Potential Theory

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Theorem 1 (Green's identity). *Let Ω be a bounded open region in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $u, v : \Omega \cup \partial\Omega \to \mathbb{R}$ be smooth functions. Then*

$$\int_D (u\,\Delta v - v\,\Delta u)\,dV = \int_{\partial\Omega} \left(u\,\frac{\partial v}{\partial \nu} - v\,\frac{\partial u}{\partial \nu} \right) dS$$

where dV is the volume element, dS is the surface area element on $\partial\Omega$, Δ is the Laplacian, and $\partial/\partial\nu$ is the outward normal derivative.

Proof. Use the Divergence Theorem and that the Laplacian is the divergence of the gradient. □

2. The **unit charge potential** is the function $\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}_+$ defined by

$$\Phi(\xi) = \frac{|\xi|}{2} \qquad (n = 1)$$

$$\Phi(\xi) = \frac{\log|\xi|}{2\pi} \qquad (n = 2)$$

$$\Phi(\xi) = k_n |\xi|^{2-n} \qquad (n \geq 3)$$

where

$$k_n = -\frac{1}{(n-2)\sigma_n}$$

and σ_n is the $(n-1)$ dimensional volume of the unit sphere in $\mathbb{R}^n$. Note that for $n > 2$ the function Φ is negative, and for $n = 2$ it is negative near $\xi = 0$. The function Φ satisfies $\Delta\Phi = 0$ and

$$\int_{\partial B_a} \frac{\partial \Phi}{\partial \nu}\,dS = 1, \qquad \int_{\partial B_a} \Phi\,dS = O(a), \qquad \int_{B_a} \Phi\,dV = O(a^2)$$

where B_a is the ball of radius a about the origin. For each $p \in \mathbb{R}^n$ define a function $\Phi_p : \mathbb{R}^n \setminus \{p\} \to \mathbb{R}_+$ by

$$\Phi_p(q) = \Phi(q - p).$$

Theorem 3. *If u is smooth on $\Omega \cup \partial\Omega$*

$$u(p) = \int_\Omega \Phi_p \Delta u \, dV + \int_{\partial\Omega} u \frac{\partial \Phi_p}{\partial \nu} \, dS - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \Phi_p \, dS \qquad (\#)$$

Proof. Let B be a small ball centered at p and read $\Omega \setminus B$ for Ω and Φ_p for v in Green's Identity. We get

$$-\int_\Omega \Phi_p \Delta u \, dV = \int_{\partial\Omega} \left(u \frac{\partial \Phi_p}{\partial \nu} - \Phi_p \frac{\partial u}{\partial \nu} \right) dS - \int_{\partial B} \left(u \frac{\partial \Phi_p}{\partial \nu} - \Phi_p \frac{\partial u}{\partial \nu} \right) dS$$

where the normal derivative on the right is out of B (i.e. into $\Omega \setminus \partial B$). But

$$\int_{\partial B} \left(u \frac{\partial \Phi_p}{\partial \nu} - \Phi_p \frac{\partial u}{\partial \nu} \right) dS \to u(p)$$

as the radius of B tends to 0. □

4. Take $\Omega = \mathbb{R}^n$ and let u have compact support. Then (#) reduces to

$$u(p) = \int \Phi_p \, \Delta u \, dV.$$

In the language of Schwartz distributions this says

$$\delta_p = \Delta \Phi_p$$

where δ_p is the Dirac distribution at p.

5. For $n = 3$ (#) becomes

$$u(p) = -\frac{1}{4\pi} \int_\Omega \frac{\Delta u}{r} \, dV - \frac{1}{4\pi} \int_{\partial\Omega} u \frac{\partial}{\partial \nu} \frac{1}{r} \, dS + \frac{1}{4\pi} \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \frac{1}{r} \, dS$$

for $p \in \Omega$ where $r(q) = |q - p|$ for $q \in \Omega \cup \partial\Omega$. This has the following physical interpretation. The quantity

$$\Phi_p = -\frac{1}{4\pi r}$$

is the potential at the point p due to a unit charge at the point q. The quantity

$$h\frac{\partial \Phi_p}{\partial \nu} = -h\frac{\partial}{\partial \nu}\frac{1}{4\pi r}$$

corresponds (to first order in h) to the potential p produced by a **dipole** at $q \in \partial\Omega$, i.e. the superposition of a unit charge at q with an equal and opposite charge located at a small distance h from q along the unit normal. The potential U produced by a charge distribution ρ satisfies

$$\rho = -\Delta U.$$

Hence (#) represents the smooth function u as the sum of three potentials: the one produced by the charge density Δu on the interior of Ω, the one produced by the charge distribution $\partial u/\partial\nu$ on the boundary of Ω, and the one produced by the dipole distribution u/h on the boundary of Ω. (See [2] page 219.)

6. For a bounded connected region Ω with smooth boundary the problem

$$\Delta H = 0, \qquad H|\partial\Omega = -\Phi_p.$$

is a special case of Dirichlet's problem and has a unique solution (see **??** below); the function

$$G_p = \Phi_p + H$$

is called the **Green's function** at the point p. The function H is smooth on $\Omega \cup \partial\Omega$ if the boundary $\partial\Omega$ is smooth. This follows from more general regularity theorems for the Laplacian. See [1] page 457.

Proposition 7. *Assume $n \geq 2$, $\Omega \subset \mathbb{R}^n$ is a bounded connected region with smooth boundary, and $p \in \Omega$. Then the Green's function at p is uniquely characterized by the following axioms.*

(i) *G_p is harmonic on $\Omega \setminus \{p\}$;*

(ii) *$G_p - \Phi_p$ extends to a continuous function on $\Omega \cup \partial\Omega$;*

(iii) *$G_p < 0$;*

(iv) *If G satisfies (i-iii) then $G \leq G_p$.*

Proof. Maximum principle. □

8. The Green's function G_p has the following physical interpretation. Place a point charge at p and let the boundary $\partial\Omega$ be a metallic conductor. There is no current in the boundary: the charges have arranged themselves so that the net force on each charge is normal to the boundary. But the force is the negative gradient of the potential so the potential is constant on the boundary. Subtract a constant so the potential vanishes on the boundary and the resulting potential is the Green's function.

9. Repeat the proof of (#) with G_p in place of Φ_p. We get

$$u(p) = \int_\Omega G_p \, \Delta u \, dV + \int_{\partial\Omega} u \frac{\partial G_p}{\partial \nu} \, dS$$

It follows that if u is the solution to the **Dirichlet Problem**

$$\Delta u = 0, \qquad u|\partial\Omega = \phi$$

then

$$u(p) = \int_{\partial\Omega} \phi \, \frac{\partial G_p}{\partial \nu} \, dS,$$

and if u is the solution to the **Poisson Problem**

$$\Delta u = f, \qquad u|\partial\Omega = 0$$

then

$$u(p) = \int_\Omega G_p \, f \, dV.$$

10. To justify these formulas rigorously we must first prove that the Dirichlet problem and the Poisson problem have unique solutions and that these solutions are C^2 on $\Omega \cup \partial\Omega$. It is not true that $f \in C^r \implies u \in C^{r+2}$ if r is an integer (see [1] page 290), however this does hold if r is not an integer (see [1] page 291). In particular, $f \in C^r \implies u \in C^{r+1}$ for $r > 0$.

11. The Green's function is symmetric:

$$G_p(q) = G_q(p)$$

see e.g. [1] page 345. This can also be proved as follows. Green's identity implies that $\langle \Delta u, v \rangle_{L^2(\Omega)} = \langle u, \Delta v \rangle_{L^2(\Omega)}$ for $u, v \in H_0^2(\Omega)$. The operator

$$\Delta : H_0^2(\Omega) \to L^2(\Omega)$$

is an isomorphism and the integral operator with kernel $G(p, q) := G_p(q)$ is its inverse.

12. The Green's function for the unit ball B in $\mathbb{R}^n$ $(n \geq 3)$ is

$$G_p(q) = k_n \left(\left|p - q\right|^{2-n} - \left| \frac{q}{|q|} - |q|p \right|^{2-n} \right)$$

(It is easy to check that $G_p(q) = G_q(p)$ so that $G_p(q) = 0$ for $|q| = 1$.) The function $P : B \times \partial B \to \mathbb{R}_+$ defined by

$$P(p, q) = \frac{\partial G_p}{\partial \nu}(q) = (2 - n)k_n \frac{|q|^2 - |p|^2}{|q - p|^n} = \frac{1 - |p|^2}{\sigma_n |q - p|^n}$$

is called the **Poisson kernel**. Thus the following **Poisson integral formula**

$$u(p) = \int_{\partial B} \phi \, P_p \, dS, \qquad P_p(q) = P(p, q)$$

defines the unique harmonic function on B which agrees with ϕ on the unit sphere ∂B. It is an immediate consequence of Poisson integral formula that a uniform limit of harmonic functions is harmonic.

13. The Green's function for the unit disk $\mathbb{D}$ in $\mathbb{C} = \mathbb{R}^2$ is

$$G_z(\zeta) = \frac{1}{2\pi} \log \left| \frac{\zeta - z}{\bar{z}\zeta - 1} \right|.$$

(Note that G_z is the composition of G_0 with an automorphism of $\mathbb{D}$ which sends z to 0.) The Poisson kernel is

$$P(z, \zeta) = \frac{\partial G_z}{\partial \nu}(\zeta) = \frac{1}{2\pi} \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2}$$

as in higher dimensions. The function

$$u(z) = \int_0^{2\pi} P(z, e^{i\theta}) \phi(e^{i\theta}) \, d\theta$$

is the harmonic function which agrees with ϕ on $|z| = 1$. Exercise: Take $\phi = e^{in\theta}$ and $u(z) = z^n$ for $n \geq 0$ or $u(z) = \bar{z}^n$ for $n < 0$ and check the last formula by elementary integration.

References

[1] R. Dautray & J. L. Lions: *Mathematical Analysis and Numerical Methods for Science and Tecchnology, Volume 1: Physical Origins and Classical Methods*, Springer, 1990.

[2] O. D, Kellogg: *Foundations of Potential Theory*, Springer 1929, Dover 1953.