Riemann Surfaces

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1 The Classifying Map

A reference for the material in this section is [3].

1.1. Let V be a vector space over $\mathbb{C}$. We denote by $\mathbb{G}_k(V)$ the **Grassmannian** of k-dimensional subspaces of V and by

$$\mathbb{P}(V) = \mathbb{G}_1(V)$$

the projective space of V. Two vector bundles over the Grassmannian $\mathbb{G}_k(V)$ are the **tautological bundle**

$$T \to \mathbb{G}_k(V), \qquad T_\lambda := \lambda \subset V$$

and the **co-tautological bundle**

$$H \to \mathbb{G}_k(V), \qquad H_\lambda := T_\lambda^* = V^*/\lambda^\perp$$

where V^* is the dual space to V and $\lambda^\perp = \{\alpha \in V^* : \alpha|\lambda = 0\}$. In case $k = 1$ the bundle $H \to \mathbb{P}(V)$ is also called the **hyperplane bundle**. Note the canonical isomorphism

$$\mathbb{G}_{N-k}(V) \to \mathbb{G}_k(V^*) : \lambda \mapsto \lambda^\perp$$

where $N = \dim_{\mathbb{C}}(V)$. For any holomorphic bundle $E \to X$ we denote by $\mathcal{O}(X, E)$ the vector space of its holomorphic sections.

Theorem 1.2. *Each functional $\alpha \in V^*$ determines a section s_α of the cotautological bundle $H \to \mathbb{G}_k(V)$ via*

$$s_\alpha(\lambda) = \alpha|\lambda.$$

The map

$$V^* \to \mathcal{O}(\mathbb{G}_k(V), H) : \alpha \mapsto s_\alpha$$

is an isomorphism.

Proof. It is clear that the map is injective; to show it is surjective we choose $s \in \mathcal{O}(X, H)$; we must find $\alpha \in V^*$ with $s = s_\alpha$. This can be found via a power series argument using the standard affine coordinates on the Grassmannian as follows. ... □

1.3. Let $E \to X$ be a holomorphic vector bundle of rank k over a compact complex manifold X. The vector space $\mathcal{O}(X, E)$ of holomorphic sections of E is finite dimensional by elliptic theory. A **base point** of E is a point $p \in X$ where the space $\{s(p) : s \in \mathcal{O}(X, E)\}$ is a proper subspace of the fiber E_p; the bundle is called **base point free** iff it has no base points. If p is not a base point of E we have

$$\{s \in \mathcal{O}(X, E) : s(p) = 0\} \in \mathbb{G}_{N-k}(\mathcal{O}(X, E))$$

here $N := \dim_{\mathbb{C}}(\mathcal{O}(X, E))$ and hence, via the canonical isomorphism of paragraph 1.1,

$$f(p) := \{s \in \mathcal{O}(X, E) : s(p) = 0\}^{\perp} \in \mathbb{G}_k(\mathcal{O}(X, E)^*).$$

For a base point free bundle $E \to X$ this defines a map

$$f : X \to \mathbb{G}_k(\mathcal{O}(X, E)^*)$$

called the **classifying map** of E.

Theorem 1.4. *Let $E \to X$ a base point free holomorphic bundle and let $H \to \mathbb{G}_k(\mathcal{O}(X, E)^*)$ be the cotautological bundle. Then the pull back of H by the classifying map is E.*

Proof. For each $p \in X$ we have a linear isomorphism

$$E_p \to H_{f(p)} := \{s \in \mathcal{O}(X, E) : s(p) = 0\}^{\perp} : v \mapsto \eta_v$$

where $\eta_v(\alpha) = \alpha(s)$ for $\alpha \in \{s \in \mathcal{O}(X, E) : s(p) = 0\}^{\perp} \subset \mathcal{O}(X, E)^*$ and $s \in \mathcal{O}(X, E)$ with $s(p) = v$. □

Theorem 1.5. *A compact Riemann surface X can be (1) embedded in $\mathbb{P}^3$ and (2) immersed in $\mathbb{P}^2$ so that its image has only transverse double points.*

Sketch of Proof. Theorem 1.4 says that vector bundles without base point correspond to maps to the Grassmannian. In particular, line bundles without base point correspond to maps to projective space. If X is a Riemann surface, Theorem 9.5 below says that a holomorphic line bundle of sufficiently high degree has no base point and that the classifying map is injective and the derivative of the classifying map is never zero. Thus shows that any Riemann surface can be embedded in projective space $\mathbb{P}^N$. Suppose $\mathbb{P}^{N-k}$ and $\mathbb{P}^k$ are transverse projective subspaces of $\mathbb{P}^N$ of the indicated dimensions. By transversality theory, for $k \geq 2$, a generic $\mathbb{P}^{N-k}$ misses the image of X. Then projection onto $\mathbb{P}^k$ along $\mathbb{P}^{N-k}$ gives a map from X into $\mathbb{P}^k$. By transversality theory this projection is generically an embedding of X for $k = 3$ and generically an immersion with at worst transverse double points for $k = 2$. $\square$

2 Degree

2.1. Let $E \to X$ be a smooth fiber oriented vector bundle over a compact smooth oriented manifold X. Assume that the **rank** (=fiber dimension) of E is the same as the dimension n of X. For an isolated zero $p \in X$ of a smooth section s of E define the **local degree** $\deg_p(s)$ of s at p by

$$\deg_p(s) = \text{degree}\left(S_p \to S(E_p) : q \mapsto \frac{s(q)}{|s(q)|}\right)$$

where S_p is the boundary of a small disk D in X centered at p and $S(E_p)$ is the boundary of the unit disk of the fiber E_p in some trivialization of E over D. Here the disk D is small in the sense that the only zero of s in its closure is the point p. Because the degree of a map between spheres of the same dimension is a homotopy invariant, the local degree of a smooth section s at an isolated zero is independent of the choice of the small disk $D \subset X$ about p and of the choice of the local trivialization of $E|D$ used in the definition.

Definition 2.2. By transversality theory (see Milnor, *Topology from the differentiable viewpoint*) the number

$$\deg(E) = \sum_{s(p)=0} \deg_p(s)$$

is independent of the choice of the smooth section with isolated zeros used to defined it. This number is called the **degree** or **Euler number** of the bundle $E \to X$. The cohomology class $e(E) \in H^n(X)$ defined by

$$\langle e(E), [X] \rangle = \deg(E)$$

is called the **Euler class** of the bundle $E \to X$; here $[X] \in H_n(X)$ is the fundamental class. In the case X is a surface (i.e. a smooth manifold of real dimension two) and $L \to X$ is a complex line bundle the Euler class of L is called the **Chern class** and denoted by $c_1(L)$.

Theorem 2.3. *The Euler number of the cotangent bundle $T^*X \to X$ (and hence also of the tangent bundle $TX \to X$) is the Euler characteristic $\chi(X)$.*

Proof. Let $f : X \to \mathbb{R}$ be a Morse function. Then the section df of T^*X has isolated zeros. At a critical point p the Morse lemma tells us that there are coordinates $x_1, \ldots, x_n$ such that

$$f(q) = -x_(q)^2 - \cdots - x_k(q)^2 + x_{k+1}(q)^2 + \cdots + x_n(q)^2$$

so

$$\frac{df(q)}{|df(q)|} = -x_1\, dx - \cdots - x_k\, dx_k + x_{k+1}\, dx_{k+1} + \cdots + x_n\, dx_n.$$

Hence $\deg_p(df) = (-1)^k$. The result now follows by Morse theory. □

3 Line Bundles

3.1. Assume that X is a Riemann surface and $E = L \to X$ is a holomorphic line bundle over X. Let s be a meromorphic section of L not identically zero. Then near a singularity (i.e. zero or pole) of s we may choose a local trivialization of L and a holomorphic coordinate $z = re^{i\theta}$ such that

$$s(q) = z(q)^k = r^k e^{ik\theta}.$$

The integer k is independent of the choice of the local trivialization and local coordinate and is called the **order** of s at p and denoted $\mathrm{Ord}_p(s)$. The order is the degree of the map $q \mapsto |s(q)|^1 s(q)$ from a small circle about the singularity to the unit circle of the fiber. Hence

$$\deg_p(s) = \mathrm{Ord}_p(s).$$

The formula

$$\deg(L) = \sum_p \deg_p(s)$$

holds for meromorphic sections since we may modify s near each pole so as to produce a smooth s with a zero of the same degree via the formula $\tilde{s}(q) = \phi(r)e^{ik\theta}$ where $\phi(r) = r^k$ for r near the boundary of the domain of z, $\phi(r) > 0$ for $r > 0$, and $\phi(0) = 0$.

Definition 3.2. The **canonical bundle** over a Riemann surface X is the bundle $K \to X$ whose fiber K_p over a point $p \in X$ is the vector space

$$K_p = L_{\mathbb{C}}(T_pX, \mathbb{C})$$

of $\mathbb{C}$-linear maps from the tangent space T_pX to $\mathbb{C}$. This bundle should be distinguished from the cotangent bundle $T^*X \to X$ whose fiber is the real dual space

$$T_p^*X = L_{\mathbb{R}}(T_pX, \mathbb{R})$$

for $p \in X$. Each holomorphic coordinate z gives a nonzero local section dz of K and on the overlap of the domains of two holomorphic coordinates z and w we have

$$dw = \phi' \, dz$$

where where ϕ is the holomorphic function such that $w(q) = \phi(z(q))$. A meromorphic section of K is called a **meromorphic differential**. A holomorphic section of K is called a **holomorphic differential** or (in some books) an **abelian differential**.

Theorem 3.3. *Let X be a compact Riemann surface. Then the degree of the canonical bundle over X is*

$$\deg(K) = -\chi(X)$$

where $\chi(X)$ is the Euler characteristic of X.

Proof. Let ω be a meromorphic differential on X. The real valued form $\xi = \Re(\omega)$ is a section of the cotangent bundle. Near a singularity $\omega = z^k \, dz$ in a suitable holomorphic coordinate. Now $z^k = r^k(\cos k\theta + i \sin k\theta)$ and $dz = dx + idy$ so $\xi = r^k \cos(k\theta) \, dx - r^k \sin(k\theta) \, dy$ and hence $\deg(K) = \deg_p(\omega) = -\deg_p(\xi) = -\chi(X)$. $\square$

4 Divisors

4.1. A **divisor** on a compact[1] Riemann surface X is a $\mathbb{Z}$ valued function on X with finite support. We represent a divisor as a formal finite sum

$$D = \sum_{k=1}^{m} n_k p_k$$

where n_k is the value of D at the point p_k. A meromorphic section s of a holomorphic line bundle $L \to X$ (in particular a meromorphic function) determines a divisor

$$(s) = \sum_{p \in X} \deg_p(s) p$$

whose support is the set of all singularities (zeros and poles) of s. The **degree** of the divisor D is the integer

$$\deg(D) = \sum_{k=1}^{m} n_k;$$

thus

$$\deg((s)) = \deg(L)$$

for a meromorphic section s of a holomorphic line bundle $L \to X$. A **principal divisor** is one of form (f) where f is a meromorphic function. two divisors are called **linearly equivalent** iff they differ by a principal divisor. The notation $D \geq 0$ means that D takes only nonnegative values. A divisor D is called **positive** or **effective** iff $D \geq 0$. For any divisor D we define the complex vector space

$$\mathcal{L}(D) := \{f \in \mathcal{M}(X) : f = 0 \text{ or } (f) + D \geq 0\}$$

and

$$\ell(D) := \dim_{\mathbb{C}}(\mathcal{L}(D)).$$

Here $\mathcal{M}(X)$ is the function field of X, i.e. the field of meromorphic functions on X and $\mathcal{M}^*(X) = \mathcal{M}(X) \setminus \{0\}$ is the multiplicative group of this field.

Theorem 4.2. *A divisor and a meromorphic section of a holomorphic line bundle are essentially the same thing. More precisely*

[1] In paragraph 11.1 we extend this definition to non compact Riemann surfaces.

(i) *Every holomorphic line $L \to X$ admits a meromorphic section s.*

(ii) *A meromorphic section s is holomorphic if and only if its divisor (s) is effective.*

(iii) *For every divisor D there is a holomorphic line bundle $L_D \to X$ and a meromorphic section s_D of L_D with $D = (s_D)$.*

(iv) *Assume that s and s' are meromorphic sections of holomorphic line bundles L and L' respectively. Then $(s) = (s')$ if and only if there is an isomorphism $L \to L'$ of holomorphic line bundles which carries s to s'.*

(v) *Two divisors are linearly equivalent if and only if the corresponding holomorphic line bundles are isomorphic.*

(vi) *Let D be the divisor of a meromorphic section s of a holomorphic line bundle $L \to X$. Then the map*

$$\mathcal{L}(D) \to \mathcal{O}(X, L) : f \mapsto fs$$

is an isomorphism from the vector space $\mathcal{L}(D)$ onto the vector space $\mathcal{O}(X, L)$ of holomorphic sections of L.

Proof. For the proof of (i) see Theorem 8.3 below. Part (ii) is obvious; a meromorphic section is holomorphic if and only if it has no poles. Given a divisor $D = \sum_{k=1}^m n_k p_k$ we will construct a line bundle L_D and a meromorphic section s_D. Choose a cover $\mathcal{U} = \{U_k\}_{0 \le k \le m}$ of X such that U_k is the domain of a holomorphic coordinate z_k centered at p_k for $k = 1, 2, \ldots, m$ and $U_0 = X \setminus \{p_1, p_2, \ldots, p_m\}$. Define a meromorphic function s_k on U_k by $s_k = z_k^{n_k}$ for $k = 1, 2, \ldots, m$ and $s_0 = 1$. The holomorphic functions $g_{jk} : U_j \cap U_k \to \mathbb{C}^*$ defined by

$$g_{jk} = \frac{s_j}{s_k}$$

satisfy $g_{ij} g_{jk} g_{ki} = 1$ and thus form the transition functions for a line bundle $L = L_D$. The formula $s_j = g_{jk} s_k$ says that the functions s_k fit together to form a (meromorphic) section s_D of L_D and by construction $(s_D) = D$. This proves (iii).

For part (iv) note that trivializations of L and L' over a common open set U determine a unique function ϕ on U with $s' = \phi s$ and ϕ is holomorphic and nowhere zero on U since $(s) = (s')$. For part (v) assume that s and s'

are meromorphic sections of L. Then the unique function ψ on X such that $s' = \psi s$ is meromorphic and satisfies $(s') = (\psi) + (s)$. Conversely assume that s and s' are meromorphic sections of L and L' respectively and that $(s') = (\psi) + (s)$ for some meromorphic function ψ on X. Choose an open cover $\{U_k\}_k$ such that both $L|U_k$ and $L'|U_k$ are trivial and let $s_k = s|U_k$, $s'_k = s'|U_k$, and $\psi_k = \psi|U_k$. Then s'_k and $\psi_k s_k$ have the same divisor in U_k so $s'_k = \phi_k \psi_k s_k$ where $\phi_k : U_k \to \mathbb{C}^*$. Thus

$$g_{jk} := \frac{s_j}{s_k} = \frac{s'_j}{s'_k} =: g'_{jk}$$

which shows that the corresponding line bundles L and L' are isomorphic.

For part (vi) first note that the condition $(f) + (s) \geq 0$ implies that fs is holomorphic so the map $f \mapsto fs$ carries $\mathcal{L}(D)$ to $\mathcal{O}(X, L)$. This map is clearly injective. Let $g_{jk} = s_j/s_k$ be the transition functions of part (iii). Then a holomorphic section of L is a collection of holomorphic functions σ_k such that $\sigma_j = g_{jk}\sigma_k$. It follows that $\sigma_j/s_j = \sigma_k/s_k$ on $U_j \cap U_k$, i.e. there is a meromorphic function f defined on X with $f|U_k = \sigma_k/s_k$, i.e. $\sigma = fs$. Since σ is holomorphic we have $(f) + (s) = (\sigma) \geq 0$, i.e. $f \in \mathcal{L}(D)$. This shows that the map $f \mapsto fs$ is surjective. □

4.3. The isomorphism classes of holomorphic line bundles over a Riemann surface X form an abelian group called the **Picard group** and denoted by $\mathrm{Pic}(X)$. The trivial line bundle is the identity element of $\mathrm{Pic}(X)$, the group operation is tensor product, and the inverse of a line bundle is its dual bundle. Theorem 4.2 defines an isomorphism

$$\mathrm{Pic}(X) = \mathrm{Div}(X)/\sim$$

where $\mathrm{Div}(X)$ is the group of divisors on X and $\sim$ denotes linear equivalence of divisors. This isomorphism is an isomorphism of groups since

$$(s_1 \otimes s_2) = (s_1) + (s_2)$$

for meromorphic sections s_1 and s_2 of L_1 and L_2 respectively. The degree of a holomorphic bundle defines a homomorphism of groups

$$\mathrm{Pic}(X) \to \mathbb{Z} : L \mapsto \deg(L).$$

Remark 4.4. Some authors mean by the term *Picard group* the subgroup $\mathrm{Pic}_0(X) = \mathrm{Div}_0(X)/\sim$ of line bundles of degree zero.

5 Sheaves

In this section $\mathcal{C}$ denotes the category of abelian groups but much of the theory described here works for any category $\mathcal{C}$ which admits inverse limits.

5.1. Let X be a topological space. A **presheaf** on X is contravariant functor $\mathcal{S}$ from the category of open subsets of X and inclusions to the category $\mathcal{C}$. Usually the value $\mathcal{S}(U)$ of $\mathcal{S}$ on an open subset U is a space of functions defined on U (or more generally a space of sections of a bundle over U) so we use the notation

$$s|V = \mathcal{S}(\iota_{VU})(s)$$

for open sets $V \subset U \subset X$, $s \in \mathcal{S}(U)$, and where $\iota_{VU} : V \to U$ is the inclusion; we call $s|V$ the **restriction** of s to V. A **sheaf** is a presheaf $\mathcal{S}$ which satisfies the following **completeness axiom**: *For every open set $U \subset X$ and every indexed open cover $\{U_i\}_{i \in I}$ of U the map*

$$\mathcal{S}(U) \to \left\{ s \in \prod_{i \in I} \mathcal{S}(U_i) : s|U_i \cap U_j = s_j|U_i \cap U_j \right\} : s \mapsto (s|U_i)_{i \in I}$$

is a bijection.

5.2. Let $\mathcal{S}$ be a presheaf over X. Consider triples (s, U, x) where $U \subset X$ is open, $s \in \mathcal{S}(U)$, and $x \in U$. Define an equivalence relation on these triples by $[s_1, U_1, x_1] = [s_2, U_2, x_2]$ iff $x_1 = x_2$ and there exists an open set $U \subset X$ with $x \in U \subset U_1 \cap U_2$ and $s_1|U_1 = s_2|U_2$. The equivalence class $[s, U, x]$ is called the **germ** of s at x; the **stalk** $\mathcal{S}_x$ is the set of germs $[s, U, x]$ at x. The natural projections

$$\mathcal{S}(U) \to \mathcal{S}_x : s \mapsto [s, U, x]$$

commute with restrictions so each stalk $\mathcal{S}_x$ is an object of $\mathcal{C}$.

Remark 5.3. To any presheaf $\mathcal{S}$ over X we can associate the disjoint union $S := \coprod_{x \in X} \mathcal{S}_x$. Denote by $\pi : S \to X$ the obvious projection $\pi([s, U, x]) = x$. Each $s \in \mathcal{S}(U)$ determines a section γ_s of π over U via the formula $\gamma_s(x) = [s, U, x]$. Equip S with the topology that makes each of these sections a homeomorphism onto its image. Then π is a local homeomorphism. For each open set $U \subset X$ let $\mathcal{O}(X, U)$ denote the set of continuous sections of π over U, i.e. the set of continuous maps $\gamma : U \to \pi^{-1}(U)$ such that $\pi \circ \gamma(x) = x$ for $x \in U$. It is not hard to prove that

- the functor $U \mapsto \mathcal{O}(X, U)$ is a sheaf, and
- a presheaf $\mathcal{S}$ is a sheaf if and only if the map $\mathcal{S}(U) \to \mathcal{O}(X, U) : s \mapsto \gamma_s$ is a bijection for every open set $U \subset X$.

A local homeomorphism is called an **etale space**. Some authors (e.g.[4]) use the term *sheaf* to signify an etale space where each $\pi^{-1}(x)$ is an abelian group, and the term *complete presheaf*, for what we have called a sheaf. The topology on S is confusing (it is often not Hausdorff) and we will avoid it.

5.4. Let $\mathcal{S}$ and $\mathcal{S}'$ be sheaves over X. A **morphism** from the sheaf $\mathcal{S}$ to the sheaf $\mathcal{S}'$ is a natural transformation $T : \mathcal{S} \to \mathcal{S}'$, i.e. an operation which assigns to each open set $U \subset X$ a morphism $T : \mathcal{S}(U) \to \mathcal{S}'(U)$ which intertwines the restriction morphisms. The morphism induces a morphism $T_x : \mathcal{S}_x \to \mathcal{S}'_x$ of stalks for each $x \in X$.

5.5. A sequence of sheaves and morphisms between them is called **exact** iff induces an exact sequence on stalks for each $x \in X$. A sheaf $\mathcal{S}$ is a **subsheaf** of the sheaf $\mathcal{T}$ iff $\mathcal{S}(U)$ is a subgroup of $\mathcal{T}(U)$ for every open set $U \subset X$. It is easy to prove that when $\mathcal{S}$ is a subsheaf of $\mathcal{T}$ there is an exact sequence

$$0 \to \mathcal{S} \to \mathcal{T} \to \mathcal{T}/\mathcal{S} \to 0$$

where the morphism $\mathcal{S} \to \mathcal{T}$ is the inclusion; the sheaf $\mathcal{T}/\mathcal{S}$ is unique up to isomorphism. WARNING: It can happen that $(\mathcal{T}/\mathcal{S})(U) \neq \mathcal{T}(U)/\mathcal{S}(U)$. For example, this happens for the exponential sequence (see below)

$$0 \to \mathbb{Z} \to \mathcal{E} \xrightarrow{\exp} \mathcal{E}^* \to 0$$

when U is not simply connected.

5.6. Here is a list of some important sheaves.

$\mathcal{E}$	smooth functions;
$\mathcal{E}^p$	smooth p-forms ($\mathcal{E} = \mathcal{E}^0$);
$\mathcal{E}^*$	nowhere zero smooth functions;
$\mathcal{E}^{p,q}$	smooth (p, q)-forms;
$\mathcal{O}$	holomorphic functions;
$\mathcal{O}^*$	nowhere zero holomorphic functions;
Ω^p	holomorphic p-forms ($\mathcal{O} = \Omega^0$);
$\mathcal{M}$	meromorphic functions;
$\mathcal{M}^*$	not identically zero meromorphic functions;
$\mathcal{P}$	principal parts, $\mathcal{P} = \mathcal{M}/\mathcal{O}$;
$\mathcal{D}$	divisors, $\mathcal{D} = \mathcal{M}^*/\mathcal{O}^*$.

The sheaves $\mathcal{E}$, $\mathcal{E}^p$, and $\mathcal{E}^*$ are defined on any smooth manifold X and the sheaf $\mathcal{M}^*$ is defined on an almost complex manifold; the remaining sheaves are defined only on complex manifolds. Unless otherwise specified the functions are complex valued. The group operation in all these is addition of functions except for $\mathcal{E}^*$, $\mathcal{O}^*$, and $\mathcal{M}^*$ where the operation is multiplication. When $E \to X$ is a vector bundle and $\mathcal{S}$ is one of the above sheaves we denote by $\mathcal{S}(E)$ the corresponding sheaf of sections of E and we use the abbreviation

$$\mathcal{S}(U, E) := \mathcal{S}(E)(U)$$

so that the elements of $\mathcal{S}(X, E)$ are sections of $E \to X$. For example, $\mathcal{E}(E)$ denotes the sheaf which assigns to the open set U the space of smooth sections of $E|U$ and $\mathcal{E}^{p,q}(E)$ denotes the sheaf which assigns to the open set U the space of smooth E valued forms of type (p, q) defined on U. These sheaves are defined for smooth complex vector bundles; the subsheaves $\mathcal{O}(E) \subset \mathcal{E}(E)$ and $\Omega^p(E) \subset \mathcal{E}^{p,0}(E)$ are defined only for holomorphic bundles.

Remark 5.7. For any holomorphic line bundle the sheaf $\mathcal{M}$ of meromorphic functions and the sheaf $\mathcal{M}(L)$ of meromorphic sections of L are isomorphic. Each meromorphic section $s \in \mathcal{M}(X, L)$ determines an isomorphism via the formula

$$\mathcal{M}(U) \to \mathcal{M}(U, L) : f \mapsto fs.$$

5.8. For each abelian group G we denote the corresponding **constant sheaf** by the same symbol, i.e. $G(U) = G$. For each point p we define the **skyscraper sheaf** G_p by $G_p(U) = G$ if $p \in U$ and $G_p(U) = 0$ if $p \notin U$.

5.9. Here are some important exact sequences of sheaves.

the smooth exponential sequence:

$$0 \to \mathbb{Z} \to \mathcal{E} \xrightarrow{\exp} \mathcal{E}^* \to 0$$

the holomorphic exponential sequence:

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$

the principal part sequence:

$$0 \to \mathcal{O} \to \mathcal{M} \to \mathcal{P} \to 0$$

the divisor sequence:

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{D} \to 0$$

the de Rham complex:

$$0 \to \mathbb{R} \to \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \ldots.$$

the pth Dolbeault complex:

$$0 \to \Omega^p \to \mathcal{E}^{p,0} \xrightarrow{\bar\partial} \mathcal{E}^{p,1} \xrightarrow{\bar\partial} \ldots.$$

6 Sheaf Cohomology

A good reference for the material in this section is [4].

6.1. Let X be a topological space. An **open cover** of X is an indexed collection $\mathcal{U} = \{U_i\}_{i\in I}$ of open subsets of X such that $X = \bigcup_{i\in I} U_i$. A **refinement** of the open cover $\mathcal{U}$ is open cover $\mathcal{V} = \{V_j\}_{j\in J}$ such that there is a map and $\tau : J \to I$ is a map such that $V_j \subset U_{\tau(j)}$ for $j \in J$; the map τ is called a **refining map** from $\mathcal{V}$ to $\mathcal{U}$. The **nerve** of the open cover $\mathcal{U}$ is the set

$$\mathrm{Nerve}(\mathcal{U}) = \bigcup_{k\ge 0} N^k(\mathcal{U})$$

where $N^k(\mathcal{U})$ consists of all finite subsequences

$$\sigma = (i_0, i_1, \ldots, i_k) \in I^{k+1}$$

such that the open set

$$U_\sigma := \bigcup_{r=0}^{k} U_{i_r}$$

is nonempty.

6.2. Let $\mathcal{S}$ be a sheaf over X and $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. We define a chain complex

$$0 \xrightarrow{\delta} C^0(\mathcal{U},\mathcal{S}) \xrightarrow{\delta} C^1(\mathcal{U},\mathcal{S}) \xrightarrow{\delta} C^2(\mathcal{U},\mathcal{S}) \xrightarrow{\delta} \cdots$$

by

$$C^k(\mathcal{U},\mathcal{S}) := \bigoplus_{\sigma\in N^k(\mathcal{U})} \mathcal{S}(U_\sigma)$$

and

$$(\delta\mu)_\sigma = \sum_{r=0}^{k} (-1)^r \mu_{\sigma(r)} | U_\sigma$$

where $\sigma = (i_0, i_1, \ldots, i_k)$ and $\sigma(r) = (i_0, i_1, \ldots, i_{r-1}, i_{r+1}, \ldots i_k)$. The kth **cohomology group** of the sheaf $\mathcal{S}$ for the cover $\mathcal{U}$ is

$$H^k(\mathcal{U}, \mathcal{S}) := \frac{\mathrm{Kernel}(\delta : C^k(\mathcal{U}, \mathcal{S}) \to C^{k+1}(\mathcal{U}, \mathcal{S}))}{\mathrm{Image}(\delta : C^{k-1}(\mathcal{U}, \mathcal{S}) \to C^k(\mathcal{U}, \mathcal{S}))}$$

Remark 6.3. Our use of the term nerve is not quite standard; the standard definition takes sets $\{i_0, i_1, \ldots, i_k\}$ of cardinality $k+1$ rather than sequences. With this definition one must linearly order I to define δ; changing the linear order changes the sign of δ. The kernel and image and (hence also the cohomology) are independent of the choice and the two notions of nerve lead to the same cohomology.

Remark 6.4. If $\delta\mu = 0$ then μ_σ is a skew symmetric function of the indices.

6.5. Let $\mathcal{S}$ be a sheaf, $\mathcal{U}$ be an open cover, and $\mathcal{V}$ be a refinement of $\mathcal{U}$. Each refining map τ induces a chain map $\tau_* : C^*(\mathcal{V}, \mathcal{S}) \to C^*(\mathcal{U}, \mathcal{S})$ and two refining maps induce chain homotopic chain maps. Hence there is a well defined map $H^*(\mathcal{V}, \mathcal{S}) \to H^*(cU, \mathcal{S})$. The open covers of X form a directed set under refinement. The kth cohomology group of $\mathcal{S}$ is defined by

$$H^k(\mathcal{S}) := \lim_{\mathcal{U} \to \infty} H^k(\mathcal{U}, \mathcal{S}).$$

When (as is usually the case) the sheaf $\mathcal{S}$ takes values in the category of complex vector spaces, we denote the dimension of the cohomology group by

$$h^k(\mathcal{S}) := \dim_{\mathbb{C}} H^k(\mathcal{S}).$$

Theorem 6.6. *For any sheaf $\mathcal{S}$ over a topological space X we have that $H^k(\mathcal{S}) = 0$ for $k > n$ where n is the covering dimension of X.*

Proof. By the definition of covering dimension any open cover has a refinement $\mathcal{U}$ with $N^k(\mathcal{U}) = \emptyset$ for $k > n$. □

6.7. Let $\mathcal{S}$ be a sheaf. A **partition of unity** on $\mathcal{S}$ is a collection $\{\eta_i\}_{i \in I}$ of sheaf homomorphisms $\eta_i : \mathcal{S} \to \mathcal{S}$ which is locally finite (i.e. for every point of X has a neighborhood which intersects the supports of only finitely many η_i) and satisfies $\sum_{i\in I}\eta_i = 1$ the identity. The partition of unity is **subordinate** to the cover $\mathcal{U} = \{U_i\}_{i\in I}$ if the support of η_i is a subset if U_i. A sheaf $\mathcal{S}$ is called a **fine sheaf** iff for any open cover $\mathcal{U}$ there is partition of unity subordinate to it.

Theorem 6.8. *Let $\mathcal{S}$ be a fine sheaf. Then $H^k(\mathcal{S}) = 0$ for $k > 0$.*

Corollary 6.9. *Let G_p be a skyscraper sheaf. Then $H^0(G_p) = G$ and $H^k(G_p) = 0$ for $k > 0$.*

Proof. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover. Choose i_0 with $p \in U_{i_0}$ and take η_{i_0} the identity and $\eta_i = 0$ for $i \neq i_0$. □

6.10. Let

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

be a short exact sequence of sheaves. Then there is a long exact sequence

$$0 \to H^0(\mathcal{A}) \to H^0(\mathcal{B}) \to H^0(\mathcal{C}) \xrightarrow{\delta}$$

$$H^1(\mathcal{A}) \to H^1(\mathcal{B}) \to H^1(\mathcal{C}) \xrightarrow{\delta}$$

$$H^2(\mathcal{A}) \to H^2(\mathcal{B}) \to H^2(\mathcal{C}) \xrightarrow{\delta} \cdots$$

(When X is a Riemann surface the $\cdots$ may be replaced by 0 according to Theorem 6.6.)

6.11. Let $L \to X$ be line bundle, $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover such that $L|U_i$ is trivial and $s_i \in \mathcal{O}^*(U_i, L)$ be a nowhere zero holomorphic section of L over U_i. Then the **transition functions** $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ defined by $s_i = g_{ij}s_j$ form a cocycle in $C^1(\mathcal{U}, \mathcal{O}^*)$ and hence a cohomology class in $H^1(\mathcal{O}^*)$ where $\mathcal{O}^*$ is the sheaf of nowhere zero holomorphic functions. It is not hard to see that this defines a group isomorphism

$$\mathrm{Pic}(X) = H^1(\mathcal{O}^*)$$

where $\mathrm{Pic}(X)$ is the **Picard group**, i.e. the multiplicative group (under tensor product) of isomorphism classes of line bundles over X.

6.12. The sheaf $\mathcal{O}^*$ fits into two exact sequences, namely the **divisor exact sequence**

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{D} \to 0$$

and the **exponential exact sequence**

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

where the map $\mathcal{O}(U) \to \mathcal{O}^*(U)$ is $s \mapsto \exp(2\pi i s)$. A common notation for the global divisors on X is

$$\mathrm{Div}(X) := H^0(\mathcal{D}) = \mathcal{D}(X).$$

When X is compact, it is not hard to see a global section is a divisor in the sense of paragraph 4.1 and that where $L = L_D$ is the line bundle constructed in part (iii) of Theorem 4.2 we have $L_D = \delta(D)$ where $\delta : H^0(\mathcal{D}) \to H^1(\mathcal{O}^*)$ is the boundary operator in the long exact sequence associated to the divisor sequence. (See paragraph 11.2 below.)

6.13. A **double complex** consists of a collection $\{C^{p,q}\}_{p,q\geq 0}$ of abelian groups indexed by the nonnegative integers together with homomorphisms $d : C^{p,q} \to C^{p,q+1}$ and $\delta : C^{p,q} \to C^{p+1,q}$ such that $d^2 = 0$, $\delta^2 = 0$, and $d\delta = \delta d$. For notational convenience define $C^{-1,q} = C^{p,-1} = 0$ and define d and δ as the inclusion of zero. Define cohomology groups

$$H_d^{p,q} = \frac{\mathrm{Kernel}(d : C^{p,q} \to C^{p,q+1})}{\mathrm{Image}(d : C^{p,q-1} \to C^{p,q})}$$

and

$$H_\delta^{p,q} = \frac{\mathrm{Kernel}(\delta : C^{p,q} \to C^{p+1,q})}{\mathrm{Image}(\delta : C^{p-1,q} \to C^{p,q})}$$

for $p, q \geq 0$.

6.14. We may represent the double complex as an array as follows

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
0 & \to & C^{0,2} & \xrightarrow{\delta} & C^{1,2} & \xrightarrow{\delta} & C^{2,2} & \xrightarrow{\delta} & \ldots \\
 & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
0 & \to & C^{0,1} & \xrightarrow{\delta} & C^{1,1} & \xrightarrow{\delta} & C^{2,1} & \xrightarrow{\delta} & \ldots \\
 & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
0 & \to & C^{0,0} & \xrightarrow{\delta} & C^{1,0} & \xrightarrow{\delta} & C^{2,0} & \xrightarrow{\delta} & \ldots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We will say that an $y \in C^{0,k}$ with $dy = 0$ and an element $x \in C^{k,0}$ with $\delta x = 0$ are **zig zag related** iff there exist $z_j \in C^{j,k-j}$ with $z_0 = y$, $z_k = x$, and $dz_j = \delta z_{j+1}$ for $j = 0, 1, \ldots, k-1$. When $k = 2$ this may be represented by

$$\begin{array}{ccccccc}
0 & & & & & & \\
\uparrow d & & & & & & \\
z_0 = y & \xrightarrow{\delta} & \delta z_0 = dz_1 & & & & \\
 & & \uparrow d & & & & \\
 & & z_1 & \xrightarrow{\delta} & \delta z_1 = dz_2 & & \\
 & & & & \uparrow d & & \\
 & & & & z_2 = x & \xrightarrow{\delta} & 0
\end{array}$$

Theorem 6.15. *Assume that the rows and columns of the double complex are exact except for the leftmost column and the bottom row, i.e. that $H_d^{p,q} = 0$ for $p > 0$ and $q \geq 0$ and that $H_\delta^{p,q} = 0$ for $q > 0$ and $p \geq 0$. Then the groups $H_d^{0,k}$ and $H_\delta^{k,0}$ are isomorphic. More precisely there is an isomorphism $T : H_d^{0,k} \to H_\delta^{k,0}$ such that for $y \in C^{0,k}$ with $dy = 0$ and $x \in C^{k,0}$ with $\delta x = 0$ we have $T[y] = [x]$ if and only if y and x are zig zag related.*

Corollary 6.16. *Suppose that*

$$0 \to \mathcal{S}^0 \to \mathcal{S}^1 \to \mathcal{S}^2 \to \cdots$$

is a **fine resolution** *of $\mathcal{S}_0$, i.e. an exact sequence of sheaves with $\mathcal{S}^k$ a fine sheaf for $k > 0$. Then there is an isomorphism*

$$H^p(X, \mathcal{S}_0) = \frac{\text{Kernel}(\mathcal{S}^p(X) \to \mathcal{S}^{p+1}(X))}{\text{Image}(\mathcal{S}^{p-1}(X) \to \mathcal{S}^p(X))}$$

Proof. Choose an open cover $\mathcal{U}$ of X. Let

$$C^{p,q} := C^p(\mathcal{U}, \mathcal{S}^q)$$

denote the space of Cêch p-cochains on the cover $\mathcal{U}$ with values in $\mathcal{S}^q$ and apply the Zig Zag Theorem. □

Corollary 6.17 (De Rham). *Let X be a compact smooth manifold. Then there is an isomorphism*

$$H^*_{DR}(X, \mathbb{C}) = \hat{H}^*(X, \mathbb{C})$$

between the de Rham cohomology and the Cêch cohomology.

Proof. Let $\mathcal{E}^p$ be the sheaf of smooth p-forms on X. Then we have a fine resolution

$$0 \to \mathbb{C} \to \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \cdots.$$

That the sequence is exact is the Poincaré Lemma. □

Corollary 6.18 (Dobeault). *Let $E \to X$ be a holomorphic vector bundle over a complex manifold X and $\Omega^p(E)$ be the sheaf of holomorphic E valued p-forms on X. Then there is an isomorphism*

$$H^{p,q}_{\bar\partial}(X, E) = H^q(X, \Omega^p(E))$$

between the Dolbeault cohomology of E-valued forms of type (p, q) and the sheaf cohomology of $\Omega^p(E)$.

Proof. Let $\mathcal{E}^{p,q}(E)$ be the sheaf of smooth E-valued forms of type (p, q). Then we have a fine resolution

$$0 \to \Omega^p(E) \to \mathcal{E}^{p,0}(E) \xrightarrow{\bar\partial} \mathcal{E}^{p,1}(E) \xrightarrow{\bar\partial} \cdots.$$

That the sequence is exact is the $\bar\partial$-Poincaré Lemma. □

7 Serre Duality

7.1. Let X be a compact complex manifold of complex dimension n, $E \to X$ be a holomorphic vector bundle, and $E^* \to X$ denote the dual bundle. For $p, q, = 0, 1, \ldots, n$ define a pairing

$$\mathcal{E}^{n-p,n-q}(X, E^*) \times \mathcal{E}^{p,q}(X, E) \to \mathbb{C} : (\alpha, \beta) \mapsto \int_X \alpha \wedge \beta.$$

By integration by parts

$$\int_X \bar\partial\gamma \wedge \beta = (-1)^p \int_X \gamma \wedge \bar\partial\beta$$

for $\alpha \in \mathcal{E}^{n-p,n-q-1}(X, E^*)$ and $\beta \in \mathcal{E}^{p,q}(X, E)$ so

$$\bar\partial^* := (-1)^p \bar\partial : \mathcal{E}^{n-p,n-q-1}(X, E^*) \to \mathcal{E}^{n-p,n-q}(X, E^*)$$

is a formal adjoint to the $\bar\partial$ operator

$$\bar\partial : \mathcal{E}^{p,q}(X, E) \to \mathcal{E}^{p,q+1}(X, E).$$

This defines a map

$$H_{\bar\partial}^{n-p,n-q}(X,E^*) \to H_{\bar\partial}^{p,q}(X,E)^* \tag{$*$}$$

between the Dolbeault cohomology groups. Using the Dolbeault isomorphism of Corollary 6.18 this induces a map

$$H^{n-q}(\Omega^{n-p}(E^*)) \to H^q(\Omega^p(E))^*$$

between the sheaf cohomology groups.

Theorem 7.2 (Serre Duality). *The pairing is nondegenerate, i.e. the map* $(*)$ *is an isomorphism.*

Sketch of proof. Let $\mathcal{E}^{p,q}(X,E)^*$ denote the dual space of the Frechét space $\mathcal{E}^{p,q}(X,E)$. The above pairing gives an inclusion

$$\mathcal{E}^{n-p,n-q}(X,E^*) \to \mathcal{E}^{p,q}(X,E)^*.$$

If, in the statement of Serre duality, the cohomology group $H_{\bar\partial}^{n-p,n-q}(X,E^*)$ of the formal dual complex

$$\cdots \to \mathcal{E}^{n-p,n-(q+1)}(X,E^*) \xrightarrow{\bar\partial^*} \mathcal{E}^{n-p,n-q}(X,E^*) \xrightarrow{\bar\partial^*} \mathcal{E}^{n-p,n-(q-1)}(X,E^*) \to \cdots$$

is replaced by the cohomology group of the dual complex

$$\cdots \to \mathcal{E}^{p,q+1}(X,E^*)^* \xrightarrow{\bar\partial^*} \mathcal{E}^{p,q}(X,E^*)^* \xrightarrow{\bar\partial^*} \mathcal{E}^{p,q-1}(X,E^*)^* \to \cdots$$

the Serre theorem becomes an exercise in linear algebra. The replacement is justified by elliptic regularity; i.e. a linear functional in the kernel of $\bar\partial^*$ is given by integration against smooth form. For the details see [4]. □

7.3. In case X is a Riemann surface, its complex dimension is $n = 1$ and simpler notations are used: the sheaf of germs of holomorphic one forms is denoted by Ω rather than Ω^1 and the sheaf of germs of holomorphic aero forms (functions) is denoted by $\mathcal{O}$ rather than Ω^0. In this case

$$H^1(\mathcal{O}(E))^* = H_{\bar\partial}^{0,1}(X,E)^* = H_{\bar\partial}^{1,0}(X,E^*) = H^0(\Omega(E^*))$$

where the middle isomorphism is Serre duality and the outer ones are the Dolbeault isomorphisms. Taking dimensions we get

$$h^1(\mathcal{O}(E)) = h^0(\Omega(E^*)) = h^0(\mathcal{O}(K \otimes E^*))$$

where K is the canonical bundle.

Remark 7.4. A line bundle L of negative degree can have no holomorphic sections so

$$\deg(L) < 0 \implies h^0(\mathcal{O}(L)) = 0.$$

Since $\deg(K \otimes L^*) = \deg(K) - \deg(L) = 2g - 2 - \deg(L)$ where g is the genus of X we have

$$\deg(L) > 2g - 2 \implies h^1(\mathcal{O}(L)) = h^0(\mathcal{O}(K \otimes L^*)) = 0.$$

8 Existence Theorem

In this section we will prove part (i) of Theorem 4.2. We follow [4]. We derive the Riemann Roch Theorem for holomorphic line bundles as a corollary. Throughout X is a compact Riemann surface.

8.1. For any line bundle L define an integer

$$\nu(L) := h^0(\mathcal{O}(L)) - h^1(\mathcal{O}(L)) - \deg(L).$$

The Riemann Roch Theorem (Theorem 8.5 below) is that $\nu(L) = 1 - g$ where g is the genus of X,

Lemma 8.2. *Let L_D be the line bundle associated with the divisor D as in Theorem 4.2. Then*

$$\nu(L \otimes L_D) = \nu(L)$$

for any divisor D and any line bundle L.

Proof. By induction and the fact that $L_{D+p} = L_D \otimes L_p$ it suffices to prove this in case $D = p$. There is a short exact sequence

$$0 \to \mathcal{O}(L) \to \mathcal{O}(L \otimes L_p) \to \mathbb{C}_p \to 0;$$

the map $\mathcal{O}(L) \to \mathcal{O}(L \otimes L_p)$ is given by $s \mapsto s \otimes s_p$. Hence (by 6.9) there is a long exact sequence

$$0 \to H^0(\mathcal{O}(L)) \to H^0(\mathcal{O}(L \otimes L_p)) \to \mathbb{C} \xrightarrow{\delta}$$
$$H^1(\mathcal{O}(L)) \to H^1(\mathcal{O}(L \otimes L_p)) \to 0.$$

Now use the fact that the alternating sums of the dimensions in an exact sequence is zero and the fact that $\deg(L \otimes L_p) = \deg(L) + 1$. □

Theorem 8.3. *A holomorphic line bundle admits a meromorphic section.*

Proof. Let L be a holomorphic line bundle. Since the line bundle L_D has a holomorphic section, to show that L has a meromorphic section it suffices to show that $L \otimes L_D$ has a holomorphic section for some divisor D. If not, then $h^0(\mathcal{O}(L \otimes L_D)) = 0$ for all D and so by Lemma 8.2 we have $\nu(L) = \nu(L \otimes L_D) = -h^1(\mathcal{O}(L \otimes L_D)) - \deg(L \otimes L_D)$. By Serre Duality this gives

$$\nu(L) = -h^0(\mathcal{O}(K \otimes L^* \otimes L_{-D})) - \deg(L \otimes L_D).$$

Now $\deg(K \otimes L^* \otimes L_{-D}) = \deg(K) - \deg(L) - \deg(L_D)$ which is negative for D large. If the degree of a holomorphic bundle is negative it has no holomorphic sections so $h^0(\mathcal{O}(K \otimes L^* \otimes L_{-D})) = 0$ for D large and hence $\nu(L) = -\deg(L \otimes L_D) = -\deg(L) - \deg(L_D)$ for all D which is clearly absurd. □

Corollary 8.4. *Every line bundle is the line bundle of a divisor.*

Proof. Let s be a meromorphic section of L and $D = (s)$. Then L is isomorphic to L_D. □

Theorem 8.5 (Riemann Roch for Line Bundles). *For any holomorphic line bundle $L \to X$ over a Riemann surface X of genus g we have*

$$h^0(\mathcal{O}(L)) - h^0(\Omega(L)) = \deg(L) + 1 - g.$$

In particular, $h^0(\Omega) = g$.

Proof. In other words, we must show $\nu(L) = 1 - g$ where $\nu(L)$ was defined in Lemma 8.2. By Corollary 8.4 it is enough to show that $\nu(L_D) = 1 - g$ for every divisor D. By Lemma 8.2 $\nu(L_D) = \nu(L_0)$ where L_0 denotes the trivial bundle; so it is enough to show $\nu(L_0) = 1 - g$. Now $h^0(\mathcal{O}(L_0)) = 1$ (a holomorphic function is constant) and $\deg(L_0) = 0$ so

$$\nu(L_0) = h^0(\mathcal{O}(L_0)) - h^0(\mathcal{O}(K \otimes L_0^*)) - \deg(L_0) = 1 - h^0(\mathcal{O}(K)).$$

Also $K \otimes K^* = L_0$ and by Theorem 3.3 $\deg(K) = 2g - 2$ so

$$\nu(K) = h^0(\mathcal{O}(K)) - h^0(\mathcal{O}(K \otimes K^*)) - \deg(K) = h^0(\mathcal{O}(K)) - 1 - (2g - 2).$$

By Lemma 8.2 $\nu(L_0) = \nu(K)$ so $h^0(\mathcal{O}(K)) = g$ so $\nu(L_0) = 1 - h^0(\mathcal{O}(K)) = 1 - g$ as required. □

9 Riemann Roch for Vector Bundles

9.1. Let $E \to X$ be a holomorphic vector bundle over a compact Riemann surface. By the definition of cohomology the sequence

$$0 \to H^0_{\bar\partial}(X,E) \to \mathcal{E}^0(X,E) \xrightarrow{\bar\partial} \mathcal{E}^{0,1}(X,E) \to H^{0,1}_{\bar\partial}(X,E) \to 0$$

is exact, i.e. $H^0_{\bar\partial}(X,E)$ is the kernel of $\bar\partial$ and $H^{0,1}_{\bar\partial}(X,E)$ is the cokernel. The operator $\bar\partial$ is elliptic so these spaces are finite dimensional. The number

$$\mathrm{ind}(\bar\partial) := \dim_{\mathbb{C}} H^0(X,E) - \dim_{\mathbb{C}} H^{0,1}(X,E)$$

is called the **index** of the $\bar\partial$ operator of E. By Serre duality the exact sequence my be written

$$0 \to \mathcal{O}(X,E) \to \mathcal{E}^0(X,E) \xrightarrow{\bar\partial} \mathcal{E}^{0,1}(X,E) \to \mathcal{O}(X, K \otimes E^*)^* \to 0$$

where $K \to X$ is the canonical bundle and so the index may be written as

$$\mathrm{ind}(\bar\partial) = h^0(\mathcal{O}(E)) - h^0(\Omega(E^*)).$$

9.2. By a transversality argument, any complex vector bundle E of rank great than one over a Riemann surface X has a smooth nowhere vanishing section. Hence E has decomposition $E = L_1 \oplus \cdots \oplus L_n$ as a direct sum of line bundles. (Warning: It is not true that a holomorphic vector bundle is a direct sum of holomorphic line bundles.) The number

$$c(E) := \sum_{k=1}^{n} \deg(L_k)$$

is called the **Chern number** of the vector bundle E. The Chern number is the value of the first Chern class $c_1(E) \in H^2(X)$ on the fundamental cycle $[X] \in H_2(X)$ so it is independent of the decomposition used to define it.

Theorem 9.3 (Riemann Roch). *The index of the $\bar\partial$ operator of E of a holomorphic vector bundle $E \to X$ over a compact Riemann surface X of genus g is given by*

$$\mathrm{ind}(\bar\partial) = n(1-g) + c(E).$$

Proof. ... □

Remark 9.4. By Theorem 3.3 the degree of the canonical bundle $K \to X$ is

$$\deg(K) = -\chi(X) = 2g - 2$$

Since $c(E^*) = -c(E)$ and the Chern number is additive and agrees with the degree for line bundles, the Riemann Roch theorem has the suggestive form

$$\dim_{\mathbb{R}} \Gamma(E) - \dim_{\mathbb{R}} \Gamma(K \otimes E) = c(E) - c(K \otimes E^*).$$

Theorem 9.5 (Kodaira Embedding for Riemann Surfaces). *Let $L \to X$ be a holomorphic line bundle over a Riemann surface X of genus g and assume that* $\deg(L) > 2g + 1$*. Then*

(i) *For $p, q \in X$ with $p \neq q$ there is a holomorphic section $s \in \mathcal{O}(X, L)$ with $s(p) = 0$ and $s(q) \neq 0$; and*

(ii) *For $p \in X$ there is a holomorphic section $s \in \mathcal{O}(X, L)$ with $s(p) = 0$ and $ds(p) \neq 0$. Here $ds(p) : T_pX \to L_p$ is the canonical derivative.*

Hence the corresponding map $X \to \mathbb{P}^N = \mathbb{P}(\mathcal{O}(X, L))^*)$, $N = h^0(\mathcal{O}(L)) - 1$ is an embedding.

Proof. ...See [1] page 144. ☐

10 The Canonical Map

Lemma 10.1. *If X has positive genus, then the canonical bundle $K \to X$ has no base point.*

Proof. Suppose every $\omega \in \mathcal{O}(X, K) = \Omega(X)$ vanishes at some point $p \in X$; we will show that X has genus zero. Let L_p denote the line bundle corresponding to the divisor $D = p$ as in Theorem 4.2. Then $\mathcal{O}(X, K \otimes L_p^*)$ and $\mathcal{O}(X, K)$ are isomorphic so $h_0(\mathcal{O}(K) = h_0(\mathcal{O}(K \otimes L_p^*)$. Hence by Riemann Roch, $h_0(\mathcal{O}(L_p)) = h_0(\mathcal{O}(K \otimes L_p^*) + \deg(L_p) + 1 - g = 2$. Hence there is a meromorphic function with a simple pole at p and no other pole so X is isomorphic to $\mathbb{P}$ and hence has genus zero. ☐

Definition 10.2. The classifying map $X \to \mathbb{P}^{g-1} = \mathbb{P}(\Omega(X))$ for the canonical bundle $K \to X$ of a Riemann surface is called the **canonical map** of X. The automorphism group $\mathrm{Aut}(X)$ of X acts on K and hence on the target projective space of the canonical map; the canonical map is equivariant.

11 Divisors and Principal Parts

11.1. Throughout this section X is a Riemann surface, not necessarily compact. Let $\mathcal{M}$ be the sheaf of germs of meromorphic functions, $\mathcal{O}$ the subsheaf of germs of holomorphic functions, and $\mathcal{P}$ the quotient sheaf. We call $\mathcal{P}$ the **sheaf of principal parts** in X and the exact sequence

$$0 \to \mathcal{O} \to \mathcal{M} \to \mathcal{P} \to 0$$

the **principal part exact sequence**. Let $\mathcal{M}^*$ be the sheaf of germs of nonzero meromorphic functions, $\mathcal{O}^*$ the subsheaf of germs of nowhere zero holomorphic functions, and $\mathcal{D}$ the quotient sheaf. We call $\mathcal{P}$ the **sheaf of divisors** on X and the exact sequence

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{D} \to 0$$

the **divisor exact sequence**. If $U \subset X$ is connected and open then $\mathcal{M}^*(U)$ is the multiplicative group the field $\mathcal{M}(U)$ and $\mathcal{O}^*(U)$ is the multiplicative group the integral domain $\mathcal{O}(U)$. The group operation in $\mathcal{M}^*$ is multiplication but we write the group operation in $\mathcal{D}$ additively.

11.2. The sheaf $\mathcal{D}$ assigns to each open set U the space $\mathcal{D}(U)$ of all formal sums

$$D = \sum_{p \in U} n_p p$$

where n_p is an integer and the support (i.e. the closure in U of the set of points p where $n_p \neq 0$) is discrete. In particular, when X is compact the support of a global section is finite, i.e.

$$H^0(X, \mathcal{D}) = \mathrm{Div}(X)$$

where $\mathrm{Div}(X)$ is as in paragraph 6.12. The map $H^0(\mathcal{M}^*) \to H^0(\mathcal{D})$ in the cohomology exact sequence of the divisor exact sequence assigns to each global meromorphic function $f \in \mathcal{M}^*(X)$ its divisor

$$(f) = \sum_{p \in X} \mathrm{Ord}_p(f) p.$$

The sheaf $\mathcal{P}$ assigns to each open set U the space $\mathcal{P}(U)$ of all formal sums

$$P = \sum_{p \in U} h_p p$$

with discrete support (as for divisors) and where h_p a finite Laurent series

$$h_p = \sum_{k=1}^{n_p} a_k(p) z^{-k}.$$

(Here z is a holomorphic coordinate centered at p so the values $a_k(p)$ depend on the choice of Z.) The map $H^0(\mathcal{M}) \to H^0(\mathcal{P})$ in the cohomology exact sequence of the principal part exact sequence assigns to each meromorphic function f the formal sum P, supported at the poles of f where h_p is the sum of the negative terms in the Laurent expansion for f at p. We next interpret the boundary operators $H^0(\mathcal{D}) \to H^1(\mathcal{O}^*)$ and $H^0(\mathcal{P}) \to H^1(\mathcal{O})$ in a parallel fashion.

11.3. A **Weierstrass distribution** is a pair $(f, \mathcal{U})$ where $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover of X, $f_i \in \mathcal{M}^*(U_i)$, and $f_i/f_j \in \mathcal{O}^*(U_i \cap U_j)$; a solution to $(f, \mathcal{U})$ is a meromorphic function $f \in \mathcal{M}^*(X)$ with $f_i = f|U_i$. For a Weierstrass distribution the number $\mathrm{Ord}_p(f_i)$ is independent of the choice if i with $p \in U_i$; this gives an cocycle $D(f, \mathcal{U}) \in C^0(\mathcal{U}, \mathcal{D})$ and every cocycle arises as a $D(f, \mathcal{D})$. The boundary operator $\delta : H^0(\mathcal{D}) \to H^1(\mathcal{O}^*)$ assigns to the cocycle D the cocycle f_i/f_j where $D = D(f, \mathcal{U})$. Hence by the cohomology exact sequence a Weierstrass distribution $(f, \mathcal{U})$ has a solution $\iff \delta[D(f, \mathcal{U})] = 0$ in $H^1(\mathcal{O}^*)$. Note that under the identification $\mathrm{Pic}(X) = H^1(\mathcal{O}^*)$ of paragraph 6.11 we have

$$\delta(D) = L_D$$

where L_D is the line bundle associated with the divisor D as in Theorem 4.2.

11.4. A **Mittag Leffler distribution** is a pair $(f, \mathcal{U})$ where $f_i \in \mathcal{M}(U_i)$ and $f_i - f_j \in \mathcal{O}(U_i \cap U_j)$; a solution to $(f, \mathcal{U})$ is a meromorphic function $f \in \mathcal{M}(X)$ with $f_i = f|U_i$. A Mittag Leffler distribution determines a cocycle $P(f, \mathcal{U}) \in C^0(\mathcal{U}, \mathcal{P})$ and every cocycle arises as a $P(f, \mathcal{U})$. The boundary operator $\delta : H^0(\mathcal{P}) \to H^1(\mathcal{O})$ assigns to the cocycle P the cocycle $f_i - f_j$ where $P = P(f, \mathcal{U})$. Hence by the cohomology exact sequence a Mittag Lefler distribution $(f, \mathcal{U})$ has a solution $\iff \delta[P(f, \mathcal{U})] = 0$ in $H^1(\mathcal{O})$.

Remark 11.5. In the theory of several complex variables a Mittag Lefler distribution is called a *Cousin-I* distribution and a Weierstrass distribution is called a *Cousin-II* distribution. The problem of finding f with $f|U_i = f_i$ is called the **Cousin problem**.

Lemma 11.6. *The sheaves $\mathcal{D}$ and $\mathcal{P}$ are fine. Hence $H^k(\mathcal{D}) = H^k(\mathcal{P}) = 0$ for $k > 0$.*

Proof. Choose any open cover $\{U_i\}_i$. Then write X as a disjoint union of sets X_i with $X_i \subset U_i$; the sets X_i are neither closed nor open. Define $\eta_i : \mathcal{D} \to \mathcal{D}$ by

$$\eta_i(D) = \sum_{p \in X_i \cap U} n_p p$$

for $U \subset X$ open and $D \in \mathcal{D}(U)$ a local section of $\mathcal{D}$. Then $\sum \eta_i = 1$. □

Theorem 11.7. $H^2(\mathcal{O}) = H^2(\mathcal{O}^*) = H^2(\mathcal{M}^*) = H^2(\mathcal{M}) = 0$.

Proof. By Dolbeault $H^2(\mathcal{O}) = 0$. In the exponential cohomology exact sequence $H^2(\mathcal{O}^*)$ lies between $H^2(\mathcal{O})$ and $H^3(\mathbb{Z})$ so $H^2(\mathcal{O}^*) = 0$. In the divisor cohomology exact sequence $H^2(\mathcal{M}^*)$ lies between $H^2(\mathcal{O}^*)$ and $H^2(\mathcal{D})$ so $H^2(\mathcal{M}^*) = 0$. In the principal part cohomology exact sequence $H^2(\mathcal{M})$ lies between $H^2(\mathcal{O})$ and $H^2(\mathcal{P})$ so $H^2(\mathcal{M}) = 0$. □

Theorem 11.8. *If X noncompact, $H^1(\mathcal{O}) = H^{0,1}(X, \mathbb{C}) = 0$.*

Proof. The Dolbealt isomorphism $H^1(\mathcal{O}) = H^{0,1}(X, \mathbb{C})$ follows from the exactness of the sheaf sequence

$$0 \to \mathcal{O} \to \mathcal{E}^0 \xrightarrow{\bar{\partial}} \mathcal{E}^{0,1} \to 0$$

and holds whether or not X is compact. To show that $H^{0,1}(X, \mathbb{C}) = 0$ we must show that for every $\omega \in \mathcal{E}^{0,1}(X)$ there is an $f \in \mathcal{E}^0(X)$ with $\omega = \bar{\partial} f$. The proof uses Runge approximation and an exhaustion argument. See [1] page 200. □

Corollary 11.9. *If X noncompact, $H^1(\mathcal{O}^*) = 0$, i.e. every holomorphic line bundle over X is trivial.*

Proof. In the exponential exact sequence $H^1(\mathcal{O}^*)$ is between $H^1(\mathcal{O}) = 0$ and $H^2(X, \mathbb{Z}) = 0$. □

Corollary 11.10 (Weierstrass). *Assume that X is not compact. Then every divisor is the divisor of a meromorphic function. Hence every Weierstrass distribution $(f, \mathcal{U})$ has a solution.*

Proof. By the sequence $H^0(\mathcal{O}^*) \to H^0(\mathcal{M}^*) \to H^0(\mathcal{D}) \to H^1(\mathcal{O}^*) = 0$ is exact so $H^0(\mathcal{M}^*) \to H^0(\mathcal{D})$ is onto and $\delta F(f, \mathcal{U}) = 0$. □

Corollary 11.11 (Mittag Lefler). *Assume that X is not compact. Then every principal part is the principal part of a meromorphic function. Hence every Mittag Lefler distribution has a solution.*

Proof. The sequence $H^0(\mathcal{O}) \to H^0(\mathcal{M}) \to H^0(\mathcal{P}) \to H^1(\mathcal{O}) = 0.$ is exact so $H^0(\mathcal{M}) \to H^0(\mathcal{P})$ is surjective and $\delta P(f, \mathcal{U}) = 0$. □

Theorem 11.12. $H^1(\mathcal{M}^*) = H^1(\mathcal{M}) = 0$.

Proof. In the noncompact case the group this follows immediately from the corresponding cohomology exact sequences and the results already proved: $H^1(\mathcal{M}^*)$ lies between $H^1(\mathcal{O}^*) = 0$ and $H^1(\mathcal{D}) = 0$ and $H^1(\mathcal{M})$ lies between $H^1(\mathcal{O}) = 0$ and $H^1(\mathcal{P}) = 0$. Assume the compact case. By Corollary 8.4 the first map in the sequence $H^0(\mathcal{M}^*) \to H^1(\mathcal{O}^*) \to H^1(\mathcal{M}^*) \to H^1(\mathcal{D}) = 0$ is surjective so second map is zero and hence $H^1(\mathcal{M}^*) = 0$. Next assume f_{ij} represents an element of $H^1(\mathcal{M})$. Refine the cover if necessary so that the number of poles of the f_{ij} is finite. Choose a divisor D of degree $> 2g-2$ such that f_{ij} is a cycle in $C^1(\mathcal{U}, \mathcal{O}(L_D))$. By Remark 7.4 $H^1(\mathcal{O}(L_D)) = 0$. Hence the cycle f_{ij} is cohomologous to zero in $C^*(\mathcal{U}, \mathcal{O}(L_D)))$ and thus certainly in $C^*(\mathcal{U}, \mathcal{M})$. □

Remark 11.13. A complex manifold (of any dimension) which is isomorphic to a closed submanifold of $\mathbb{C}^N$ for some N is called a **Stein manifold**. One can prove that a noncompact Riemann surface is a Stein manifold. The various cohomology vanishing theorems proved in this section are special cases of more general theorems for Stein manifolds. If X is a submanifold of projective space then X has a finite open cover $\mathcal{U} = \{U_i\}_{i \in I}$ where each U_σ for $\sigma \in \text{Nerve}(\mathcal{U})$ is Stein. Thus for the various sheaves $\mathcal{S} = \mathcal{O}, \Omega^p, \ldots$ the homologies $H^k(U_\sigma, \mathcal{S})$ vanish for $k > 1$, i.e. $\mathcal{U}$ is what is called a **Leray cover** for the sheaf $\mathcal{S}$. A theorem of Leray (see [4] page 44) implies that $H^k(\mathcal{S}) = H^k(\mathcal{U}, \mathcal{S})$, i.e. in computing the sheaf cohomology we don't need to take the limit over all covers.

12 Serre Duality and Divisors

12.1. Now suppose that L_D is the line bundle associated to a divisor D. Then Theorem 4.2 gives isomorphisms

$$\mathcal{O}_D := \mathcal{O}(X, L_D) = \mathcal{L}(D), \qquad \Omega_{-D} := \Omega(X, L_D^*) = \mathcal{L}(K - D)$$

where, as is traditional, we have used the same letter K for the canonical line bundle $K \to X$ and the divisor of a meromorphic section of K, i.e. a meromorphic one form. Serre duality takes the form

$$H^1(\mathcal{O}_D) = H^0(\Omega_{-D})^*.$$

Our aim is to find a formula for the pairing

$$H^1(\mathcal{O}_D) \times H^0(\Omega_{-D}) \to \mathbb{C}.$$

We will use the principal part exact sequence

$$0 \to \mathcal{O}(K) \to \mathcal{M}(K) \to \mathcal{P}(K) \to 0$$

for one forms.

12.2. Where $D = \sum_p n_p p$ and $U \subset X$ is open, an element of $\mathcal{O}_D(U)$ is a meromorphic function $f \in \mathcal{M}(U)$ such that for $p \in U$ with $n_p < 0$ the pole of f has order $\leq -n_p$ at p; similarly an element of $\Omega_{-D}(U)$ is a meromorphic one form $\omega \in \mathcal{M}(U, K)$ with zero of order $\geq -n_p$ at each p where $n_p < 0$. Thus for each open set U there is a bilinear map

$$\mathcal{O}_D(U) \times \Omega_{-D}(U) \to \Omega(U) : (f, \omega) \mapsto f\omega$$

defined by pointwise multiplication. In particular this induces a pairing

$$H^0(\mathcal{U}, \mathcal{O}_D) \times H^1(\mathcal{U}, \Omega_{-D}) \to H^1(\mathcal{U}, \Omega) : ((f_i)_i, (\omega_{ij})_{ij}) \mapsto (f_i\omega_{ij})_{ij}.$$

for each open cover $\mathcal{U}$ and hence a pairing

$$H^0(\mathcal{O}_D) \times H^1(\Omega_{-D}) \to H^1(\Omega).$$

12.3. The sum of the residues of a meromorphic one form in $H^0(\mathcal{M}(K))$ is zero but there is a map

$$\mathrm{Res} : H^1(\Omega) = H^1(\mathcal{O}(K)) \to \mathbb{C}$$

defined as follows. Choose $\mu \in H^1(\Omega)$. Since $H^1(\mathcal{M}(K)) = 0$ the boundary map $\delta : H^0(\mathcal{P}) \to H^1(\mathcal{O}(K)) = H^1(\Omega)$ is surjective. Therefore there is a Mittag Lefler distribution $(\nu, \mathcal{U})$ representing μ, i.e. $\nu_i \in \mathcal{M}(U_i, K)$ and the cocycle $\nu_i - \nu_j \in \mathcal{O}(U_i \cap U_j, K)$ represents μ. Since the $\nu_i - \nu_j$ is holomorphic on $U_i \cap U_j$ we have that $\mathrm{res}_p(\mu) := \mathrm{res}_p(\nu_i)$ is independent of the choice of i with $p \in U_i$ used to define it. Define

$$\mathrm{Res}(\mu) = \sum_{p \in X} \mathrm{res}_p(\mu).$$

Theorem 12.4. *The composition*

$$H^0(\mathcal{O}_D) \times H^1(\Omega_{-D}) \to H^1(\Omega) \xrightarrow{\text{Res}} \mathbb{C}$$

is the Serre Duality pairing.

Proof. ... □

Theorem 12.5. *Let $f \in C^0(\mathcal{U}, \mathcal{M})$ be a Mittag Lefler distribution of meromorphic functions on a compact Riemann surface X. Then f has a solution if and only if*

$$\text{Res}(f\omega) = 0 \qquad \textit{for all } \omega \in \Omega(X).$$

Proof. ...See [1] page 147, □

Theorem 12.6 (Abel). *Let $D \in \text{Div}_0(X)$ be a divisor of degree zero on a compact Riemann surface X. Then $D = (f)$ for some $f \in \mathcal{M}(X)$ if and only if there is a singular one chain c on X with $\partial c = D$ and*

$$\int_c \omega = 0 \qquad \textit{for all } \omega \in \Omega(X).$$

Proof. ...See [1] page 163. □

13 Abel Jacobi

13.1. Let X be a compact Riemann surface, $\mathcal{M}^*(X)$ be the multiplicative group of nonzero meromorphic functions on X, $\text{Div}(X)$ be the group of divisors on X, $\text{Div}_0(X)$ be the subgroup of divisors of degree zero, $\text{Pic}(X)$ be the Picard group of holomorphic line bundles on X, $\text{Pic}_0(X)$ be the subgroup line bundles of degree zero, $\mathcal{M}(X, K)$ be the space of meromorphic differentials in X, $\Omega(X) = \mathcal{O}(X, K)$ be the space of holomorphic differentials on X, and $\Omega(X)^*$ be its dual as a complex vector space. Each $\gamma \in H_1(X, \mathbb{Z})$ determines a functional $I_\gamma \in \Omega(X)^*$ via the formula

$$I_\gamma(\omega) = \int_\gamma \omega.$$

Define $\Lambda \subset \Omega(X)^*$ by

$$\Lambda = \{I_\gamma : \gamma \in H_1(X, \mathbb{Z})\}.$$

It is easy to see that the map $\gamma \to I_\gamma$ is a homomorphism of groups so that Λ is a subgroup of the additive group of the vector space $\Omega(X)^*$. The **Jacobian** of X is the quotient group

$$\mathrm{Jac}(X) := \Omega(X)^*/\Lambda.$$

Lemma 13.2. *Fix $o \in X$ and let $D = \sum_k n_k p_k \in \mathrm{Div}(X)$. Then the class $u(D) \in \mathrm{Jac}(X)$ of the linear functional defined by*

$$\Omega(X) \to \mathbb{C} : \omega \mapsto \sum_k n_k \int_o^{p_k} \omega$$

is independent of the choice of the arcs from o to p_k used to define the integrals. If $D \in \mathrm{Div}_0(X)$ then $u(D)(\omega)$ is also independent of the choice of $o \in X$. The map $u : \mathrm{Div}_0(X) \to \mathrm{Jac}(X)$ thus defined is called **Abel Jacobi map** *of X.*

Theorem 13.3 (Abel-Jacobi). *The sequence*

$$\mathcal{M}^*(X) \to \mathrm{Div}_0(X) \xrightarrow{u} \mathrm{Jac}(X) \to 0$$

is exact. Here the map $\mathcal{M}^*(X) \to \mathrm{Div}_0(X)$ assigns to each meromorphic function f its divisor (f). Moreover, Λ is a lattice so $\mathrm{Jac}(X)$ is a (compact) torus.

Remark 13.4. The cohomology exact sequence of the divisor exact sequence restricts to

$$\mathcal{M}^*(X) \to \mathrm{Div}_0(X) \to \mathrm{Pic}_0(X) \to 0$$

so as a corollary we have that that the Abel Jacobi map induces an isomorphism

$$\mathrm{Pic}_0(X) = \mathrm{Jac}(X)$$

also denoted by u. The assertion that the map u is injective is *Abel's Theorem*; the assertion that it is surjective is the *Jacobi Inversion Theorem.*

13.5. Let $P \subset \mathbb{C}$ be a fundamental polygon for X, i.e. the sides of P are given by

$$\partial P = \alpha_1 \cup \beta_1 \cup \alpha_1' \cup \beta_1' \cup \cdots \cup \beta_g'$$

(in the indicated order) where g is the genus of X and there is a holomorphic map from a neighborhood of P onto X which is injective in the interior of P and identifies α_i' and α_i^{-1} and also identifies β_i' and β_i^{-1}. Define

$$\alpha_{g+i} = \beta_i, \qquad \beta_{g+1} = \alpha_i^{-1}.$$

Fix a point o in the interior of P.

Lemma 13.6. *Let ω be a meromorphic differential in X having no pole on ∂P, ϕ be a holomorphic differential on P, and $f : P \to \mathbb{C}$ be holomorphic function defined by*

$$f(p) = \int_o^p \phi.$$

Then

$$2\pi i \sum_{p\in X} \mathrm{res}_p(f\omega) = -\sum_{i=1}^{2g} w_i b_i, \quad \textit{where} \quad w_i = \int_{\alpha_i} \omega, \quad b_i = \int_{\beta_i} \phi.$$

Proof. By the Residue Theorem

$$2\pi i \sum_{p\in X} \mathrm{res}_p(f\omega) = \int_{\partial P} f\omega.$$

If $p \in \alpha_i$ is identified with $p' \in \alpha_i^{-1}$ then $f(p') - f(p) = b_i$ so $\int_{\alpha_i \cup \alpha_i'} f\omega = b_i w_i$. □

13.7. Now choose a basis $\phi_1, \ldots, \phi_g$ for $\Omega(X)$ and use vector notation

$$\Phi = (\phi_1, \ldots, \phi_g), \qquad F(p) = \int_o^p \Phi \in \mathbb{C}^g,$$

and for $i = 1, 2, \ldots, 2g$ define $A_i, B_i \in \mathbb{C}^g$ by

$$A_i = \int_{\alpha_i} \Phi, \qquad B_i = \int_{\beta_i} \Phi.$$

For $x = (x_1, \ldots, x_{2g} \in \mathbb{C}^{2g}$ define $A(x), B(x) \in \mathbb{C}^g$ by

$$A(x) = \sum_{i=1}^{2g} x_i A_i, \qquad B(x) = \sum_{i=1}^{2g} x_i B_i,$$

and for $y \in \mathbb{C}^g$ define $A^\#(y), B^\#(y) \in \mathbb{C}^{2g}$ by

$$A^\#(y) = (y \cdot A_1, \ldots, y \cdot A_{2g}), \qquad B^\#(y) = (y \cdot B_1, \ldots, y \cdot B_{2g})$$

where $(y_1, \ldots, y_g) \cdot (z_1, \ldots, z_g) = \sum_i y_i z_i$. Thus

$$A^\#(y)(x) = y \cdot A(x), \qquad B^\#(y)(x) = y \cdot B(x)$$

for $x \in \mathbb{C}^{2g}$, $y \in \mathbb{C}^g$.

Lemma 13.8. *The sequence*

$$0 \to \mathbb{C}^g \xrightarrow{A^\#} \mathbb{C}^{2g} \xrightarrow{B} \mathbb{C}^g \to 0$$

is exact (and similarly if A and B are reversed).

Proof. To see that $B : \mathbb{C}^{2g} \to \mathbb{C}$ is surjective we show that $B^\#$ is injective. Choose $x = (x_1, \ldots, x_g) \in \mathbb{C}^g$ with $B^\#(x) = 0$; we will show that $x = 0$. Let $\omega = \sum_k x_k \phi_k$. Then ω is holomorphic and $\int_{\beta_i} \omega = 0$ for all i. Hence $p \mapsto \int_o^p \omega$ is single valued and thus constant, so $\omega = 0$ so (as the ϕ_i are independent) $x = 0$.

To see that $B \circ A^\# = 0$ assume $x = A^\#(y)$. Let $\omega = \sum_I y_i \phi_i$. The ω is holomorphic so, by the Lemma, $\sum_i w_i B_i = 0$ where $w_i = \int_{\alpha_i} \omega = A^\#(y)_i = x_i$, i.e. $B(w) = B(x) = 0$.

Exactness at $\mathbb{C}^{2g}$ follows since the proof that B is surjective showed that $B^\#$ injective and the same argument shows that $A^\#$ is injective. □

13.9. Now we prove Abel's Theorem, i.e. that u is injective. Suppose that $D \in \mathrm{Div}_0(X)$ and that $u(D) = 0$; we must find $f \in \mathcal{M}^*(X)$ with $D = (f)$. Let $D = \sum_{p \in X} n_p p$. As $\sum_p n_p = 0$ there is a meromorphic differential ω with $\mathrm{res}_p(\omega) = n_p$. Suppose first that $\int_{\alpha_j} \omega \in 2\pi i \mathbb{Z}$. Then $f(q) = \int_o^q \omega$ is well defined, meromorphic, and satisfies $(f) = D$. It remains to reduce to the case where $w_j := \int_{\alpha_j} \omega \in 2\pi i \mathbb{Z}$. By the Lemma

$$\frac{-1}{2\pi i} \sum_j w_j B_j = \sum_p \mathrm{res}_p(F\omega) = \sum_p n_p F(p) = u(D) \in \Lambda$$

and thus it has the form $\sum_j m_j B_j$ where $m_j \in \mathbb{Z}$. Hence $\sum_j (w_j + 2\pi i m_j) B_j = 0$ so $w_j + 2\pi i m_j = A^\#(y)_j$ for some $y \in \mathbb{C}^g$, Now $\psi = \omega - \sum_j y_j \phi_j$ has the same poles and residues as ω and $\int_{\alpha_j} \psi \in 2\pi i \mathbb{Z}$.

13.10. Now we prove the Jacobi Inversion Theorem, i.e. that u is surjective. It is enough to show that the image of u contains an open set since u is a homomorphism of groups. For this choose $q_1, \ldots, q_g \in X$ such that $\omega(q_1) = \cdots \omega(q_j) = 0$, $\omega \in \Omega(X) \implies \omega = 0$. Then the derivative of u at $D = \sum_j q_j$ is injective and hence by the Inverse Function Theorem there are neighborhoods V_i of $q)i$ such that u maps $V_1 \times V_g$ onto an open set.

Lemma 13.11. *Fix $o \in X$ and define $\iota : X^g \to \mathrm{Pic}_0(X)$ by*

$$\iota(p_1, \ldots, p_g) = \sum_{j=1}^{g} (p_j - o).$$

Then ι is surjective.

Proof. Choose $D \in \mathrm{Div}_0(X)$. Riemann Roch gives

$$\ell(-\iota(p_1, \ldots, p_g) + D) \geq \ell(go + D) \geq \deg(go + D) + 1 - g = 1$$

so there is an f with $(f) \geq \iota(p_1, \ldots, p_g) - D$. Since both sides have degree zero we must have equality, i.e. $\iota(p_1, \ldots, p_g) = D$ in $\mathrm{Pic}_0(X)$. □

13.12. Finally we show that $A_1, \ldots A_{2g}$ are independent over $\mathbb{R}$, i.e. that Λ is a lattice. If this fails, Λ lies in a hyperplane in $\mathbb{R}^{2g-1}$ and $\mathbb{C}^q/\Lambda$ is not compact. Both ι and u are surjective and hence also the composition $u \circ \iota$. But X^g is compact so it follows that that the image $\mathbb{C}^g/\Lambda$ of $u \circ \iota$ is compact as required.

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