to \mathbb{R}$ be continuous. Then the following conditions are equivalent*

**(i)** *The function $u$ satisfies the following form of the* **maximum principle***: For every connected open subset $W \subset X$ and every harmonic function $v$ on $W$ either $u - v$ is constant or else it does not assume its maximum in $W$;*

**(ii)** *For every conformal disk* $(z, D)$ *we have* $u \leq u_D$*;*

**(iii)** $u$ *satisfies* **mean value inequality**, *i.e.* $u(p) \leq M(u, z, \partial D)$ *for every conformal disk* $(z, D)$ *centered at* $p$*;*

*A function* $u$ *which satisfies these conditions is called* **subharmonic**. *A function* $u$ *is called* **superharmonic** *iff* $-u$ *is subharmonic.*

*Proof.* We prove (i) $\Longrightarrow$ (ii). Choose $(z, D)$ and let $v = u_D$. Then $u - u_D$ is zero on $\partial D$ and thus either the constant 0 on $D$ or else nowhere positive. In either case $u \leq u_D$.

We prove (ii) $\Longrightarrow$ (iii). Choose $(z, D)$ centered at $p$. Then by (ii) we have $u(p) \leq u_D(p) = M(u_D, z, \partial D) = M(u, z, \partial D)$.

We prove (iii) $\Longrightarrow$ (i). Suppose that $v$ is harmonic on a connected open subset $W \subset X$ and that $u - v$ assumes its maximum $M$ at some point of $W$, i.e. that the set

$$W_M := \{p \in W : u(p) - v(p) = M\}$$

is nonempty. We must show that $u - v = M$ on all of $W$, i.e. that $W_M = W$. The set $W_M$ is closed in $W$ so it is enough to show that $W_M$ is open. Choose $p \in W_M$ and let $(z, D)$ be a conformal disk centered at $p$. Then

$$M = (u - v)(p) \leq M(u - v, z, \partial D) \leq M.$$

But $u - v \leq M$ so $u - v = M$ on $\partial D$. By varying the radius of $D$ we get that $u - v = M$ near $p$. □

**Corollary 2.2.** *Subharmonic functions satisfy the following properties.*

**(i)** *The max and sum of two subharmonic functions is subharmonic and a positive multiple of a subharmonic function is subharmonic.*

**(ii)** *The subharmonic property is local: if* $X = X_1 \cap X_2$ *where* $X_1$ *and* $X_2$ *are open and* $u \in C^0(X, \mathbb{R})$ *is subharmonic on* $X_1$ *and on* $X_2$*, then it is subharmonic on* $X$*.*

**(iii)** *If* $u$ *is subharmonic so is* $u_D$*.*

**(iv)** *If* $u : X \to \mathbb{R}$ *is continuous, positive and harmonic on an open set* $V$*, and vanishes on* $X \setminus V$*, then* $u$ *is subharmonic on* $X$*.*

*Proof.* Part (i) is immediate and part (ii) follows easily from part (i) of 2.1. For (iii) suppose that $v$ is harmonic and $u_D - v$ assumes its maximum $M$ at $p$. Since $u_D - v$ is harmonic in $D$ it follows by the maximum principle that the maximum on $D$ is assumed on $\partial D$ so we may assume that $p \in X \setminus D$. But $u = u_D$ on $X \setminus D$ and $u - v \leq u_D - v \leq M$ so $u - v$ also assumes its maximum at $p$. Hence $u - v = M$ and hence $u$ is harmonic. For (iv) suppose $u - v$ assumes its maximum at a point $p$ and $v$ is harmonic. We derive a contradiction. After subtracting a constant we may assume that this maximum is zero. Then $u \leq v$ so $0 \leq v$ on $X \setminus V$ and $0 < u \leq v$ on $X$. If $p \notin X$ then $v(p) = u(p) = 0$ and $v$ assumes its minimum at $p$ which contradicts the fact that $v$ is harmonic. If $p \in X$ then $u - v$ assumes its maximum at $p$ and this contradicts the fact that $u$ is subharmonic on $V$. □

**Remark 2.3.** The theory of subharmonic functions works in all dimensions. In dimension one, condition (ii) of lemma 2.1 says that $u$ is a convex function.

**Exercise 2.4.** A $C^2$ function $u$ defined on an open subset of $\mathbb{C}$ is subharmonic iff and only if $\Delta u \geq 0$.

**2.5.** A **Perron family** on a Riemann surface $X$ is a collection $\mathcal{F}$ of functions on $X$ such that

**(P-1)** $\mathcal{F}$ is nonempty;

**(P-2)** every $u \in \mathcal{F}$ is subharmonic;

**(P-3)** if $u, v \in \mathcal{F}$, then there exists $w \in \mathcal{F}$ with $w \geq \max(u, v)$;

**(P-4)** for every conformal disk $(z, D)$ in $X$ and every $u \in \mathcal{F}$ we have $u_D \in \mathcal{F}$;

**(P-5)** the function

$$u_{\mathcal{F}}(q) := \sup_{u \in \mathcal{F}} u(q)$$

is everywhere finite.

**Theorem 2.6 (Perron's Method).** *If $\mathcal{F}$ is a Perron family, then $u_{\mathcal{F}}$ is harmonic.*

*Proof.* Choose a conformal disk $(z, D)$. Diagonalize on a countable dense subset to construct a sequence $u_n$ of elements of $\mathcal{F}$ which converges pointwise to $u_{\mathcal{F}}$ on $D$ on a dense set. By (P-3) choose $w_n \in \mathcal{F}$ with $w_1 = u_1$ and $w_{n+1} \geq$

$\max(w_1, \dots, w_n)$. The new sequence is pointwise monotonically increasing. By (P-2) and (P-4) we may assume that each $w_n$ is harmonic on $D$. Then $u_{\mathcal{F}}$ is harmonic in $D$ by Harnack's Principle and (P-4). □

**2.7.** Assume $Y \subseteq X$ open with $\partial Y \neq \emptyset$. Define

$$\mathcal{F}_\phi = \{u \in C^0(\bar{Y}, \mathbb{R}) : u \text{ subharmonic on } Y, \ \ u \leq \sup_{\partial Y} \phi, \ u|\partial Y \leq \phi\}$$

and

$$u_\phi(q) = \sup_{u \in \mathcal{F}_\phi} u(q).$$

**Lemma 2.8.** *If $\phi : \partial Y \to \mathbb{R}$ is continuous and bounded then the family $\mathcal{F}_\phi$ is a Perron family so $u_\phi$ is harmonic on $Y$.*

*Proof.* Maximum principle. □

**2.9.** A **barrier function** at $p \in \partial Y$ is a function $\beta$ defined in a neighborhood $U$ of $p$ which is continuous on the closure $\overline{Y \cap U}$ of $Y \cap U$, superharmonic on $Y \cap U$, such that $\beta(p) = 0$ and $\beta > 0$ on $\overline{Y \cap U} \setminus \{p\}$. A point $p \in \partial Y$ is called **regular** iff there is a barrier function at $p$

**Lemma 2.10.** *If $\partial Y$ is a $C^1$ submanifold, it is regular at each of its points.*

*Proof.* Suppose w.l.o.g that $Y \subset \mathbb{C}$ and that $\partial Y$ is transverse to the real axis at 0, and that $Y$ lies to the right. Then

$$\beta(z) = \sqrt{r}\cos(\theta/2) = \Re(\sqrt{z}), \qquad z = re^{i\theta}$$

is a barrier function at 0. □

**Lemma 2.11.** *If $p$ is regular, then $\lim_{y \to p} u_\phi(y) = \phi(p)$.*

*Proof.* The idea is that $u_\phi(p) \leq \phi(p)$ and if we had strict inequality we could make $u_\phi$ bigger by adding $\varepsilon - \beta$. See [1] page 203. □

**Corollary 2.12.** *If $\partial Y$ is a $C^1$ submanifold of $X$ then $u_\phi$ solves the Dirichlet problem with boundary condition $\phi$, i.e. it extends to a continuous function on $Y \cup \partial Y$ which agrees with $\phi$ on $\partial Y$.*

# 3 Green Functions

**Definition 3.1.** Let $X$ be a Riemann surface and $p \in X$. A **Green function** at $p$ is a function $g : X \setminus \{p\} \to \mathbb{R}$ such that

**(G-1)** $g$ is harmonic;

**(G-2)** for some (and hence every) holomorphic coordinate $z$ centered at $p$ the function $g(z) + \log(z)$ is harmonic near $p$;

**(G-3)** $g > 0$;

**(G-4)** if $g' : X \setminus \{p\} \to \mathbb{R}$ satisfies (G-1),(G-2),(G-3) then $g \leq g'$.

Condition (G-4) implies that the Green function at $p$ is unique (if it exists) so we denote it by $g_p$. Warning: When $X$ is the interior of a manifold with boundary, the Green function defined here differs from the usual Green's function by a factor of $-1/(2\pi)$.

**Definition 3.2.** A Riemann surface $X$ is called **elliptic** iff it is compact, **hyperbolic** iff it admits a nonconstant negative subharmonic function, and **parabolic** otherwise. By the maximum principle for subharmonic functions (in the elliptic case) and definition (in the parabolic case) a nonhyperbolic surface admits no nonconstant negative subharmonic function. In particular, it admits no nonconstant negative harmonic function and hence (add a constant) no nonconstant bounded harmonic function.

**Theorem 3.3.** *For a Riemann surface the following are equivalent.*

**(i)** *there is a Green function at every point;*

**(ii)** *there is a Green function at some point;*

**(iii)** *$X$ is hyperbolic;*

**(iv)** *for each compact set $K \subset X$ such that $\partial K$ smooth and $W := X \setminus K$ is connected, there is a continuous function $\omega : W \cup \partial W \to \mathbb{R}$ such that $\omega \equiv 1$ on $\partial K = \partial W$ and on $W$ we have both that $0 < \omega < 1$ and that $\omega$ is harmonic on $W$.*

*Proof. That (i) $\implies$ (ii) is obvious; we prove (ii) $\implies$ (iii).* Suppose $g_p$ is a Green function at $p$. Then $u = \max(-2, -g_p)$ is negative and subharmonic. Now $u = -2$ near $p$, so either $u$ is nonconstant or else $-g_p \le -2$ everywhere. The latter case is excluded, since otherwise $g' = g_p - 1$ would satisfy (G-1), (G-2), (G-3) but not (G-4).

*We prove (iii) $\implies$ (iv).* Assume $X$ is hyperbolic. Then there is a superharmonic $u : X \to \mathbb{R}$ which is nonconstant and everywhere positive. Choose a compact $K$ $K$ and let $W = X \setminus K$. After rescaling we may assume that $\min_K u = 1$. By the Maximum principle (for $-u$) and the fact that $u$ is not constant there are points (necessarily in $W$) where $u < 1$ so after replacing $u$ by $\min(1, u)$ we may assume that $u \equiv 1$ on $K$. The family

$$\mathcal{F}_K = \{v \in C^0(W \cup \partial W, \mathbb{R}) : v \le u \text{ and } v \text{ subharmonic on } W\}$$

is a Perron family: (P-1) $\mathcal{F}_K \neq \emptyset$ as the restriction of $-u$ to $X \setminus K$ is in $\mathcal{F}_K$; (P-2) $v \in \mathcal{F}_K \implies v$ subharmonic by definition; (P-3) $v_1, v_2 \in \mathcal{F}_K \implies \max(v_1, v_2) \in \mathcal{F}$; (P-4) $v \in \mathcal{F}_K$ and $D$ a conformal disk in $X \setminus K$ implies that $v_D \le u_D \le u$ as $v \le u$ on $\partial D$ and $u$ is superharmonic; and (P-5) the function $\omega := \sup_{v \in \mathcal{F}} v$ is finite as $v \in \mathcal{F} \implies v \le u$. It remains to show that $0 < \omega < 1$ on $W$ and $\omega = 1$ on $\partial W$. Suppose $Y \subset X$ is open, with $\partial Y$ smooth and $Y \cup \partial Y$ compact, and $\partial K \subset \partial Y$. Let $w$ be the solution of the Dirichlet problem with $w = 1$ on $\partial K$ and $w = 0$ on $(\partial Y) \setminus (\partial K)$. Extend $w$ by zero on $W \setminus Y$. The extended function $w$ is subharmonic by Corollary 2.2 part (iv). Thus $w_Y - u$ is subharmonic and $\le 0$ on $X \setminus Y$ and on $\partial Y$. Hence $w_Y | W \in \mathcal{F}$. As the sets $Y$ exhaust $X$ and $w > 0$ on $Y$ it follows that $\omega$ satisfies $0 < \omega$ on $W$ and $\omega = 1$ on $\partial W$. Since $\partial W$ is smooth it follows that $\omega$ is continuous on $\partial W$. Since $\omega \le u$ and $u < 1$ on $W$ we have that $\omega \le u < 1$ on $Y$ so $\omega$ satisfies (iv).

*We prove (iv) $\implies$ (i).* Choose $p \in X$ and a conformal disk $(z, D)$ centered at $p$. Let $\mathcal{F}$ be the set of all continuous functions $v : X \setminus \{p\} \to \mathbb{R}$ satisfying the following conditions:

**(a)** $\operatorname{supp}(v) \cup \{p\}$ is compact;

**(b)** $v$ is subharmonic on $X \setminus \{p\}$, and

**(c)** $v + \log|z|$ extends to a subharmonic on function on $D$.

We show that $\mathcal{F}$ is a Perron family. The set $\mathcal{F}$ is not empty since it contains the function $-\ln|z|$ (extended by 0). Properties (i-iii) in 2.5 are immediate.

It remains to show (iv), i.e. that $u_{\mathcal{F}}$ is finite. For $0 < r \leq 1$ let

$$K_r = \{q \in D : z(q) \leq r\},$$

define $\omega_r$ as in (iv) reading $K_r$ for $K$ and $\omega_r$ for $\omega$, and let

$$\lambda_r = \max_{|z|=1} \omega_r.$$

We will show that for $v \in \mathcal{F}$ we have

$$v \leq \frac{\log r}{\lambda_r - 1} \tag{$*$}$$

$X \setminus K_r$ and this shows that $u_{\mathcal{F}} < \infty$ on $X \setminus \{p\} = \bigcup_{r>0} X \setminus K_r$. Choose $v \in \mathcal{F}$ and let $c_r = \max_{|z|=r} v$. The function $v + \log|z|$ is subharmonic so its maximum on $K_1$ must occur on $\partial K_1$, i.e.

$$c_r + \log r \leq c_1.$$

But $c_r\omega - v \geq 0$ on $\partial K_r$ and off the support of $v$ so $v \leq c_r\omega$ on $X \setminus K_r$ and hence

$$c_1 \leq c_r\lambda_r.$$

It follows that

$$c_r \leq \frac{\log r}{\lambda_r - 1}$$

i.e. that $(*)$ holds on $\partial K_r$. But $v$ has compact support so $(*)$ holds on $X \setminus K_r$.

The desired Green function is

$$g_p = u_{\mathcal{F}}.$$

From 2.6 we conclude that $g_p$ is harmonic on $X \setminus \{p\}$ and hence that $g_p + \ln|z|$ is harmonic on $D \setminus \{p\}$. From $(*)$ we conclude that the inequality

$$v + \log|z| \leq \frac{\log r}{\lambda_r - 1} + \log r$$

holds on $\partial K_r$ and hence (as the left hand side is subharmonic) on $K_r$. Thus the function $g_p + \ln|z|$ is bounded on $D$ and therefore harmonic on $D$ by the Removable Singularity Theorem 1.6. Moreover $g_p > 0$ because $g_p \geq 0$ and $g_p$ is nonconstant. Suppose $g'$ also satisfies these properties; we must show $g_p \leq g'$. If $v \in \mathcal{F}$ then $v - g'$ is subharmonic on $X \setminus p$ (because $v$ is) and on $D$ (because it equals $(v + \ln|z|) - (g' + \ln|z|)$) and hence on all of $X$. But $v - g' < 0$ off the support of $v$ and hence $v < g'$ everywhere. Thus $g = u_{\mathcal{F}} \leq g'$. □

# 4 Nonhyperbolic surfaces

**Theorem 4.1 (Extension Theorem).** *Assume $X$ is a nonhyperbolic connected Riemann surface. Suppose $p \in X$ and that $f$ is a holomorphic function defined on $D \setminus \{p\}$ where $(z, D)$ is a conformal disk centered at $p \in X$. Then there is a unique harmonic function $u : X \setminus \{p\} \to \mathbb{R}$ bounded in the complement of any neighborhood of $p$ such that $u - \Re(f)$ is harmonic in $D$ and vanishes at $p$.*

**Remark 4.2.** Suppose that $X = \mathbb{C}$, that $p = 0$, and that the function $f$ has a Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

valid in $0 < |z| < 1$. The function $u$ is given by $u = \Re(w)$ where

$$w(z) = \sum_{n=-\infty}^{-1} c_n z^n.$$

The latter series converges for all $z \neq 0$.

*Proof of 4.1.* The proof of uniqueness is easy. If $u_1$ and $u_2$ are two functions as in the theorem, then $u_1 - u_2$ is bounded in the complement of every neighborhood of $p$ (as $u_1$ and $u_2$ are) and near $p$ (as $u_1 - u_2 = (u_1 - \Re(f)) - (u_2 - \Re(f))$ and is thus bounded, hence constant (as $X$ is nonhyperbolic) hence zero (as it vanishes at $p$). For existence we need two preliminary lemmas. By the definition of conformal disk the open set $z(D)$ contains the closed unit disk in $\mathbb{C}$; for $r \leq 1$ let

$$D_r = \{q \in D : |z(q)| < r\}.$$

The following lemma is an immediate consequence of Stoke's Theorem if $X$ is compact.

**Lemma 4.3.** *If $r < 1$ and $u$ is harmonic and bounded on $X \setminus D_r$, then*

$$\int_{\partial D} *du = 0.$$

*Proof.* By adding a large positive constant we may assume w.l.o.g. that $u$ is nonnegative on $\partial D_r$. Choose an increasing sequence of open subsets $X_n \subset X$ such that $X_n \cup \partial X_n$ is compact, $\partial X_n$ is smooth, and the closure of $D$ is a subset of $X_n$. Let $u_n$ and $v_n$ be the solutions of the Dirichlet problem on $X_n \setminus D_r$ with boundary conditions $u_n = v_n = 0$ on $\partial X_n$, $u_n = u$ on $\partial D_r$ and $v_n = 1$ on $\partial D_r$. By the maximum principle we have that $0 \leq u_n \leq u_m \leq \max_{\partial D} u$ and $0 \leq v_n \leq v_m \leq 1$ on $X_n$ for $m \geq n$. Hence by Harnack and 1.9 $u_n$ and $v_n$ converge in $C^k$ uniformly on compact subsets of $X \setminus D$ (in fact on compact subsets of the complement of the closure of $D_r$). Moreover $\lim_n v_n = 1$ on $\partial D$ and $\lim_n v_n \leq 1$ on $X \setminus D$ so we must have $\lim_n v_n = 1$ on $X \setminus D$ by (iv) of Corollary 2.2 and the fact that $X$ is nonhyperbolic. Hence

$$\int_{\partial D} *du = \lim_{n\to\infty} \int_{\partial D} v_n * du_n - u_n * dv_n$$

But $u_n = v_n = 0$ on $\partial X_n$ on $\partial D$ so this may be written

$$\int_{\partial D} *du = -\lim_{n\to\infty} \int_{\partial(X_n \setminus D)} v_n{*}du_n - u_n{*}dv_n.$$

Now by Stokes

$$\int_{\partial(X_n\setminus D)} v_n{*}du_n - u_n{*}dv_n = \int_{X_n \setminus D} v_n\,\Delta u_n - u_n\,\Delta v_n = 0.$$

□

**Lemma 4.4.** *For $0 < \rho < 1$ and let $u_\rho$ be the solution of the Dirichlet problem on $X \setminus D_\rho$ with $u = \Re(f)$ on $\partial D_\rho$. Then for $0 < r < 1/20$ there is a constant $c(r)$ such that for $0 < \rho < r$ we have*

$$\max_{\partial D_r} |u_\rho| \leq c(r).$$

*Proof.* By Lemma 4.3 the 1-form $*\,du_\rho$ is exact on on the interior of $D \setminus D_\rho$ so there is a holomorphic function $F_\rho$ with $u_\rho = \Re(F_\rho)$. The function $F_\rho - f$ has a Laurent expansion about 0; its real part is

$$(u_\rho - \Re f)(te^{i\theta} = \sum_{n=-\infty}^{\infty} (\alpha_n \cos(n\theta) + \beta_n \sin(n\theta) t^n$$

valid for $\rho \leq t \leq 1$. (The coefficients $\alpha_n$ and $\beta_n$ depend on $\rho$.) Then

$$\frac{1}{\pi}(u_\rho - \Re f)(te^{i\theta}\cos(k\theta)\,d\theta = \alpha_k t^k + \alpha_{-k}t^{-k}$$

and

$$\frac{1}{\pi}(u_\rho - \Re f)(te^{i\theta}\sin(k\theta)\,d\theta = \beta_k t^k + \beta_{-k}t^{-k}.$$

For $t = \rho$ the integrand vanishes so

$$\alpha_{-k}(\rho) = -\alpha_k(\rho)\rho^{2k}, \qquad \beta_{-k}(\rho) = -\beta_k(\rho)\rho^{2k}. \tag{1}$$

For $t = 1$ we have

$$|\alpha_k|(1-\rho^{2k}) = |\alpha_k + \alpha_{-k}| \leq 2M_\rho, \qquad |\beta_k|(1-\rho^{2k}) = |\beta_k + \beta_{-k}| \leq 2M_\rho$$

where

$$M_\rho = \max_{|z|=1}|u_\rho| + \max_{|z|=1}|\Re(f)|;$$

Hence for $\rho < 1/2$ we have $|\alpha_k|, |\beta_k| \leq 4M_\rho$ so

$$\max_{|z|=r}|u_\rho| \leq \max_{|z|=r}|\Re(f)| + 4M_\rho \sum_{n=0}^{\infty} r^n + \rho^{2n}r^{-n}.$$

Since $\rho < r$ sum on the right is less than $2\sum r^n = 1/(1-r)$ so we get

$$\max_{|z|=r}|u_\rho| \leq \max_{|z|=r}|\Re(f)| + \frac{8M_\rho}{1-r}. \tag{2}$$

The function $u_\rho$ is harmonic and bounded on $X \setminus D_\rho$ so we have

$$\max_{|z|=1}|u_\rho| \leq \max_{|z|=r}|\Re(f)| + \frac{8M_\rho}{1-r}.$$

and hence

$$M_\rho \leq \max_{|z|=1}|\Re(f)| + \max_{|z|=r}|\Re(f)| + \frac{8M_\rho}{1-r}.$$

Since $8/(1-r) < 1/2$ this gives the bound

$$M_\rho \leq 2\left(\max_{|z|=1}|\Re(f)| + \max_{|z|=r}|\Re(f)\right)$$

on $M_\rho$ independent of $\rho < r$. hence a bound of $|u_\rho|$ on $|z| = 1$ (i.e. $\partial D$), and hence a bound on $|u_\rho|$ on $X \setminus D$. □

*We complete the proof of Theorem 4.1.* Let $r_n = 1/(21n)$ so that $1/20 > r_1 > r_2 > \cdots$ and $\lim_{n\to\infty} r_n = 0$. Let $u_k = u_{\rho_k}$ where $\rho_k = r_k/2$. By the Compactness Theorem 1.10 and the fact that $|u_k| < c(r_1)$ for $\rho_k < r_1$ there is a subsequence of the $u_k$ (still denoted by $u_k$) converging uniformly to a harmonic function $u$ on $X \setminus D_{r_1}$. For the same reason there is a subsequence converging uniformly on $X \setminus D_{r_2}$, and a subsequence converging uniformly on $X \setminus D_{r_3}$, etc. Diagonalize and we get a sequence converging uniformly on compact subsets to a harmonic function $u$ on $X \setminus \{p\}$. □

# 5 Maps to $\mathbb{P}$

The material in this section is not required for the proof of the Uniformization Theorem.

**Theorem 5.1.** *Let $X$ be a Riemann surface, $p_1, p_2, \ldots, p_n \in X$ be distinct, and $a_1, a_2, \ldots, a_n \in \mathbb{P} := \mathbb{C} \cup \{\infty\}$. Then there is a meromorphic function $f$ (i.e. a holomorphic map $f : X \to \mathbb{P}$) with $f(p_j) = a_j$ for $j = 1, 2, \ldots, n$.*

*Proof.* First suppose $n = 2$, $a_1 = \infty$, and $a_2 = 0$ amd Then choose holomorphic coordinates $z_j$ centered at $p_j$. In case that $X$ is nonhyperbolic there are functions $u_j : X \setminus \{p_j\} \to \mathbb{R}$ with $u_j - \Re(1/z_j)$ bounded and harmonic. In case that $X$ is hyperbolic there are functions $u_j : X \setminus \{p_j\} \to \mathbb{R}$ with $u_j - \log |z_j|$ bounded and harmonic. In either case by the Cauchy Riemann equations the function

$$f(z) = \frac{u_{1x} - iu_{1y}}{u_{2x} - iu_{2y}}$$

locally a ratio of two holomorphic differentials and is independent of the choice of local coordinates $z = x + iy$ used to defined it. (Note: No need to assume $X$ is simply connected.) Now $g_{12} = f/(f + 1)$ takes the value 1 at $p_1$ and 0 at $p_2$. For general $n$ the function $h_j = \prod_{k\neq j} g_{kj}$ satisfies $h_j(p_k) = \delta_{jk}$. Take $f = \sum_j a_j h_j$. □

**Theorem 5.2.** *Let $X$ be a compact Riemann surface and $p \in X$. Then there is a meromorphic function $F : X \to \mathbb{P}$ having $p$ as its only pole.*

*Proof.* Let $g$ be the genus of $X$ so that $\dim_{\mathbb{R}} H^1(X, \mathbb{R}) = 2g$. For $k = 1, 2, \ldots, 2g + 1$ let $u_k$ be the harmonic function on $X \setminus \{p\}$ given by Theorem 4.1 with $f = 1/z^k$, i.e. $u_k$ is harmonic on $X \setminus \{p\}$ and $u_k - \Re(1/z_k)$ is harmonic near $p$. Then $*du_k$ is a closed 1-form on $X$ so some nontrivial linear combination is exact; i.e. $dv = \sum_k a_k *du_k$. The function $v$ is the imaginary part of a holomophic function $F = u + iv$ whose real part $\Re(F) = \sum_k a_k u_k$ is bounded in the complement of every neighborhood of $p$. Thus $p$ is the only pole of $F$. □

# 6 The Uniformization Theorem

**Theorem 6.1 (Uniformization Theorem).** *Suppose that $X$ is connected and simply connected. Then*

1. *if $X$ is elliptic, it is isomorphic to $\mathbb{P}^1$;*
2. *if $X$ hyperbolic, it is isomorphic to the unit disk $\mathbb{D}$;*
3. *if $X$ is parabolic, it is isomorphic to $\mathbb{C}$.*

**Definition 6.2.** A holomorphic function $F$ on $X$ is called a **holomorphic Green function** at the point $p \in X$ iff

$$|F| = e^{-g_p}$$

where $g_p$ is the Green function for $X$ at $p$.

**Lemma 6.3.** *Assume $X$ is simply connected and hyperbolic and $p \in X$. Then there is a holomorphic Green function $F$ at $p$.*

*Proof.* Choose a holomorphic coordinate $z = x + iy$ centered at $p$ and let $h$ be a holomorphic function defined near $p$ with $\Re(h) = g_p + \log|z|$. Let $F_p = e^{-h}z$. Then $F$ is holomorphic and $\log|F| = -\Re(h) + \log|z| = -g_p$. Now the condition $g_p = -\log|F|$ defines a holomorphic function $F$ (unique up to a multiplicative constant) in a neighborhood of any point other than $p$ so $F$ extends to $X$ by analytic continuation. □

**Lemma 6.4.** *Let $F$ be holomorphic Green function $p$. Then*

**(i)** *$F$ is holomorphic;*

**(ii)** *$F$ has a simple zero at $p$;*

**(iii)** *$F$ has no other zero;*

**(iv)** *$F : X \to \mathbb{D}$;*

**(v)** *If $F'$ satisfies (i-iv) then $|F'| \le |F|$.*

*By (v) the holomorphic Green function at $p$ is unique up to a multiplicative constant of absolute value one.*

*Proof.* Since $F_p = e^{-h}z$ the function $F$ has a simple zero at $p$. Since $g_p > 0$ we have that $F : X \to \mathbb{D}$ and $F$ has no other zero. □

**Lemma 6.5.** *A holomorphic Green function is injective.*

*Proof.* Choose $q \in X$ and let

$$\phi(r) = \frac{F_p(q) - F_p(r)}{1 - \bar{F}_p(q)F_p(r)}.$$

Then $\phi$ is the composition of $F_p$ with an automorphism of $\mathbb{D}$ which maps $F_p(q)$ to 0. Suppose that $\phi$ has a zero of order $n$ at a point $q$. Let $u = -\log|\phi|/n$. Let $z$ be a holomorphic coordinate at centered at $q$. Then $u + \log|z|$ is bounded near $p$ and hence (by Bôcher) harmonic near $p$. The Green function $g_q$ at $q$ is defined by $g_q = u_{\mathcal{F}}$ where $\mathcal{F}$ is the set of all $v$ of compact support, with $v$ subharmonic on $X \setminus \{q\}$ and $v + \log|z|$ subharmonic near $q$. By the maximum principle, and because $v$ has compact support we have $v \le u$ for $v \in \mathcal{F}$. Hence $g_q \le u$ so

$$|F_q(r)| \ge |\phi(r)|^{1/n} \ge |\phi(r)|. \qquad (\#)$$

Since $F_p(p) = 0$ we have $\phi(p) = F_p(q)$ so $F_q(p)| \ge |F_p(q)|$. Reversing $p$ and $q$ gives $|F_q(p)| = |F_p(q)|$. By (#) $|F_q(r)/\phi(r)| \le 1$ with equality at $r = p$. Hence $F_q = c\phi$ where $c$ is a constant with $|c| = 1$. But $f_q(r) \ne 0$ for $r \ne q$ so $\phi(r) \ne 0$ for $r \ne q$ so $F_p(r) \ne F_p(q)$ for $r \ne q$, i.e. $F_p$ is injective. □

The proof that $X$ is isomorphic to $\mathbb{D}$ now follows from the Riemann Mapping Theorem. However we can also prove that $F_p$ is surjective as follows.

**Lemma 6.6.** *Suppose $W$ is a simply connected open subset of the unit disk $\mathbb{D}$ such that $0 \in W$ but $W \ne \mathbb{D}$. Then there is a injective holomorphic map $H : W \to \mathbb{D}$ with $H(0) = 0$ and $|H'(0)| > 1$.*

*Proof.* Suppose $a^2 \in \mathbb{D} \setminus W$. Then the unction $(z - a^2)/(1 - \bar{a}^2 z)$ is holomorphic and nonzero on $W$. Since $W$ is simply connected this function has a square root, i.e. there is a holomorphic function $h$ on $W$ such that

$$h(z)^2 = \frac{z - a^2}{1 - \bar{a}^2 z}$$

and $h(0) = ia$. Consider the function $H : W \to D$ defined by

$$H(z) = \frac{h - ia}{1 + iah}$$

Then $H'(0) = (1 + |a|^2)/(2ia)$ so $|H'(0)| > 1$. This map is injective as $H(z) = H(w) \implies h(z) = h(w) \implies h(z)^2 = h(w)^2 \implies \dfrac{z - a^2}{1 - \bar{a}^2 z} = \dfrac{w - a^2}{1 - \bar{a}^2 w} \implies z = w$. □

**Lemma 6.7.** *A holomorphic Green function is surjective.*

*Proof.* Assume not. Read $F_p(X)$ for $W$ in Lemma 6.6. Note that both $F_p$ and $H \circ F_p$ has a simple zero at $p$. The function $-\log|H \circ F_p|$ has all the properties of the Green function so

$$-\log|F_p| = g_p \leq -\log|H \circ F_p|$$

by the minimality of the Green function. Hence $|H \circ F_p| \leq |F_p|$ so $|H| \leq |z|$ near zero contradicting $|H'(0)| > 1$. □

This proves the Uniformization Theorem in the hyperbolic case. To prove the Uniformization Theorem in the nonhyperbolic case we introduce a class of functions to play the role of the holomorphic Green function of 6.3.

**Definition 6.8.** Let $X$ be a Riemann surface and $p \in X$. A function $F : X \to \mathbb{P} := \mathbb{C} \cup \{\infty\}$ is called **unipolar** at $p$ iff it it is meromorphic, has a simple pole at $p$, and is bounded (hence holomorphic) in the complement of every neighborhood of $p$. In other words, a unipolar function is a holomorphic map $F : X \to \mathbb{P}$ such that $\infty$ is a regular value, $F^{-1}(\infty)$ consists of a single point, and $F$ is **proper at infinity** in the sense that for any sequence $q_n \in X$ we have $\lim_{n\to\infty} F(q_n) = \infty \implies \lim_{n\to\infty} q_n = p$. By the Extension Theorem 4.1 for any point $p$ in a nonhyperbolic Riemann surface $X$ there is a unique function $u$ which is unipolar at $p$ (take $f = 1/z$).

**Lemma 6.9.** *Assume that $X$ is nonhyperbolic and that $F'$ and $F$ are both unipolar at $p$. Then $F' = aF + b$ for some $a, b \in \mathbb{C}$.*

*Proof.* For some constant $a$, $F' - aF$ has no pole at $p$ and is hence bounded and holomorphic on $X$. On a nonhyperbolic surface the only bounded holomorphic functions are the constant functions. □

**Lemma 6.10.** *Assume that $X$ is nonhyperbolic, that $p \in X$, and that $F : X \to \mathbb{P} := \mathbb{C} \cup \{\infty\}$ is meromorphic, has a simple pole at $p$, and that $\Re(F)$ bounded in the complement of every neighborhood of $p$. Then for $q$ sufficiently near (but distinct from) $p$ the function $G(r) = 1/(F(r) - F(q))$ is unipolar at $q$. In particular, $G$ is unipolar at $q$ if $F$ is unipolar at $p$.*

*Proof.* Since $F$ has a simple pole at $p$ it maps a neighborhood $U$ of $p$ diffeomorphically to a neighborhood of infinity by the Inverse Function Theorem. Let $M = \sup_{r \notin U} u(r)$. For $q$ sufficiently near $p$ we have $|F(q)| > 2M$. For such $q$ we have that $q$ is the only pole of $G$ in $U$ (as $F$ is injective on $U$) and that $|G(r)| < M$ for $r \notin U$ (since $|G(r)| = 1/|F(r) - F(q)| \leq 1/|u(r) - F(q)| < 1/M$). Thus $q$ is the only pole of $G$. Since $G = L \circ F$ where $L(w) = 1/(w - F(q))$ we have that $G$ maps $U$ diffeomorphically to a neighborhood of infinity so $G$ is proper at infinity so $G$ is unipolar as required. □

**Lemma 6.11.** *Assume $X$ is simply connected and nonhyperbolic and that $p \in X$. Then there is a function $F$ unipolar at $p$.*

*Proof.* Use Theorem 4.1 with $f(z) = 1/z$. As $X$ is simply connected the resulting function $u$ is the real part of a meromorphic function $F = u + iv$ with $u$ bounded in the complement of every neighborhood of $p$, and $F - 1/z$ holomorphic in a neighborhood of $p$ and vanishing at $p$. We must show that $v$ is also bounded in the complement of every neighborhood of $p$. Apply Theorem 4.1 with $f(z) = i/z$. We get a meromorphic function $\tilde{F} = \tilde{u} + i\tilde{v}$ with $\tilde{u}$ bounded in the complement of every neighborhood of $p$ and $\tilde{F} - i/z$ holomorphic in a neighborhood of $p$. Thus to prove that $v$ is bounded in the complement of every neighborhood of $p$ it is enough to show that $\tilde{F} = iF$ for then $v = -\tilde{u}$.

By Lemma 6.10 the functions $G(r) = 1/(F(r) - F(q))$ $\tilde{G}(r) = 1/(\tilde{F}(r) - \tilde{F}(q))$ are unipolar for $q$ sufficiently near $q$. Then for suitable constants $a$ and $\tilde{a}$ the function $aG(r) + \tilde{a}\tilde{G}(r)$ has no pole at $q$ (and hence no pole at all) and hence, as $X$ is nonhyperbolic, must be constant. solve the equation

$aG(r) + \tilde{a}\tilde{G}(r) = c$ for $\tilde{F}$ in terms of $F$. Then $\tilde{F} = (\alpha F + \beta)/(\gamma F + \delta)$. But $F(z) = 1/z + R(z)$ and $\tilde{F}(z) = i/z + \tilde{R}(z)$ where $R$ and $\tilde{R}$ vanish at $p$. Hence $\tilde{F} = iF$ as claimed. $\square$

**Lemma 6.12.** *If $F$ is unipolar at $p$ and $F'$ is unipolar at $q$, then $F' = L \circ F$ for some automorphism $L$ of $\mathbb{P}$.*

*Proof.* Fix $p$ and let $S$ be the set of points $q$ where the lemma is true. By Lemma 6.9 $p \in S$ so it suffices to show that $S$ is open and closed. Choose $q_0 \in S$ and let $F_0$ be unipolar at $q_0$. By Lemma 6.10 the function $F(r) = 1/(F_0(r) - F_0(q))$ is unipolar at $q$ for $q$ sufficiently near $q_0$. Now $F = L \circ F_0$ where $L(w) = 1/(w - F_0(q))$ so by Lemma 6.9 (and the fact that the automorphisms form a group) the lemma holds for $q$ sufficiently near $q_0$, i.e. $S$ is open. Now choose $q \in X$ and assume that $q = \lim_{n\to\infty} q_n$ where $q_n \in S$. By Lemma 6.11 let $F'$ be unipolar at $p$. For $n$ sufficiently large $G'(r) = 1/(F'(r) - F'(q_n))$ is unipolar at $q_n$ by Lemma 6.10. Hence $G' = L \circ F$ for some $L$ so $F' = F'(q_n) + 1/(L \circ F) = L' \circ F$ so $q \in S$. Thus $S$ is closed. $\square$

**Lemma 6.13.** *Assume $X$ is simply connected and nonhyperbolic. Then a unipolar function is injective.*

*Proof.* Suppose that $F$ is unipolar at some point $o \in X$ and assume that $F(p) = F(q)$. Let $F_p$ be unipolar at $p$. Then there is an automorphism $L$ with $F_p = L \circ F$. Thus $F_p$ has a pole at $q$ so $q = p$. $\square$

*Proof of the Uniformization Theorem continued.* By Lemma 6.13 we may assume that $X$ is an open subset of $\mathbb{P} = \mathbb{C} \cup \{\infty\}$. If $X$ is elliptic we must have $X = \mathbb{P}$. If $X = \mathbb{P} \setminus \{a\}$ then a suitable automorphism of $\mathbb{P}$ maps $X$ to $\mathbb{P} \setminus \{\infty\} = \mathbb{C}$. Hence it suffices to show that a simply connected open subset of $\mathbb{P}$ which omits two points admits a bounded nonconstant holomorphic function and is hence hyperbolic. By composing with an automorphism of $\mathbb{P}$ we may assume that $X \subset \mathbb{C} \setminus \{0, \infty\}$. As $X$ is simply connected there is a square root function $f$ defined on $X$,i.e. $f(z)^2 = z$ for $z \in X$. Hence $X \cap (-X) = \emptyset$ else $z = f(z)^2 = f(-z)^2 = -z$ for some $z \in X$ so either 0 or $\infty$ is in $X$, a contradiction. As $-X$ is open the function $g(z) = 1/(z - a)$ is bounded on $X$ for $a \in -X$. $\square$

# 7 Surfaces with abelian fundamental group

**7.1.** The Uniformization Theorem classifies all connected Riemann surfaces $X$ whose fundamental group $\pi_1(X)$ is trivial. In this section we extend this classification to surfaces $X$ whose fundamental group us abelian. We also determine the automorphism group of each such $X$. Note that the upper half plane $\mathbb{H}$ and the unit disk $\mathbb{D}$ are isomorphic via the diffeomorphism $f : \mathbb{H} \to \mathbb{D}$ defined by $f(z) = (1 + zi)/(1 - zi)$.

**Theorem 7.2.** *The connected Riemann surfaces with abelian fundamental group are*

**(i)** *the plane* $\mathbb{C}$*,*

**(ii)** *the upper half plane* $\mathbb{H}$*,*

**(iii)** *the Riemann sphere* $\mathbb{P} = \mathbb{C} \cup \{\infty\}$*,*

**(iv)** *the punctured plane* $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$*,*

**(v)** *the punctured disk* $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$*,*

**(vi)** *the annulus* $\mathbb{D}_r = \{z \in \mathbb{D} : r < |z|\}$ *where* $0 < r < 1$*,*

**(vii)** *the torus* $\mathbb{C}/\Lambda_\tau$ *where* $\tau \in \mathbb{H}$ *and* $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$*.*

**Theorem 7.3.** *No two of these are isomorphic except that* $\mathbb{C}/\Lambda_\tau$ *and* $\mathbb{C}/\Lambda_{\tau'}$ *are isomorphic if and only if* $\mathbb{Q}(\tau) = \mathbb{Q}(\tau')$*, i.e. if and only if* $\tau' = g(\tau)$ *for some* $g \in \mathrm{SL}_2(\mathbb{Z})$*.*

**Theorem 7.4.** *The automorphism groups of these surfaces* $X$ *are as follows.*

**(i)** *The group* $\mathrm{Aut}(\mathbb{C})$ *of automorphisms of* $\mathbb{C}$ *is the group consisting of transformations* $\phi$ *of form*

$$\phi(z) = az + b$$

*where* $a, b \in \mathbb{C}$ *and* $a \neq 0$*.*

**(ii)** *The group* $\mathrm{Aut}(\mathbb{P})$ *of automorphisms of the Riemann sphere* $\mathbb{P}$ *is the group* $\mathrm{PGL}(2, \mathbb{C})$ *of all transformations* $\phi$ *of form*

$$\phi(z) = \frac{az + b}{cz + d}$$

*where* $a, b, c, d \in \mathbb{C}$ *and* $ad - bc \neq 0$*.*

**(iii)** *The group* $\mathrm{Aut}(\mathbb{H})$ *of automorphisms of the upper half plane* $\mathbb{H}$ *is the group* $\mathrm{PGL}(2, \mathbb{R})$ *of all transformations* $\phi$ *of form*

$$\phi(z) = \frac{az + b}{cz + d}$$

*where* $a, b, c, d \in \mathbb{R}$ *and* $ad - bc \neq 0$.

**(iv)** *The group* $\mathrm{Aut}(\mathbb{C}^*)$ *of automorphisms of the punctured plane* $\mathbb{C}^*$ *is the group of all transformations* $\phi$ *of one of the forms*

$$\phi(z) = az \qquad \text{or} \qquad \phi(z) = \frac{a}{z}$$

*where* $a \in \mathbb{C}$ *and* $a \neq 0$.

**(v)** *The group* $\mathrm{Aut}(\mathbb{C}/\Lambda_\tau)$ *of automorphisms of the torus* $\mathbb{C}/\Lambda_\tau$ *is the group of all transformations* $\phi$ *of form*

$$\phi(z + \Lambda_\tau) = az + b + \Lambda_\tau$$

*where* $b \in \mathbb{C}$ *and* $a = 1$ *if* $\tau \notin \mathbb{Q}(i) \cup \mathbb{Q}(j)$, $a^4 = 1$ *if* $\tau \in \mathbb{Q}(i)$, *and* $a^6 = 1$ *if* $a \in \mathbb{Q}(j)$. *(Here* $j$ *is the intersection point in* $\mathbb{H}$ *of the two circles* $|z| = 1$ *and* $|z - 1| = 1$.*)*

**(vi)** *The group* $\mathrm{Aut}(\mathbb{D}^*)$ *of automorphisms of the punctured disk* $\mathbb{D}^*$ *is the group of all transformations* $\phi$ *of form*

$$\phi(z) = az$$

*where* $a \in \mathbb{C}$ *and* $|a| = 1$.

**(vii)** *The group* $\mathrm{Aut}(\mathbb{D}_r)$ *of automorphisms of the annulus* $\mathbb{D}_r$ *is the group of all transformations* $\phi$ *of one of the forms*

$$\phi(z) = az \qquad \text{or} \qquad \phi(z) = \frac{ar}{z}$$

*where* $a \in \mathbb{C}$ *and* $|a| = 1$.

**Theorem 7.5.** *A Riemann surface has abelian fundamental group if and only if its automorphism group is not discrete.*

**7.6.** Fix a connected Riemann surface $X$. By the Uniformization Theorem the universal cover $\tilde{X}$ of a (connected) Riemann surface $X$ is one of $\mathbb{P}$, $\mathbb{C}$, or $\mathbb{H} \simeq \mathbb{D}$ and hence $X$ is isomorphic to $\tilde{X}/G$ where $G \subset \mathrm{Aut}(\tilde{X})$ is the group of deck transformations of the covering projection $\pi : \tilde{X} \to X$, i.e.

$$G = \{g \in \mathrm{Aut}(\tilde{X}) : \pi \circ g = \pi\}.$$

Note that $G$ is discrete and acts freely.

**Lemma 7.7.** *The automorphism group of $X = \tilde{X}/G$ is isomorphic to the quotient $N(G)/G$ where*

$$N(G) = \{\phi \in \mathrm{Aut}(\tilde{X}) : \phi \circ G \circ \phi^{-1} = G\}$$

*is the* **normalizer** *of $G$ in* $\mathrm{Aut}(\tilde{X})$.

*Proof of Theorem 7.4(i).* Let $\phi \in \mathrm{Aut}(\mathbb{C})$. Then $\phi$ is an entire function. It cannot have an essential singularity at infinity by Casorati-Weierstrass and the pole at infinity must be simple as $\phi$ is injective. Hence $\phi(z) = az + b$. □

*Proof of Theorem 7.4(ii).* Choose $\phi \in \mathrm{Aut}(\mathbb{P})$. After composing with an element of $\mathrm{PGL}(2, \mathbb{C})$ we may assume that infinity is fixed, i.e. that $\phi(z) = az + b$. □

*Proof of Theorem 7.4(iii).* Choose $\phi \in \mathrm{Aut}(\mathbb{H})$. Let $f : \mathbb{H} \to \mathbb{D}$ be the isomorphism given by $f(z) = (1 + zi)/(1 - zi)$. Then $\psi := f^{-1} \circ \phi \circ f$ is an automorphism of the disk $\mathbb{D}$. Composing with $\alpha(z) = (z - a)/(\bar{a}z - 1)$ we may assume that $\psi(0) = 0$. Then $|\psi(z)| \leq |z|$ by the Maximum Principle ($\psi(z)/z$ is holomorphic) and similarly $|\psi^{-1}(z)| \leq |z|$. Hence $|\psi(z)| = |z|$ so $\psi(z) = cz$ where $|c| = 1$ by the Schwartz lemma. Hence $\phi \in \mathrm{PGL}(2, \mathbb{C})$. The coefficients must be real as the real axis is preserved so $\phi \in \mathrm{PGL}(2, \mathbb{R})$. □

*Proof of Theorem 7.4(iv).* The universal cover of the punctured plane $\mathbb{C}^*$ is the map

$$\mathbb{C} \to \mathbb{C}^* : z \mapsto \exp(2\pi i z).$$

The group $G$ of deck transformations is the cyclic group generated by the translation $z \mapsto z + 1$. The normalizer $N(G)$ of $G$ in $\mathrm{Aut}(\mathbb{C})$ is . . . □

*Proof of Theorem 7.4(v).* The universal cover of the torus $\mathbb{C}/\Lambda_\tau$ is the map

$$\mathbb{C} \to \mathbb{C}/\Lambda_\tau : z \mapsto z + \Lambda_\tau.$$

The group $G$ of deck transformations is the abelian group generated by the translations $z \mapsto z+1$ and $z \mapsto z+\tau$. The normalizer $N(G)$ of $G$ in $\mathrm{Aut}(\mathbb{C})$ is ... □

*Proof of Theorem 7.4(vi).* The universal cover of the punctured disk $\mathbb{D}^*$ is the map

$$\mathbb{H} \to \mathbb{D}^* : z \mapsto \exp(2\pi i z).$$

The group $G$ of deck transformations is the cyclic group generated by the translation $z \mapsto z+1$ . The normalizer $N(G)$ of $G$ in $\mathrm{Aut}(\mathbb{D})$ is ... □

*Proof of Theorem 7.4(vii).* The universal cover of the annulus $\mathbb{D}_r$ is the map

$$\mathbb{H} \to D_r : z \mapsto \exp\left(\frac{\log r \log z}{\pi i}\right).$$

Here $\log z$ denotes the branch of the logarithm satisfying $0 < \Im(\log z) < \pi$ so writing $z = \rho e^{i\theta}$ the cover takes the form

$$\mathbb{H} \to D_r : \rho e^{i\theta} \mapsto r^{\theta/\pi} \exp\left(\frac{\log r \log \rho}{\pi i}\right).$$

The group $G$ of deck transformations is the cyclic group generated by $z \mapsto az$ where $a = \exp(-2\pi^2/\log r)$. The normalizer $N(G)$ of $G$ in $\mathrm{Aut}(\mathbb{D})$ is ... □

**Lemma 7.8.** *If $\tilde{X} = \mathbb{P}$ then $G = \{1\}$ so $X = \mathbb{P}$.*

*Proof.* Any nontrivial element of $\mathrm{PSL}(2,\mathbb{C})$ has a fixed point. □

**Lemma 7.9.** *If $\tilde{X} = \mathbb{C}$ then the group $G$ consists of a discrete abelian group of translations. More precisely $G$ is the set of all transformations $f(z) = z+b$ where $b \in \Gamma$ and where the subgroup $\Gamma \subset \mathbb{C}$ is one of the following:*

**(i)** $\Gamma = \{0\}$ *in which case* $X = \tilde{X} = \mathbb{C}$*;*

**(ii)** $\Gamma = \omega\mathbb{Z}$ *in which case* $X \simeq \mathbb{C}^*$*;*

**(iii)** $\Gamma = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ *in which case* $X \simeq \mathbb{C}/\Lambda_\tau$*,* $\tau = \omega_2/\omega_1$*.*

*Proof.* Any automorphism of form $z \mapsto az + b$ where $a \neq 1$ has a fixed point so $G$ is a discrete group of translations. Kronecker's theorem says that for $\omega \in \mathbb{R}$ the group $\mathbb{Z}\omega + \mathbb{Z}$ is dense in $\mathbb{R}$ if and only if $\omega \notin \mathbb{Q}$. (Proof: Consider a minimal positive element of $\mathbb{Z}\omega + \mathbb{Z}$.) It follows easily that a discrete subgroup of the additive group $\mathbb{R}^n$ has at most $n$ generators. Hence the three possibilities. In case (ii) the group $G$ is conjugate in $\mathrm{Aut}(\mathbb{C})$ to the cyclic group generated by the translations $z \mapsto z + \tau$. In case (iii) the group $G$ is conjugate in $\mathrm{Aut}(\mathbb{C})$ to the free abelian group generated by the translations $z \mapsto z + 1$ and $z \mapsto z + \tau$. □

**Corollary 7.10 (Picard's Theorem).** *An entire function $f : \mathbb{C} \to \mathbb{C}$ which omits two points is constant.*

*Proof.* $\mathbb{C} \setminus \{a, b\}$ has a nonabelian fundamental group so its universal cover must be $\mathbb{D}$. A holomorphic map $f : \mathbb{C} \to \mathbb{C} \setminus \{a, b\}$ lifts to a map $\tilde{f} : \mathbb{C} \to \mathbb{D}$ which must be constant by Liouville. □

**Lemma 7.11.** *A fixed point free automorphism $\phi$ of $\mathbb{H}$ is conjugate in $\mathrm{Aut}(\mathbb{H})$ either to a homothety $z \mapsto az$ where $a > 0$ or to the translation $z \mapsto z + 1$.*

*Proof.* Let $A \in \mathrm{SL}(2, \mathbb{R})$ be a matrix representing the automorphism $\phi$. Since $\phi$ has no fixed points in $\mathbb{H}$ the eigenvalues of $A$ must be real. Since their product is one we may rescale so that they are positive. If there are two eigenvalues $\lambda$ and $\lambda^{-1}$ then $A$ is conjugate in $\mathrm{SL}(2, \mathbb{R})$ to a diagonal matrix and so $\phi$ is conjugate in $\mathrm{Aut}(\mathbb{H})$ to $z \mapsto \lambda^2 z$. Otherwise the only eigenvalue is 1 and $A$ is conjugate in $\mathrm{SL}(2, \mathbb{R})$ to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ so $\phi$ is conjugate in $\mathrm{Aut}(\mathbb{H})$ to $z \mapsto z + 1$. □

**Corollary 7.12.** *If $\tilde{X} = \mathbb{H}$ and $G$ is abelian, then the group $G$ is conjugate in $\mathrm{Aut}(\mathbb{H})$ to either a free abelian group generated by a homothety $z \mapsto az$ where $a > 0$ or the free abelian group generated by the translation $z \mapsto z + 1$. In the former case $X \simeq \mathbb{D}^r$ for some $r$ and in the latter case $X \simeq \mathbb{D}^*$.*

*Proof.* If $G$ contains a homethety it must be a subgroup of the group of homotheties (as it is abelian) and hence cyclic as it is discrete. Similarly, If $G$ contains a translation it must be a subgroup of the group of translations and hence cyclic. □

# References


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