# Riemann Surfaces and Algebraic Curves

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Tuesday December 11, 2001, 9:03 AM

We describe the relation between algebraic curves and Riemann surfaces. An elementary reference for this material is [1].

## 1 Riemann surfaces

**1.1.** A **Riemann surface** is a smooth complex manifold $X$ (without boundary) of complex dimension one. Let $K \to X$ denote the **canonical line bundle** so that the fiber $K_p$ over $p \in X$ is the space of complex linear maps from $T_pX$ to $\mathbb{C}$. A section of $K$ is called a **differential** on $X$. We define

$$\begin{aligned}
\mathcal{M}(X) &= \text{the field of meromorphic functions on } X,\\
\mathcal{O}(X) &= \text{the ring of holomorphic functions on } X,\\
\mathcal{M}(X,K) &= \text{the space of meromorphic differentials on } X,\\
\Omega(X) &= \text{the space of holomorphic differentials on } X.
\end{aligned}$$

If $X$ is compact, $\mathcal{O}(X) = \mathbb{C}$ the constant functions. An element of $\mathcal{M}(X)$ can be viewed as a holomorphic map to the Riemann sphere (projective line)

$$\mathbb{P} := \mathbb{C} \cup \{\infty\}$$

and the only holomorphic map which does not arise this way is the constant map which sends all of $X$ to $\infty$. The **genus** $g$ of a compact Riemann surface $X$ is defined by

$$2g = \dim_{\mathbb{R}} H^1(X, \mathbb{R})$$

so the **Euler characteristic** of $X$ is $\chi(X) = 2 - 2g$. The Riemann Roch Theorem implies that for $X$ compact we have

$$g = \dim_{\mathbb{C}}(\Omega(X))$$

the dimension of the space of holomorphic differentials.

**1.2.** Let $p \in X$ and $z$ be a local holomorphic coordinate on $X$ with $z(p) = 0$. Any $f \in \mathcal{M}(X) \setminus \{0\}$ has form

$$f(z) = z^k h(z)$$

in the coordinate $z$ where $h$ is holomorphic and $h(0) \neq 0$. The integer

$$\mathrm{Ord}_p(f) := k$$

is independent of the choice of the coordinate $z$; it is called the **order** of $f$ at $p$. A point $p$ is called a **zero** of $f$ if $\mathrm{Ord}_p(f) > 0$, a **pole** of $f$ if $\mathrm{Ord}_p(f) < 0$, and a **singularity** of $f$ if it is either a zero or pole, i.e. if $\mathrm{Ord}_p(f) \neq 0$. (One can define analogously the order of a singularity meromorphic section of any holomorphic line bundle but here we only need the notion for differentials.) Thus any $\omega \in \mathcal{M}(X, K)$ has form

$$\omega = f\, dz$$

where $f \in \mathcal{M}(X)$. The integer

$$\mathrm{Ord}_p(\omega) := \mathrm{Ord}_p(f)$$

is independent of the choice of the coordinate $z$; it is called the **order** of $\omega$ at $p$. The complex number

$$\mathrm{res}_p(\omega) = \frac{1}{2\pi i} \oint_{\gamma_p} \omega$$

is independent of the choice of the small circle $\gamma_p$ about $p$ having no pole other than $p$ in its interior; it is called the **residue** of $\omega$ at $p$.

**Theorem 1.3 (Residue Theorem).** *Let $X$ be a compact Riemann surface and $\omega \in \mathcal{M}(X, K) \setminus \{0\}$. Then*

$$\sum_{p \in X} \mathrm{res}_p(\omega) = 0.$$

*Proof.* Away from the singularities we have $\omega = f(z)\,dz$ where $f$ is holomorphic. Hence $\partial\omega = 0$ (as $dz \wedge dz = 0$) and $\bar\partial\omega = 0$ (as $f$ is holomorphic) so $d\omega = 0$. Hence for any open subset $\Omega \subset X$ with smooth boundary and such that $\Omega \cup \partial\Omega$ contains no pole we have

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega = 0.$$

Choose a tiny disk $\Delta_p$ about each pole $p$ so that

$$\int_{\partial\Delta_p} \omega = 2\pi i \mathrm{res}_p(\omega).$$

For $\Omega = X \setminus \bigcup_p \Delta_p$ we have

$$\int_{\partial\Omega} \omega = \sum_p \int_{\partial\Omega} \omega.$$

(See Theorem 4.8 on page 18 of [1].) ∎

**Corollary 1.4.** *Let $X$ be a compact Riemann surface and $f \in \mathcal{M}(X) \setminus \{0\}$. Then*

$$\sum_{p\in X} \mathrm{Ord}_p(f) = 0.$$

*Proof.* Let $\omega = df/f$. Then $\mathrm{Ord}_p(f) = \mathrm{res}_p(\omega)$. ∎

**1.5.** The **degree** of a holomorphic map $f : X \to Y$ between compact Riemann surfaces is the sum of the local degrees over the preimage of a given point $y \in Y$. The local degree at$p \in X$ of a holomorphic map is the same as the order of the zero of of the local representative of the map in any holomorphic coordinates $z$ centered at $p$ and $w$ centered at $f(p)$. Thus when $Y = \mathbb{P}$, this local degree at $p \in X$ is $\mathrm{Ord}_p(f)$ if $f(p) = 0$ and $-\mathrm{Ord}_p(f)$ if $f(p) = \infty$ so the corollary is also a corollary of the theorem that the degree of a holomorphic map $f : X \to Y$ is well defined, i.e. independent of the choice of $y \in Y$ used to defined it.

**Theorem 1.6 (Poincaré-Hopf).** *Let $X$ be a compact Riemann surface and $\omega \in \mathcal{M}(X, K) \setminus \{0\}$. Then*

$$\sum_{p\in X} \mathrm{Ord}_p(\omega) = -\chi(X)$$

*where $\chi(X)$ is the Euler characteristic of $X$.*

*Proof.* In a suitable holomorphic coordinate centered at $p$ we have

$$\omega = z^\nu \, dz$$

where $\nu = \mathrm{Ord}_p(\omega)$ so where $z = x + iy = re^{i\theta}$ we have

$$\Re\omega = r^\nu(\cos(\nu\theta)\, dx - \sin(\nu\theta)\, dy)$$

so the degree of the map

$$\frac{\Re\omega}{|\Re\omega|} : \{|z| = \varepsilon\} \to S^1$$

is $-\mathrm{Ord}_p(\omega)$. The sum of these degrees is the Euler characteristic by the Poincaré Hopf Theorem. (See Theorem 6.5 on page 24 of [1]). □

**Theorem 1.7 (Weil).** *Let $X$ be a compact Riemann surface and $f, g \in \mathcal{M}(X) \setminus \{0\}$. Assume that$(f)$ and $(g)$ are disjoint. Then*

$$\prod_{p \in X} f(p)^{\mathrm{Ord}_p(g)} = \prod_{p \in X} g(p)^{\mathrm{Ord}_p(f)}.$$

*Proof.* See [2] page 242. □

**1.8.** The restriction of nonconstant holomorphic map $f : X \to Y$ to the complement of the preimage of the set of critical values is a $d$-sheeted covering space, i.e. if $V \subset Y$ is a sufficiently small open set containing no critical value of $f$, then $f^{-1}(V)$ is a disjoint union of $d$ open sets each mapped diffeomorphicaly to $V$ by $f$. The number $d$ is the degree of $f$ as defined in paragraph 1.5. Near each a critical point $f$ has the form $z \mapsto z^k$ where $k = \deg_p(f)$ is the local degree of the critical point. For this reason a nonconstant holomorphic map is called a **ramified cover** and the critical points of $f$ are called **ramification points**. The number $e_p(f) = \deg_p(f) - 1$ is called the **ramification index** so that $e_p(f) > 0$ if and only if $p$ is a ramification point of $f$.

**Theorem 1.9 (Riemann Hurwitz).** *If $f : X \to Y$ is a holomorphic map between compact Riemann surfaces of degree $d$, then*

$$\chi(X) = d\chi(Y) - \sum_{p \in X} e_p(f)$$

*where $\chi(X)$ is the Euler characteristic of $X$.*

*Proof.* Triangulate $X$ and $Y$ so that the ramification points are vertices and the map $f$ is simplicial and use the fact that the Euler characteristic $\chi$ is the number of vertices minus the number of edges plus the number of faces in any triangulation. See [1] page 92. $\square$

## 2 Algebraic curves

**2.1.** An **projective algebraic variety** $X$ is a subset of a complex projective space $\mathbb{P}^N$ of form

$$X = \{x \in \mathbb{P}^N : F_1(x) = \cdots = F_k(x) = 0\} \tag{$*$}$$

where $F_1, \ldots, F_n$ are homogeneous polynomials. An **affine algebraic variety** is a subset of a complex affine space $\mathbb{C}^N$ of form

$$Y = \{y \in \mathbb{C}^N : f_1(y) = \cdots = f_k(y) = 0\}.$$

For every polynomial $f(y_1, \ldots, y_N)$ there is a unique homogeneous polynomial $F(x_0, x_1, \ldots, x_N)$ of the same degree such that

$$f(y_1, \ldots, y_N) = F(1, y_1, \ldots, y_N),$$

so every affine variety corresponds to a projective variety. We use the term *algebraic variety* ambiguously to mean either *projective algebraic variety* or *affine algebraic variety*. (There is an abstract notion of *algebraic variety* which embraces both projective and affine algebraic varieties as special cases.)

**2.2.** An algebraic variety is **irreducible** iff it is not the union of two distinct varieties. Every algebraic variety $X$ may be written as

$$X = X_1 \cup X_2 \cup \cdots \cup X_k$$

where the $X_i$ are irreducible and $X_i \neq X_j$ for $i \neq j$; this decomposition is unique up to a reindexing. The varieties $X_i$ are called the **irreducible components** of $X$.

**2.3.** Let $X$ be an algebraic variety. A point $p \in X$ is called a **smooth point** iff it has a neighborhood $U$ such that $U \cap X$ is a holomorphic submanifold. A point which is not smooth point is called a **singular point**. For an irreducible variety the dimension of $U \cap X$ is independent of the choice of the smooth point $p$ and is called the **dimension** of $X$. An **algebraic curve** is an algebraic variety each of whose irreducible components has dimension one; a **plane algebraic curve** is an algebraic curve of codimension one, i.e. an algebraic curve which is a subset of $\mathbb{P}^2$.

**2.4.** Every compact Riemann surface admits a holomorphic embedding into $\mathbb{P}^3$. (See [1] page 213.) A closed holomorphic submanifold of $\mathbb{P}^N$ is a smooth algebraic variety (Chow's Theorem, see [2] page 187); hence every Riemann surface is isomorphic to a smooth algebraic curve.

**2.5.** Let $C \subseteq \mathbb{P}^N$ be an algebraic curve and $S \subseteq C$ be the set of singular points of $C$. A **normalization** of $C$ is a holomorphic map

$$\sigma : X \to \mathbb{P}^N$$

from a compact Riemann surface $X$ such that $\sigma(X) = C$, $\sigma^{-1}(S)$ is finite and the restriction

$$X \setminus \sigma^{-1}(S) \to C \setminus S$$

is bijective. (Since the restriction is a holomorphic map between Riemann surfaces it follows that it is biholomorphic.)

**Theorem 2.6 (Normalization Theorem).** *Every algebraic curve admits a normalization. The normalization is unique up to isomorphism in the following sense: If $\sigma : X \to \mathbb{P}^N$ and $\sigma' : X' \to \mathbb{P}^N$ are normalizations of the same curve $C$, then the unique continuous map $\tau : X \to X'$ satisfying $\sigma' = \tau \circ \sigma$ is (a bijection and) biholomorphic.*

*Proof.* See [1] page 5 and page 68. $\square$

**Remark 2.7.** The number $k$ in equation $(*)$ of paragraph 2.1 is always greater than or equal to the codimension of $X$; a variety which has form $(*)$ with $k$ equal to the codimension is called a **complete intersection**. The **twisted cubic**

$$x_0x_3 = x_1x_2, \qquad x_0x_2 = x_1^2, \qquad x_1x_3 = x_2^2$$

(so called because its affine part may be parameterized by the equations $x_i = t^i$) is a smooth algebraic curve in $\mathbb{P}^3$ which is not a complete intersection.

**2.8.** Every plane algebraic curve $C$ is a complete intersection (see [2] page 13) and thus has form

$$C = \{[x_0, x_1, x_2] \in \mathbb{P}^2 : F(x_0, x_1, x_2) = 0\}$$

where $F$ is a complex homogeneous polynomial; the polynomial $F$ is called a **defining polynomial** for $C$. Every curve has a defining polynomial of minimal degree, i.e. one with no repeated factors; this polynomial is unique up to multiplication by a nonzero constant. It is easy to see that a point of $C$ is a smooth point if and only if it is regular point of the minimal degree defining polynomial, and that an algebraic plane curve is irreducible if and only if it has a defining polynomial which is irreducible.

**2.9.** By **affine coordinates** at a point $p \in \mathbb{P}^2$ we mean coordinates $(x, y)$ of form

$$x = \frac{a_{10}x_0 + a_{11}x_2 + a_{12}x_2}{a_{00}x_0 + a_{01}x_2 + a_{02}x_2}, \qquad y = \frac{a_{20}x_0 + a_{21}x_2 + a_{22}x_2}{a_{00}x_0 + a_{01}x_2 + a_{02}x_2},$$

where the matrix $(a_{ij})$ is invertible, the numerators vanish at $p$, and the denominators do not. (Every choice of affine coordinates establishes a correspondence between projective plane curves and affine plane curves as in paragraph 2.1.

**2.10.** Let $C \subseteq \mathbb{P}^2$ be an algebraic curve, $p \in C$, $(x, y)$ be affine coordinates at $p$, and $f(x, y)$ the defining polynomial of $C$ in these coordinates. Since $p \in C$ we have $f(0, 0) = 0$. We call $p$ a $k$**tuple point** of $C$ iff $d^j f(0, 0) = 0$ for $j = 1, 2, \ldots, k-1$ and $d^k f(0, 0) \neq 0$. A $k$tuple point is also called a **simple point** if $k = 1$, a **double point** if $k = 2$, a **triple point** if $k = 3$, etc. A point is a smooth point if and only if it is a simple point. Let $p$ be a $k$tuple point. The homogeneous polynomial

$$f_k(x, y) := \frac{d^k}{dt^k} f(tx, ty)\bigg|_{t=0}$$

factors into linear factors. The point $p$ is called an **ordinary point** iff these factors are distinct.

**Theorem 2.11.** *Let $X$ be a compact Riemann surface. Then there is an algebraic curve $C \subseteq \mathbb{P}^2$ and a normalization $\sigma : X \to C$ such that (1) the map $\sigma$ is an immersion, and (2) the only singularities of $C$ are ordinary double points.*

*Proof.* See [1] page 213. ☐

**Theorem 2.12 (The Genus Formula).** *Let $C \subset \mathbb{P}^2$ be an irreducible plane curve whose only singularities are double points. Then*

$$g = \frac{(d-1)(d-2)}{2} - \delta$$

*where $g$ is the genus of its normalization, $d$ is the degree of its irreducible defining polynomial, and $\delta$ is the number of double points.*

*Proof.* Project $C$ onto a projective line $\mathbb{P}^1$ from a point not on $C$. Using suitable affine coordinates we see that the number of critical points of this projection is $d(d-1)$. Apply the Riemann Hurwitz formula (Theorem 1.9) to the composition of this projection with the normalization map. For more details see [1] page 213. ☐

# References

[1] P. A. Griffiths: *Introduction to Algebraic Curves*, AMS Translations of Math. Monographs **76** 1989.

[2] P. A. Griffiths & J. Harris: *Principles of Algebraic Geometry*, Wiley Interscience, 1978.