# Homology Theory

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Feb 6, 2005

## 1 Abelian Groups

**1.1.** Let $\{G_\alpha\}_{\alpha\in\Lambda}$ be an indexed family of Abelian groups. The **direct product** $\prod_{\alpha\in\Lambda} G_\alpha$ of this family is the set of all functions $g$ defined on the index set $\Lambda$ such that $g(\alpha) \in G_\alpha$ for $\alpha \in \Lambda$; the direct sum is the subgroup

$$\bigoplus_{\alpha\in\Lambda} G_\alpha \subset \prod_{\alpha\in\Lambda} G_\alpha$$

of those $g$ of finite support, i.e. $g(\alpha) = 0$ for all but finitely many $\alpha$. A **free Abelian group** is a group which is isomorphic to $G = \bigoplus_{\alpha\in\Lambda} \mathbb{Z}$ for some index set $\Lambda$. The elements $e_\alpha \in \bigoplus_{\alpha\in\Lambda} G_\alpha$ defined by $e_\alpha(\alpha) = 1$ and $\alpha_\alpha(\beta) = 0$ for $\beta \neq \alpha$ have the property that every element $g \in G$ is uniquely expressible as a finite sum

$$g = \sum_\alpha n_\alpha e_\alpha$$

where the coefficients $n_\alpha$ are integers; such a system of elements of a free Abelian group is call a **free basis**. It is easy to see that the cardinality a free basis is independent of the choice of the basis; it is called the **rank** of the free group. An Abelian group $G$ is said to be **finitely generated** iff there is a surjective homomorphism $h : \mathbb{Z}^n \to G$.

**1.2.** A **graded Abelian group** is an Abelian group $C$ equipped with a with a direct sum decomposition

$$C = \bigoplus_{n\in\mathbb{Z}} C_n.$$

Unless otherwise specified we assume the grading is **nonnegative** meaning that $C_n = 0$ for $n < 0$. A **subgroup** $A$ of $C$ is called **graded** iff

$$A = \bigoplus_{n\in\mathbb{Z}} A_n \text{ where } A_n := A \cap C_n.$$

It is easy to see that the quotient is then also graded, i.e.

$$C/A = \bigoplus_{n\in\mathbb{Z}} (C_n/A_n).$$

The usage of the direct sum is a notational convenience only; it is better to think of a graded Abelian group as a sequence $\{C_n\}_n$ of Abelian groups.

**1.3.** homomorphism $h : A \to B$ between graded Abelian groups is said to **shift the grading** by $r$ iff $h(A_n) \subset B_{n+r}$ for all $n$; $h$ is said to **preserve the grading** iff it shifts the grading by 0. The kernel, image, and hence also cokernel of $h$ is graded. The graded Abelian groups and grade shifting homomorphisms form a category as do the graded Abelian groups and grade preserving homomorphisms,

**1.4.** Suppose that $\alpha : A \to B$ and $\beta : B \to C$ are homomorphisms of Abelian groups. We say that the sequence

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

is **exact at** $B$ iff the kernel of $\beta$ is the image of $\alpha$, i.e. $\alpha(A) = \beta^{-1}(0)$. A sequence

$$\cdots \longrightarrow C_{n+1} \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots$$

is called **exact** iff it is exact at each $C_n$. When the sequence terminates (at either end), no condition is placed on the group at the end. To impose a condition an extra zero is added. Thus a **short exact sequence** is an exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0,$$

i.e. it is exact at $A$, $B$, and $C$. For a short exact sequence the map $\alpha$ is injective, the map $\beta$ is surjective, and $C$ is isomorphic to the quotient $B/\alpha(A)$. Often $A$ is a subgroup of $B$ and $\alpha$ is the inclusion so $C \approx B/A$. More generally, an exact sequence

$$0 \longrightarrow K \longrightarrow A \xrightarrow{\alpha} B,$$

gives an isomorphism from $K$ to the **kernel** $\alpha^{-1}(0)$ of $\alpha$, and an exact sequence

$$A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0,$$

gives an isomorphism from $C$ to the **cokernel** $B/\alpha(A)$ of $\alpha$.

**Exercise 1.5.** Show that for a short exact sequence as in 1.4 the following are equivalent:

**(1)** There is an isomorphism $\phi : A \oplus C \to B$ with $\phi|A = \alpha$ and $\beta \circ \phi|C = \mathrm{id}_C$.

**(2)** There is a homomorphism $\lambda : B \to A$ with $\lambda \circ \alpha = \mathrm{id}_A$.

**(3)** There is a homomorphism $\rho : C \to B$ with $\beta \circ \rho = \mathrm{id}_C$.

When these three equivalent conditions hold we say that the exact sequence **splits** or that the subgroup $\alpha(A) \subset B$ **splits** in $B$.

**Exercise 1.6.** Show that a short exact sequence as in 1.4 always splits when $C$ is free but give an example of a short exact sequence which doesn't split even though $A$ and $B$ are free.

**Exercise 1.7.** The concepts of 1.4 remain meaningful for non Abelian groups although it is customary to use multiplicative notation ($1$, $ab$, $A \times B$) rather than additive notation ($0$, $a+b$, $A \oplus B$). Show that the implications (1) $\iff$ (2) and (1) $\implies$ (3) remain true in the non Abelian case but give an example where (3) $\implies$ (1) fails.

**Lemma 1.8 (Smith Normal Form).** *For any integer matrix $A \in \mathbb{Z}^{m\times n}$ there are square integer matrices $P \in \mathbb{Z}^{m\times m}$ and $Q \in \mathbb{Z}^{n\times n}$ of determinant $\pm 1$ (so $P^{-1}$ and $Q^{-1}$ are integer matrices by Cramer's rule) such that*

$$PAQ^{-1} = \begin{bmatrix} D & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}$$

*where $D$ is a diagonal integer matrix of form*

$$D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_r \end{bmatrix},$$

*$d_i > 0$, and $d_i$ divides $d_{i+1}$.*

*Proof.* Use row and column operations to transform to a matrix where the least common denominator of the entries is in the $(1,1)$ position and then use row and column operations to transform so that the other entries in the first row and the first column to vanish. Then use induction on the remaining $(m-1)\times(n-1)$ matrix. As in elementary linear algebra, the row operations give $P$ and the column operations give $Q$. See Theorem 11.3 Page 55 of [7] for more details. □

**Corollary 1.9.** *A subgroup of a free Abelian group is free Abelian of lower or equal rank.*

*Proof.* It follows from Smith Normal Form that the range of $A$ is a free subgroup. It is easy to see that any subgroup of $\mathbb{Z}^m$ is the range of $A$ for some integer matrix $A$. Hence Smith Normal Form implies that See Lemma 11.1 page 53 of [7] for a more direct argument. Lemma 11.2 page 54 of [7] shows that a subgroup of a free Abelian group is free Abelian even without the hypothesis that the ambient group is finitely generated. □

**1.10.** The **torsion subgroup** $T(G)$ of an Abelian group $G$ is the subgroup of elements of finite order. It is not hard to see that if $G$ is finitely generated the quotient $G/T(G)$ is free and hence splits by Exercise 1.6. Lemma 1.8 yields the following stronger

**Corollary 1.11 (Fundamental Theorem of Abelian Groups).** *Let A finitely generated Abelian group G has a direct sum decomposition*

$$G = F \oplus T(G), \qquad T(G) = Z/d_1 \oplus \cdots Z/d_r$$

*where $F$ is free, $d_i > 0$, and $d_i$ divides $d_{i+1}$.*

**Remark 1.12.** The rank of the free group $G/T(G)$ is also called the **rank** of $G$ itself. The tensor product $\mathbb{Q} \otimes G$ of $G$ with the rational numbers $\mathbb{Q}$ is a vector space over $\mathbb{Q}$. It is easy to see that the dimension of the vector space $\mathbb{Q} \otimes G$ is the rank of $G$.

**Exercise 1.13.** The torus $T^2$ may be viewed as the quotient $\mathbb{R}^2/\mathbb{Z}^2$ of the group $\mathbb{R}^2$ by the subgroup $\mathbb{Z}^2$. Consider the linear map $\mathbb{R}^2 \to \mathbb{R}^2$ and its inverse represented by the matrices

$$A = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix}.$$

As $A$ has integer entries it defines a map $f : T^2 \to T^2$ by

$$f(x + \mathbb{Z}^2) = Ax + \mathbb{Z}^2$$

for $x \in \mathbb{R}^2$. As $A^{-1}$ also has integer entries, this map is a homeomorphism. How many fixed points does it have? (A fixed point of $f$ is a point $p \in T^2$ such that $f(p) = p$.)

**Lemma 1.14 (Five Lemma).** *Consider a commutative diagram*

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r, "j"] \arrow[d, "\gamma"] & D \arrow[r, "k"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "\ell'"] & E'
\end{tikzcd}$$

*of Abelian groups and homomorphisms. Assume that the rows are exact and that $\alpha$, $\beta$, $\delta$, and $\varepsilon$ are isomorphisms. Then $\gamma$ is an isomorphism.*

# 2 Abstract Homology

**2.1.** A **chain complex** is a pair $(C, \partial)$ consisting of a

$$C = \bigoplus_{n \in \mathbb{Z}} C_n,$$

and a homomorphism $\partial : C \to C$ called the **boundary operator** such that $\partial(C_{n+1}) \subset C_n$ and $\partial^2 = 0$. The chain complex $(C, \partial)$ will be denoted simply by $C$ when no confusion can result. If there are several chain complexes in the discussion we write $\partial_C$ for $\partial$. A **chain map** from a chain complex $(A, \partial_A)$ to a chain complex $(B, \partial_B)$ is a group homomorphism $\phi : A \to B$ which preserves the grading and satisfies

$$\partial_B \circ \phi = \phi \circ \partial_A.$$

Chain complexes and chain maps form a category, i.e. the identity $\mathrm{id}_C : C \to C$ is a chain map and the composition of chain maps is a chain map.

**Remark 2.2.** Unless otherwise specified we assume that chain complexes are **nonnegative** meaning that $C_n = 0$ for $n < 0$.

**2.3.** Let $(C, \partial)$ be a chain complex, $Z(C) = \partial^{-1}(0)$ denote the kernel of $\partial$, and $B(C) = \partial(C)$ denote the image of $\partial$. As $\partial$ shifts the grading by $-1$ these are graded subgroups, i.e. $B(C) = \bigoplus_n B_n(C)$ and $Z(C) = \bigoplus_n Z_n(C)$ where

$$B_n(C) := \partial(C_{n+1}) = B(C) \cap C_n \qquad \text{and} \qquad Z_n(C) := Z(C) \cap C_n.$$

The condition $\partial^2 = 0$ is equivalent to the condition $B(C) \subset Z(C)$. The quotient $H(C) := Z(C)/B(C)$ is called the **homology group** of $C$. This is also graded, namely $H(C) = \bigoplus_n H_n(C)$ where

$$H_n(C) := Z_n(C)/B_n(C).$$

By Remark 2.2 $Z_0(C) = C_0$. An elements of $B(C)$ is called a **boundary**, an element of $Z(C)$ is called a **cycle**, and an element of $H(C)$ is called a **homology class**. A chain map $\phi : A \to B$ sends boundaries to boundaries and cycles to cycles and hence induces a graded homomorphism

$$\phi_* : H_n(A) \to H_n(B)$$

for each $n$. The operation which sends the chain complex $C$ to the graded group $H(C)$ and sends the chain map $\phi : A \to B$ to the graded homomorphism $\phi_* : H(A) \to H(B)$ is a **functor**, i.e. $\big(\mathrm{id}_C\big)_* = \mathrm{id}_{H(C)}$ and $(\psi \circ \phi)_* = \psi_* \circ \phi_*$.

**Remark 2.4.** A chain complex

$$\cdots \xrightarrow{\partial} C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots$$

is exact at $C_n$ (where $n > 0$) if and only if $H_n(C) = 0$. A chain complex whose homology vanishes is sometimes called **acyclic**.

**2.5.** An **augmented chain complex** is a nonnegative chain complex $C$ which is equipped with an **augmentation**, i.e. a surjective homomorphism $\varepsilon : C_0 \to \mathbb{Z}$ such that $\varepsilon \circ \partial | C_1 = 0$. We use the augmentation to produce a modified nonnegative chain complex $\tilde{C}$ defined by

$$\tilde{C}_n = C_n \text{ for } n > 0 \text{ and } \tilde{C}_0 = \varepsilon^{-1}(0).$$

For an augmented chain complex, the **reduced homology group** $\tilde{H}(C)$ is the homology of the modified complex, i.e.

$$\tilde{H}_n(C) := \tilde{Z}_n(C)/\tilde{B}_n(C)$$

where $\tilde{Z}_n(C) := Z_n(C)$ for $n > 0$, $\tilde{Z}_0(C) := \varepsilon^{-1}(0)$, and $\tilde{B}_n(C) := B_n(C)$ for $n \geq 0$. Obviously $\tilde{H}_n(C) = H_n(C)$ for $n > 0$. A chain map $\phi : A \to B$ between two augmented chain complexes is said to be **augmentation preserving** iff $\varepsilon_B \circ (\phi|A_0) = \varepsilon_A$. Just as for chain maps, an augmentation preserving chain map induces a graded homomorphism

$$\phi_* : \tilde{H}(A) \to \tilde{H}(B)$$

on reduced homology.

**Remark 2.6.** An exact sequence $0 \to A \to B \to C \to 0$ of augmented chain complexes gives rise to an exact sequence $0 \to \tilde{A} \to \tilde{B} \to \tilde{C} \to 0$ of the modified chain complexes. An augmented chain complex can be viewed as chain complex with $C_{-1} = \mathbb{Z}$ and $\partial|C_0 = \varepsilon$. This construction gives a chain complex with the same homology as $\tilde{C}$ but is inconvenient because it does not preserves exact sequence of chain complexes in the aforementioned sense: a sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0$ is *never* exact.

**2.7.** Consider short exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0,$$

of chain complexes, i.e. the homomorphisms $i$ and $j$ are chain maps. It is not hard to show that there is a unique homomorphism $\partial_* : H(C) \to H(A)$ called the **boundary homomorphism** such that for $c \in Z(C)$ and $a \in Z(A)$ we have

$$\partial_*[c] = [a] \iff \exists b \in B \text{ such that } i(a) = \partial_B(b) \text{ and } j(b) = c.$$

where the square brackets signify the homology class of the cycle it surrounds.

**Theorem 2.8 (Long Exact Homology Sequence).** *The homology sequence*

$$\cdots \xrightarrow{\partial_*} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial_*} H_{n-1}(A) \xrightarrow{i_*} \cdots$$

*associated to the short exact sequence of chain complexes of 2.7. The sequence is also* **natural** *meaning that a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A \arrow[r, "i'"] & B \arrow[r, "j'"] & C \arrow[r] & 0
\end{tikzcd}$$

*of chain complexes and chain maps gives rise to a commutative diagram*

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_*"] & H_n(A) \arrow[r, "i_*"] \arrow[d, "\alpha_*"] & H_n(B) \arrow[r, "j_*"] \arrow[d, "\beta_*"] & H_n(C) \arrow[r, "\partial_*"] \arrow[d, "\gamma_*"] & H_{n-1}(A) \arrow[r, "i_*"] \arrow[d, "\alpha_*"] & \cdots \\
\cdots \arrow[r, "\partial_*"] & H_n(A') \arrow[r, "i'_*"] & H_n(B') \arrow[r, "j'_*"] & H_n(C') \arrow[r, "\partial_*"] & H_{n-1}(A') \arrow[r, "i'_*"] & \cdots
\end{tikzcd}$$

*of long exact sequences.*

**Theorem 2.9 (Standard Basis Theorem).** *Assume that $C$ is a chain complex such that each group $C_n$ is free and of finite rank. Then there is a direct sum decomposition*

$$C_k = U_k \oplus V_k \oplus W_k$$

*such that $\partial(U_k) \subset W_{k-1}$ and $\partial(V_k) = \partial(W_k) = 0$. Moreover, there are bases for $U_k$ and $W_{k-1}$ relative to which $\partial : U_k \to W_{k-1}$ is represented by a diagonal integer matrix*

$$D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_r \end{bmatrix}$$

*where $D$ is as in Lemma 1.8. Hence $Z_k = V_k \oplus W_k$ and*

$$H_{k-1}(C) \approx V_{k-1} \oplus \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_r$$

*where $\mathbb{Z}/d = 0$ if $d = 1$.*

*Proof.* Let $Z_k \subset C_k$ denote the cycles and $B_k \subset C_k$ denote the boundaries as usual. Let $W_k$ denote the **weak boundaries**, i.e. those elements $w \in C_k$ such that $mw \in B_k$ for some $m \in \mathbb{Z}$. Then

$$B_k \subset W_k \subset Z_k \subset C_k.$$

By Smith Normal Form (Lemma 1.8) there is a basis $e_1, \ldots, e_n$ for $C_k$, a basis $f_1, \ldots, f_m$ for $C_{k-1}$, and integers $d_i$ as in the theorem such that $\partial e_i = d_i f_i$ for $i = 1, \ldots, r$ and $\partial e_i = 0$ for $i = r+1, \ldots, n$. It follows that

1. $e_{r+1}, \ldots, e_n$ is a basis for $Z_k$.
2. $f_1, \ldots, f_r$ is a basis for $W_{k-1}$.
3. $d_1 f_1, \ldots, d_k f_r$ is a basis for $B_{k-1}$.

Then one shows $W_k$ splits in $V_k$ (i.e. that there is a subgroup $V_k$ of $Z_k$ with $Z_k = V_k \oplus W_k$) and takes $U_k$ to be spanned by $e_1, \ldots, e_r$. See Theorem 11.4 page 58 of [7] for more details. $\square$

**Remark 2.10.** Smith Normal Form is a special case of Theorem 2.9. The sequence

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^m \longrightarrow 0$$

is a chain complex.

**2.11.** Here is an **algorithm for computing homology** for a free finitely generated chain complex. Assume that $C_{k+1} = \mathbb{Z}^p$, $C_k = \mathbb{Z}^n$, and $C_{k-1} = \mathbb{Z}^m$ so that $\partial : C_{k+1} \to C_k$ is represented by a matrix $B \in \mathbb{Z}^{n \times p}$ and $\partial : C_k \to C_{k-1}$ is represented by a matrix $A \in \mathbb{Z}^{m \times n}$ where $AB = 0$. The homology group $H_k$

is the quotient of the kernel of $A$ by the image of $B$. Assume w.l.o.g. that $B$ is in Smith Normal Form

$$B = \begin{bmatrix} D & 0_{r\times(p-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)\times(p-r)} \end{bmatrix}, \qquad D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_r \end{bmatrix}.$$

Then as $D$ is invertible over $\mathbb{Q}$ and $AB = 0$ the matrix $A$ must have the form

$$A = \begin{bmatrix} 0_{m\times r} & A' \end{bmatrix}, \qquad A' \in \mathbb{Z}^{m\times(n-r)}.$$

The rank of $A$ equals the rank of $A'$ so the nullity of $A'$ is $\nu := n - r - \mathrm{rank}(A)$. The homology is

$$\frac{\mathrm{Ker}(A)}{\mathrm{Im}(B)} \approx \mathbb{Z}^\nu \oplus \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_r.$$

**2.12.** The algorithm in 2.11 can easily be modified to produce the direct sum decomposition $C_k = U_k \oplus V_k \oplus W_k$ of Theorem 2.9, i.e. given a sequence $A_1, A_2, \ldots, A_n$ of matrices with $A_k \in \mathbb{Z}^{n_k\times n_{k-1}}$ and $A_{k-1}A_k = 0$, the modified algorithm constructs matrices $P_k \in \mathbb{Z}^{n_k\times n_k}$ of determinant $\pm 1$ such that

$$P_{k-1}A_kP_k^{-1} = \begin{bmatrix} 0_{u_{k-1}\times u_k} & 0_{u_{k-1}\times v_k} & 0_{u_{k-1}\times w_k} \\ 0_{v_{k-1}\times u_k} & 0_{v_{k-1}\times v_k} & 0_{v_{k-1}\times w_k} \\ D & 0_{w_{k-1}\times v_k} & 0_{w_{k-1}\times w_k} \end{bmatrix}$$

where $u_k + v_k + w_k = n_k$, $u_k = w_{k-1}$, where $D$ is as in Lemma 1.8. Assume inductively that $P_n, \ldots, P_{k-1}$ have been constructed. Then, as in 2.11,

$$A_{k-1}P_{k-1}^{-1} = \begin{bmatrix} A' & 0_{n_{k-1}\times u_k} \end{bmatrix}.$$

Apply the Smith Normal form algorithm to $A_{k-1}P_{k-1}^{-1}$ avoiding column operations which modify the last $w_{k-1}$ columns. This will modify $P_{k-1}$ but will not change $P_{k-1}A_kP_k^{-1}$. Then apply row operations to move the nonzero block in the upper left hand corner to the lower left hand corner. This modifies $P_{k-2}$ but not $P_{k-1}$.

**Exercise 2.13.** Assume $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ satisfy $AB = 0$. Show that $\mathbb{R}^n = U \oplus V \oplus W$ where $U = \mathrm{Im}(A^*)$, $W = \mathrm{Im}(B)$, and $V = \ker(A^*A + BB^*)$.

**2.14.** Two chain maps $\phi, \psi : A \to B$ are called **chain homotopic** iff there is a **chain homotopy** between them, i.e. a homomorphism $P : A \to B$ such that $P(A_n) \subset B_{n+1}$ and

$$\psi - \phi = \partial_B \circ P + P \circ \partial_A.$$

Chain homotopy is an equivalence relation and compositions of chain homotopic maps are chain homotopic so the chain complexes and chain homotopy classes form a category. An isomorphism of this category is called a **chain homotopy equivalence**. A chain complex $C$ is called **chain contractible** iff the identity map of $C$ is chain homotopy equivalent to the zero map.

**Theorem 2.15.** *If two chain maps $\phi, \psi : A \to B$ are chain homotopic, they induce the same map on homology, i.e. $\phi_* = \psi_* : H(A) \to H(B)$.*

*Proof.* Assume that $\phi - \psi = \partial_B P + P\partial_A$. If $\partial_A x = 0$ then $\phi(x) = \psi(x) + \partial_B Px$ so $\phi_*[x] = \psi_*[x]$. $\square$

# 3 Singular Homology

**3.1.** Let $X$ be a topological space. A map $\sigma : \Delta^n \to X$ is called a **singular $n$-simplex** in $X$. The free Abelian group generated by the singular $n$-simplices is denoted by $C_n(X)$ and its elements are called **singular $n$-chains**. A singular $n$-chain $c$ is a finite formal sum of singular $n$-simplices, i.e.

$$c \in C_n(X) \iff c = \sum_{k=1}^{r} c_k \sigma_k$$

where each $\sigma_k$ is a singular $n$-simplex and $c_k \in \mathbb{Z}$. The **boundary operator** $\partial : C_n(X) \to C_{n-1}(Y)$ is defined by

$$\partial\sigma = \sum_{i=0}^{n} (-1)^k \sigma \circ \iota_k$$

where $\iota_k : \Delta^{n-1} \to \Delta^n$ is defined by

$$\iota_k(x_1, \ldots, x_n) = (x_1, \ldots, x_{k-1}, 0, x_k, \ldots, x_n).$$

It is easy to see that the sequence

$$\cdots \xrightarrow{\partial} C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots$$

is a chain complex, i.e. that $\partial^2 = 0$. We write

$$C(X) := \bigoplus_{n \in \mathbb{Z}} C_n(X)$$

where $C_n(X) := 0$ for $n < 0$. The cycles of this complex are denoted by $Z(X)$ and the boundaries by $B(X)$. As usual $Z_n(X) := Z(X) \cap C_n(X)$ and $B_n(X) := B(X) \cap C_n(X)$. The quotient

$$H(X) := \bigoplus_n H_n(X), \qquad H_n(X) := Z_n(X)/B_n(X)$$

is called the **singular homology group** of the space $X$.

**3.2.** Let $A \subset X$ be a subspace of $X$. Then $C(A)$ is a subcomplex of $C(X)$. The quotient complex is denoted

$$C(X, A) := \bigoplus_n C_n(X, A), \qquad C_n(X, A) := C_n(X)/C_n(A).$$

Elements of the groups

$$Z_n(X,A) := \partial^{-1}C_n(A), \qquad B_n(X,A) := B_n(X) + C_n(A)$$

are called **relative singular cycles** and **relative singular boundaries** respectively. The homology

$$H(X,A) := \bigoplus_n H_n(Z,A), \qquad H_n(X,A) := Z_n(X,A)/B_n(X,A)$$

is called the **relative singular homology** of the pair $(X,A)$.

**3.3.** The **standard augmentation** $\varepsilon : C_0(X) \to \mathbb{Z}$ of the singular chain complex $C(X)$ is defined by

$$\varepsilon\left(\sum_i c_i p_i\right) = \sum_i c_i.$$

Here we identify the point $p \in X$ with the singular simplex $\Delta^0 \to X : 1 \mapsto p$. (Recall that $\Delta^0 = \{1\}$.) The corresponding reduced homology group is denoted $\tilde{H}(X)$ and is called the **reduced singular homology group** of $X$. Thus

$$\tilde{H}(X) = \bigoplus_n \tilde{H}_n(X)$$

where $\tilde{H}_n(X) = H_n(X)$ for $n > 0$ and

$$\tilde{H}_0(X) = \frac{\varepsilon^{-1}(0)}{B_0(X)} \subset H_0(X).$$

**3.4.** A map $f : X \to Y$ induces a homomorphism

$$f_\# : C_n(X) \to C_n(Y)$$

defined by $f_\#(\sigma) := f \circ \sigma$ for each singular $n$-simplex in $X$. This homomorphism is a chain map, i.e. $f_\# \circ \partial = \partial \circ f_\#$. This implies that $f_\#(B_n(X)) \subset B_n(Y)$ and $f_\#(Z_n(X)) \subset Z_n(Y)$. Hence $f$ induces a map

$$f_* : H(X) \to H(Y).$$

The chain map preserves the standard augmentation so $f_*(\tilde{H}_n(X)) \subset \tilde{H}_n(Y)$. Moreover, if $f : (X,A) \to (Y,B)$ then $f_\#(C_n(A)) \subset C_n(B)$ so $f$ induces a map $f_* : H_n(X,A) \to H_n(Y,B)$ on relative homology.

**Remark 3.5.** The three constructions $H(X)$, $H(X,A)$, and $\tilde{H}(X)$ determine one another as follows. The unique map $q : X \to \{*\}$ from the space $X$ to the one point space gives a set theoretic equality

$$\tilde{H}(X) = \operatorname{Ker}(q_*) \subset H(X).$$

For $p \in X$ the sequence $\{p\} \to X \to \{p\}$ gives a splitting

$$H(X) = \tilde{H}(X) \oplus H(\{p\}).$$

The composition $\tilde{C}(X) \to C(X) \to C(x, \{p\})$ induces an isomorphism

$$\tilde{H}(X) \approx H(X, \{p\}).$$

In Corollary 5.7 below we will prove that for "nice pairs" $(X, A)$ the projection $(X, A) \to (X/A, A/A)$ induces an isomorphism

$$H(X, A) \approx H(X/A, A/A) \approx \tilde{H}(X/A).$$

Note the obvious identification $X \to X/\emptyset$. The empty sum is zero so $C(\emptyset) = \{0\}$. The map $C(X) \to C(X, \emptyset) : c \mapsto c + \{0\} = \{c\}$ gives a corresponding isomorphism

$$H(X) = H(X, \emptyset).$$

**Theorem 3.6 (Dimension Axiom).** *If $X$ is a point $p$, then $H_n(X) = 0$ for $n > 0$ and $H_0(X) = \mathbb{Z}$.*

*Proof.* $C_n(p) = \mathbb{Z}$ for $n \geq 0$ and $\partial|C_n(p) = 0$ or the identity according as $n$ is odd or even. $\square$

**Proposition 3.7.** *If $X$ is pathwise connected, then $H_0(X) = \mathbb{Z}$.*

**Proposition 3.8.** *Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be the path components of $X$. Then $H_n(X) = \bigoplus_{\alpha\in\Lambda} H_n(X_\alpha)$.*

*Proof.* $Z_n(X) = \bigoplus_{\alpha\in\Lambda} Z_n(X_\alpha)$ and $B_n(X) = \bigoplus_{\alpha\in\Lambda} B_n(X_\alpha)$. $\square$

**Theorem 3.9 (Homotopy Axiom).** *If $f, g : X \to Y$ are homotopic, then the chain maps $f_\#$ and $g_\#$ are chain homotopic so $f_* = g_*$.*

*Proof.* Let $F : X \times \mathbb{I} \to Y$ satisfy $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. Define the **prism operator** $P : C_n(X) \to C_{n+1}(Y)$ by

$$P(\sigma)(x_0, \ldots, x_n) = \sum_{k=0}^{n+1} (-1)^k F(\sigma\big(x_0, \ldots, x_n), x_0 + \cdots + x_{k-1}\big)$$

for each singular $n$-simplex $\sigma$ in $X$. The prism operator is a chain homotopy. The geometric interpretation is that the prism $\Delta^n \times \mathbb{I}$ is written as a union of $(n+1)$-simplices

$$\Delta^n \times \mathbb{I} = \bigcup_{k=0}^{n} [v_0, \ldots, v_k, w_k, \ldots, w_n]$$

where $v_i = (u_i, 0)$, $w_i = (u_i, 1)$, $u_i$ are the vertices $\Delta^n$, and $[\ldots]$ denotes convex hull. The formula $\partial P = g_\# - f_\# + P\partial$ expresses the boundary of the prism as the top minus the bottom minus the prism on the faces of $\Delta^n$. The internal faces of the decomposition cancel. See [3] page 112. $\square$

## 4 Simplicial Homology

In this section we define two subcomplexes $\Delta(X)$ and $\Delta'(X)$ of the singular chain complex $C(X)$ of a $\Delta$-complex $X$ and state analogs of the basic theorems in homology theory. The analogs are easier to understand than the corresponding theorems on singular theory since the subcomplexes are finitely generated. In section 6 we present a more general theory which makes singular homology even easier to compute. The inclusions

$$\Delta(X) \to \Delta'(X) \to C(X)$$

induce isomorphisms in homology.

**4.1.** Let $\{\Phi_\alpha : D_\alpha \to X\}_{\alpha\in\Lambda}$ be a $\Delta$-complex. Each characteristic map $\Phi_\alpha : \Delta^n \to X$ may be viewed as a singular $n$-simplex: denote by $\Delta_n(X)$ the subgroup of the singular chain group $C_n(X)$ generated by the characteristic maps. By the definition of $\Delta$-complex there is a function $\beta = \beta(\alpha, k)$ which assigns to the index $\alpha$ the (unique) index $\beta$ such that $\Phi_\beta = \Phi_\alpha \circ \iota_k$ where $\iota_k : \Delta^{n-1} \to \Delta^n$ is inclusion into the $k$th face. Hence

$$\partial\Phi_\alpha = \sum_{k=0}^{n}(-1)^k\Phi_{\beta(\alpha,k)}$$

which shows that

$$\Delta(X) := \bigoplus_{n\in\mathbb{N}} \Delta_n(X)$$

is a subcomplex of the singular chain complex $C(X)$. The homology

$$H^\Delta(X) = \bigoplus_{n\in\mathbb{N}} H_n^\Delta(X)$$

of this subcomplex is called the **simplicial homology** of the $\Delta$-complex $X$. Thus

$$H_n^\Delta(X) := Z_n^\Delta(X)/B_n^\Delta(X)$$

where $Z_n^\Delta(X) := \Delta_n(X) \cap Z_n(X)$ and $B_n^\Delta(X) := \partial(\Delta_{n+1}(X))$. The inclusion $\Delta(X) \to C(X)$ is a chain map and induces a homomorphism $H^\Delta(X) \to H(X)$. Theorem 4.21 below asserts that this homomorphism is an isomorphism.

**Example 4.2.** The **torus** $T^2$ is the surface obtained from the unit square $\mathbb{I}^2$ with the identifications $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$. The **Klein bottle** $K^2$ is the surface obtained from the unit square $\mathbb{I}^2$ with the identifications $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, 1 - y)$. The **projective plane** $P^2$ is the surface obtained from the unit square $\mathbb{I}^2$ with the identifications $(x, 0) \sim (1 - x, 1)$ and $(0, y) \sim (1, 1 - y)$. For $X = T^2, K^2, P^2$ there is a $\Delta$-complex $\Phi$ as indicated in Figure 1. For $X = T^2$ and $X = K^2$ there are one 0-simplex $v$, three 1-simplices $a$, $b$, $c$, and two 2-simplices $U$ and $L$. For $X = P^2$ we must adjoin another 0-simplex $w$. In each case the bijections $\Phi_\alpha$ is determined by the diagram in

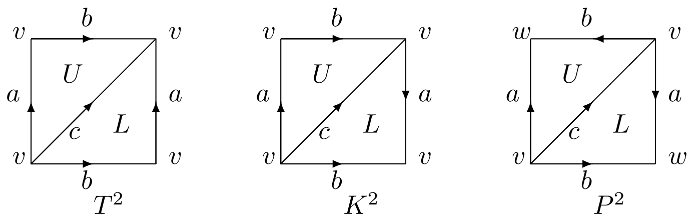


Figure 1: Three $\Delta$-complexes

the only way possible so that the corresponding maps to $\mathbb{I}^2$ are affine and the arrows on the edges match up. Note that had we reversed the arrows marked $a$ in $P^2$ the vertices of triangle $U$ would be cyclically (not linearly) ordered. The figure would still represent the projective plane but not a $\Delta$-complex.

**Proposition 4.3.** *The simplicial homology of the $\Delta$-complex for the torus $T^2$ is*

$$H_2^\Delta(T^2) = \mathbb{Z}, \qquad H_1^\Delta(T^2) = \mathbb{Z}^2, \qquad H_0^\Delta(T^2) = \mathbb{Z}$$

*with $H_n^\Delta(T^2) = 0$ for $n > 2$.*

*Proof.* The chain groups are $\Delta_2(T^2) = \mathbb{Z}^2$ with generators $U, L$, $\Delta_1(T^2) = \mathbb{Z}^3$ with generators $a, b, c$, and $\Delta_0(T^2) = \mathbb{Z}$ with generator $v$. The boundary operator is given by

$$\partial U = b - c + a, \qquad \partial L = a - c + b, \qquad \partial a = \partial b = \partial c = 0.$$

Using row and column operations we find the Smith Normal Form for the matrix representing $\partial : \Delta_2(T^2) \to \Delta_1(T^2)$ is

$$B := P \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} Q^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and the matrix representing $\partial : \Delta_1(T^2) \to \Delta_0(T^2)$ is of course

$$A := \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

In the notation of 2.11 $\nu := n - r - \mathrm{rank}(A) = 3 - 1 - 0 = 2$ so $H_1^\Delta(T^2) = \mathbb{Z}^\nu \oplus \mathbb{Z}/1 = \mathbb{Z}^2$. From $B_2 = 0$ and $\mathrm{rank}(B) = 1$ we get $H_2^\Delta(T^2) = Z_2^\Delta(T^2) = \mathbb{Z}$ and from $A = 0$ we get $H_0^\Delta(T_2) = \Delta_0(T^2) = \mathbb{Z}$. $\square$

**Proposition 4.4.** *The simplicial homology of the $\Delta$-complex for the Klein bottle $K^2$ is*

$$H_2^\Delta(K^2) = 0, \qquad H_1^\Delta(K^2) = \mathbb{Z} \oplus (\mathbb{Z}/2), \qquad H_0^\Delta(K^2) = \mathbb{Z}$$

*with $H_n^\Delta(P^2) = 0$ for $n > 2$.*

*Proof.* The chain groups are as for $T^2$ but the boundary operator is given by

$$\partial U = b - c + a, \qquad \partial L = a - b + c, \qquad \partial a = \partial b = \partial c = 0.$$

Using row and column operations we find the Smith Normal Form for the matrix representing $\partial : \Delta_2(K^2) \to \Delta_1(K^2)$ is

$$B := P \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} Q^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

and the matrix representing $\partial : \Delta_1(K^2) \to \Delta_0(K^2)$ is again $A = 0_{1\times 3}$. In the notation of 2.11 $\nu := n - r - \operatorname{rank}(A) = 3 - 2 - 0 = 1$ so $H_1^\Delta(K^2) = \mathbb{Z}^\nu \oplus (\mathbb{Z}/1) \oplus (\mathbb{Z}/2) = \mathbb{Z} \oplus (\mathbb{Z}/2)$. From $B_2 = 0$ and $\operatorname{rank}(B) = 2$ we get $H_2^\Delta(K^2) = Z_2^\Delta(K^2) = 0$ and $H_0^\Delta(K^2) = \mathbb{Z}$ as for $T^2$. □

**Proposition 4.5.** *The simplicial homology of the* $\Delta$*-complex for the projective plane* $P^2$ *is*

$$H_2^\Delta(P^2) = 0, \qquad H_1^\Delta(P^2) = \mathbb{Z}/2, \qquad H_0^\Delta(P^2) = \mathbb{Z}$$

*with* $H_n^\Delta(P^2) = 0$ *for* $n > 2$.

*Proof.* The chain groups are $\Delta_2(P^2) = \mathbb{Z}^2$ and $\Delta_1(P_2) = \mathbb{Z}^3$ as before but $\Delta_0(P^2) = \mathbb{Z}^2$ with generators $v$ and $w$. The boundary operator is given by

$$\partial U = b - a + c, \qquad \partial L = a - b + c, \qquad \partial a = \partial b = w - v, \quad \partial c = 0.$$

Using row and column operations we find the Smith Normal Form for the matrix representing $\partial : \Delta_2(T^2) \to \Delta_1(T^2)$ is

$$B := P \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} Q^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

and the matrix representing $\partial : \Delta_1(P^2) \to \Delta_0(P^2)$ is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}.$$

In the notation of 2.11 $\nu := n - r - \operatorname{rank}(A) = 3 - 2 - 1 = 0$ so $H_1^\Delta(P^2) \approx \mathbb{Z}^\nu \oplus (\mathbb{Z}/1) \oplus (\mathbb{Z}/2) = \mathbb{Z}/2$. As for $K^2$ we have $H_2^\Delta(P^2) = Z_2^\Delta(P^2) = 0$. The Smith normal form for $A$ is

$$M \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} N^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so $B_0^\Delta(P^2) \approx \mathbb{Z} \times 0 \subset \mathbb{Z}^2 = \Delta_0(P^2) =: Z_0^\Delta(P^2)$ and hence $H_0^\Delta(P^2) = \mathbb{Z}$. □

**4.6.** The standard $n$-simplex $\Delta^n$ is itself a simplicial complex with a characteristic map $\Phi_\alpha : \Delta^k \to \Delta^n$ for each subset $\alpha = \{\alpha_0, \alpha_1, \ldots, \alpha_k\} \subset \{0, 1, \ldots, n\}$, $(\alpha_0 < \alpha_1 < \cdots < \alpha_k)$, namely $\phi_\alpha(x) = y$ where $x_i = y_{\alpha_i}$ for $i = 0, 1, \ldots, k$ and $y_j = 0$ for $j \notin \alpha$. The space $\Delta^n$ is homeomorphic to the disk $\mathbb{D}^n$, its boundary

$$\Sigma^{n-1} := \{x \in \Delta^n : \text{ some } x_i \neq 0\}$$

is a subcomplex of $\Delta^n$ homeomorphic to the sphere $\mathbb{S}^{n-1}$.

**Proposition 4.7.** *The $\Delta$-homology of $\Delta^n$ is given by*

$$H_0^\Delta(\Delta^n) = \mathbb{Z}, \qquad H_k^\Delta(\Delta^n) = 0 \text{ for } k \neq 0.$$

*The $\Delta$-homology of $\Sigma^{n-1}$ is given by*

$$H_0^\Delta(\Sigma^{n-1}) = H_{n-1}^\Delta(\Sigma^{n-1}) = \mathbb{Z}, \qquad H_k^\Delta(\Delta^n) = 0 \text{ for } k \neq 0, n.$$

*The boundary $\partial \mathrm{id}_{\Delta^n}$ of the identity map of $\Delta^n$ (viewed as an element of $\Delta_{n-1}(\Sigma^{n-1})$) gives a generator of $H_{n-1}^\Delta(\Sigma^{n-1})$.*

*Proof.* Define a chain homotopy $K_k : \Delta_k(\Delta^n) \to \Delta_{k+1}(\Delta^n)$ by

$$K_k(\Phi_\alpha) = \begin{cases} \Phi_{\{0\}\cup\alpha} & \text{if } 0 \notin \alpha, \\ 0 & \text{if } 0 \in \alpha, \end{cases}$$

so $K_k\partial + \partial K_{k+1} = \mathrm{id}$ and $\partial K_0 = \mathrm{id} - \Phi_{\{0\}}$. Also $K : \Delta_k(\Sigma^{n-1}) \to \Delta_{k+1}(\Sigma^{n-1})$ for $k < n$ and the complexes $\Delta(\Delta^n)$ and $\Delta(\Sigma^{n-1})$ are the same except that $\Delta_n(\Delta^n) = \mathbb{Z}$ and $\Delta_n(\Sigma^{n-1}) = 0$. □

**4.8.** Let $\{\Phi_\alpha : D_\alpha \to X\}_{\alpha\in\Lambda}$ be a $\Delta$-complex and $\Delta'_k(X) \subset C_k(X)$ be the subgroup of the $n$th singular chain group of $X$ generated by all maps of form $\Phi_\alpha \circ g$ where $\Phi_\alpha : \Delta^n \to X$ is a characteristic map of the $\Delta$-complex $X$ and $g : \Delta^k \to \Delta_n$ is simplicial, i.e. it sends each vertex $v_i$ of $\Delta^k$ to a vertex $g(v_i)$ of $\Delta^n$ and preserves convex combinations. Then

$$\Delta'(X) := \bigoplus_{n\in\mathbb{N}} \Delta'_n(X)$$

is a subcomplex of the singular chain complex. We denote by

$$H^{\Delta'}(X) = \bigoplus_{n\in\mathbb{N}} H_n^{\Delta'}(X)$$

the homology of this subcomplex. Thus

$$H_n^{\Delta'}(X) := Z_n^{\Delta'}(X)/B_n^{\Delta'}(X)$$

where $Z_n^{\Delta'}(X) := \Delta'_n(X) \cap Z_n(X)$ and $B_n^{\Delta'}(X) := \partial(\Delta'_{n+1}(X))$. The chain map $f_\# : C(X) \to C(Y)$ induced by a $\Delta$-map $f : X \to Y$ satisfies

$$f_\#(\Delta'(X)) \subset \Delta'(Y).$$

Thus $H^{\Delta'}$ defines a functor from the category of $\Delta$-complexes to the category of chain groups. By contrast, usually $f_\#$ will not map $\Delta(X)$ to $\Delta(Y)$ an so the operation $H^\Delta$ is not obviously functorial. However $H^\Delta$ has the advantage that it is much easier to compute since the group $\Delta_n(X)$ has much lower rank than the group $\Delta'_n(X)$.

**4.9.** For a $\Delta$-complex the standard augmentation $\varepsilon : C_0(X) \to \mathbb{Z}$ of 3.3 restricts to augmentations of $\Delta(X)$ and $\Delta'(X)$. The corresponding reduced chain groups are denoted by $\tilde{\Delta}(X)$ and $\tilde{\Delta}'(X)$ so that $\tilde{\Delta}_n(X) = \Delta_n(X)$ and $\tilde{\Delta}'_n(X) = \Delta'_n(X)$ for $n > 0$ and

$$\tilde{\Delta}_0(X) = \tilde{\Delta}'_0(X) = \left\{ \sum_{x \in X_0} c_x x : \sum_{x \in X_0} c_x = 0 \right\}$$

where the sums are understood to be finite even if $X_0$ is infinite. The corresponding reduced homology groups are denoted $\tilde{H}^\Delta(X)$ and $\tilde{H}^{\Delta'}(X)$.

**4.10.** Let $(X, A)$ be a $\Delta$-pair and define the quotient chain complexes

$$C^\Delta(X, A) := C^\Delta(X)/C^\Delta(A), \qquad C^{\Delta'}(X, A) := C^{\Delta'}(X)/C^{\Delta'}(A).$$

The homology groups of these complexes are denoted respectively $H^\Delta(X, A)$ and $H^{\Delta'}(X, A)$.

**Theorem 4.11.** *Let $X$ be a $\Delta$-complex. Then the inclusion $\phi : \Delta(X) \to \Delta'(X)$ is a chain homotopy equivalence.*

*Proof.* Equip the vertices of the standard simplex $\Delta^n$ with the lexicographical ordering i.e. where $v_i$ denotes the vertex with 1 in the $i$th place and 0 elsewhere we have $v_i < v_j \iff i < j$. An injective simplicial map $g : \Delta^k \to \Delta^n$ determines a unique simplicial automorphism $\sigma$ of $\Delta^k$ such that $g \circ \sigma$ is an order preserving embedding from the vertices of $\Delta^k$ to the vertices of $\Delta^n$. Let $\operatorname{sgn}(\sigma)$ denote the sign of the corresponding permutation of the vertices. Define $\psi : \Delta'(X) \to \Delta(X)$ by

$$\psi(\Phi_\alpha \circ g) = \begin{cases} \operatorname{sgn}(\sigma)\Phi_\alpha(g \circ \sigma) & \text{if } g \text{ is injective} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\psi$ is a chain map, $\psi \circ \phi$ is chain homotopic to the identity of $\Delta(X)$, and $\phi \circ \psi$ is chain homotopic to the identity of $\Delta'(X)$. See [7] page 77. $\square$

**Exercise 4.12.** Let $p$ be a point viewed as a $\Delta$-complex in the only way possible. Show that $H^\Delta_n(p) = H^{\Delta'}_n(p) = 0$ for $n > 0$ and $H^\Delta_0(p) = H^{\Delta'}_0(p) = \mathbb{Z}$. Hence $\tilde{H}^\Delta(p) = \tilde{H}^{\Delta'}(p) = 0$.

**Exercise 4.13.** Two $\Delta$-maps $f, g : X \to Y$ are said to be **$\Delta$-homotopic** iff there is a $\Delta$-map $F : X \times \mathbb{I} \to Y$ such that $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$. Here $X \times \mathbb{I}$ is the product $\Delta$-complex described in Theorem 2.10 on page 111 of [3]. Show that If two $\Delta$-maps $f, g : X \to Y$ are $\Delta$-homotopic, then the induced maps $f_\#, g_\# : \Delta'(X) \to \Delta'(Y)$ are $\Delta$-homotopic.

**Exercise 4.14.** Let $(X, A)$ be a $\Delta$-pair, $i : A \to X$ denote the inclusion, and $p : X \to X/A$ denote the projection. Show that the sequence

$$0 \longrightarrow \tilde{\Delta}(A) \xrightarrow{i_\#} \tilde{\Delta}(X) \xrightarrow{p_\#} \tilde{\Delta}(X/A) \longrightarrow 0$$

is exact. This gives a long exact sequence

$$\cdots \xrightarrow{\partial} \tilde{H}_k^\Delta(A) \xrightarrow{i_\#} \tilde{H}_k^\Delta(X) \xrightarrow{p_\#} \tilde{H}_k^\Delta(X/A) \xrightarrow{\partial} \tilde{H}_{k-1}^\Delta(A) \xrightarrow{i_\#} \cdots$$

**Exercise 4.15.** Continue the notation of 4.14. The exact sequence

$$0 \longrightarrow \Delta(A) \xrightarrow{i_\#} \Delta(X) \xrightarrow{p_\#} \Delta(X)/\Delta(A) \longrightarrow 0$$

is exact and gives a long exact sequence

$$\cdots \xrightarrow{\partial} H_k^\Delta(A) \xrightarrow{i_\#} H_k^\Delta(X) \xrightarrow{p_\#} H_k^\Delta(X, A) \xrightarrow{\partial} H_{k-1}^\Delta(A) \xrightarrow{i_\#} \cdots$$

**Remark 4.16.** For a finite zero-dimensional $\Delta$-complex (i.e. a finite set) $X$ the exact sequence of 4.15 is

$$0 \to \mathbb{Z}^{\#(A)} \to \mathbb{Z}^{\#(X)} \to \mathbb{Z}^{\#(X)-\#(A)} \to 0$$

and the exact sequence of Theorem 4.14 is

$$0 \to \mathbb{Z}^{\#(A)-1} \to \mathbb{Z}^{\#(X)-1} \to \mathbb{Z}^{\#(X)-\#(A)} \to 0.$$

**Exercise 4.17.** Let $(X, A)$ be a $\Delta$-pair and $Z \subset A$ be an open subset of $X$ such that $A \setminus Z$ is a subcomplex of $A$. Then $X \setminus Z$ is a subcomplex of $X$ and the inclusion $(X \setminus Z, A \setminus Z) \subset (X, A)$ induces an isomorphism

$$H^\Delta(X \setminus Z, A \setminus Z) \approx H^\Delta(X, A).$$

This might be called the **Excision** Theorem for simplicial homology.

**Exercise 4.18.** In the situation of 4.17 the map

$$(X \setminus Z)/(A \setminus Z) \to X/A$$

induced by the inclusion is a $\Delta$-isomorphism and thus induces an isomorphism $\tilde{H}^\Delta\big((X \setminus Z)/(A \setminus Z)\big) \to \tilde{H}^\Delta(X/A)$. Show that this is the same as the isomorphism obtained by combining the isomorphisms of 4.17 and 4.15.

**Exercise 4.19.** Let $A$ and $B$ be subcomplexes of a $\Delta$-complex $X$. Then $A \cap B$ and $A \cup B$ are also subcomplexes. Show that there is an exact sequence

$$\cdots \to H_n^\Delta(A \cap B) \to H_n^\Delta(A) \oplus H_n^\Delta(B) \to H_n^\Delta(A \cup B) \to H_{n-1}^\Delta(A \cap B) \to \cdots$$

This might be called the **Mayer Vietoris** Theorem for simplicial homology. Hint: The sequence

$$0 \to \Delta(A) \cap \Delta(B) \to \Delta(A) \oplus \Delta(B) \to \Delta(A \cup B) \to 0$$

is exact. The map $\Delta(A) \cap \Delta(B) = \Delta(A \cap B) \to \Delta(A) \oplus \Delta(B)$ is the direct sum of the inclusions and the map $\Delta(A) \oplus \Delta(B) \to \Delta(A \cup B)$ is the difference of the inclusions of the summands.

**Remark 4.20.** For a finite zero-dimensional $\Delta$-complex (i.e. a finite set) $X$ the exact sequence of 4.19 is

$$0 \to \mathbb{Z}^{\#(A\cap B)} \to \mathbb{Z}^{\#(A)} \oplus \mathbb{Z}^{\#(A)} \to \mathbb{Z}^{\#(A\cup B)} \to 0$$

which shows that the Mayer Vietoris Theorem is a generalization of the **Inclusion-Exclusion Principle** of combinatorics.

**Theorem 4.21.** *Let $X$ be a $\Delta$-complex. Then the inclusion $\Delta(X) \to C(X)$ induces an isomorphism $H^\Delta(X) \to H(X)$ between simplicial homology and singular homology.*

*Proof.* This is proved for the $k$-skeleton $X^{(k)}$ by induction on $k$, the long exact sequences (in both homology theories) for the pair $(X^{(k)}, X^{(k-1)})$, and the Five Lemma 1.14. See [3] Page 128. $\square$

# 5 Subdivision

**5.1.** For a $\Delta$-pair $(X, A)$ the sequence

$$0 \to \Delta(A) \to \Delta(X) \to \tilde{\Delta}(X/A) \to 0$$

is exact and induces a long exact sequence in simplicial homology. The analogous sequence

$$0 \to C(A) \to C(X) \to \tilde{C}(X/A) \to 0$$

in singular theory is not exact. For example, if $X = \mathbb{I}^2$, $A = \{\frac{1}{2}\} \times \mathbb{I}$, and $\sigma \in C_1(X/A)$ is defined by $\sigma(t) = (0, t)$ for $t < \frac{1}{2}$ and $\sigma(t) = (1, t)$ for $t > \frac{1}{2}$ then $\sigma$ does not lift to $X$. This nonexactness is the main reason why the relative homology groups $H(X, A)$ are used in place of the reduced homology groups $\tilde{H}(X/A)$. Corollary 5.7 asserts that $H(X, A)$ and $\tilde{H}(X/A)$ are isomorphic. The proof requires the following

**Theorem 5.2.** *Let $X$ be a topological space and $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ be a collection of subsets of $X$ whose interiors cover $X$. Let $C^{\mathcal{U}}(X)$ be the subcomplex of the singular chain complex of $X$ generated by the singular simplices $\sigma : \Delta^n \to X$ such that $\sigma(\Delta^n) \subset U_\lambda$ for some $\lambda \in \Lambda$ and let $H^{\mathcal{U}}(X)$ denote the homology of this subcomplex. Then the inclusion $C^{\mathcal{U}}(X) \to C(X)$ induces an isomorphism $H^{\mathcal{U}}(X) \to H(X)$.*

**5.3.** Fix a convex subset $K$ of $\mathbb{R}^m$. Denote by $LC(K)$ the subcomplex of the singular chain complex $C(K)$ generated by all singular simplices $\sigma : \Delta^n \to \mathbb{R}^m$ of form

$$\sigma(t_0, t_1, \ldots, t_n) = \sum_{i=0}^{n} t_i w_i$$

where $w_0, w_1, \ldots, w_n \in K$. The singular simplex $\sigma$ is called the **linear simplex** with vertices $w_0, w_1, \ldots, w_n$. The elements of $LC(K)$ are called **linear singular chains**. Each point $b \in K$ determines a **cone operator** $b : C_n(K) \to C_{n+1}(K)$ denoted by the same symbol via the formula

$$b(\sigma)(s_{-1}, s_0, s_1, \ldots, s_n) = s_{-1}b + \sum_{i=0}^{n} s_i w_i.$$

For each linear $n$-simplex $\sigma$ the point

$$b_\sigma := \frac{1}{n+1} \sum_{i=0}^{n} w_i$$

is called the **barycenter** of $\sigma$. The map $S : LC_n(K) \to LC_n(K)$ defined inductively by

$$S([w]) = [w], \qquad S(\sigma) = b_\sigma(S\partial\sigma)$$

is called **linear barycentric subdivision**.

**Lemma 5.4.** *The map $S$ is a chain map and is chain homotopic to the identity, i.e. there are maps $T : LC_n(K) \to LC_{n+1}(K)$ such that*

$$\partial T + T\partial = \mathrm{id} - S.$$

*Proof.* See [3] page 121-2. □

**5.5.** We now specialize to the $K = \Delta^n$ the standard $n$-simplex. Denote the vertices of $\Delta^n$ of $v_0, v_1, \ldots, v_n$. For a set $I \subset \{0, 1, \ldots, n\}$ of indices let $\Delta^I$ denote the convex hull of $\{v_i : i \in I\}$. Each permutation $\alpha$ of $\{0, 1, \ldots, n\}$ determines a linear simplex $\sigma_\alpha \in LC_n(\Delta^n)$ whose $k$th vertex $w_k$ is the barycenter of $\Delta^{I_k}$ where $I_k = \{\alpha(0), \ldots, \alpha(k)\}$. Let $\mathrm{id}_n$ denote the identity map of $\Delta^n$ viewed as a linear simplex. Then

$$\mathrm{id}_n = \bigcup_\alpha \varepsilon_\alpha \sigma_\alpha$$

where $\varepsilon_\alpha = \pm 1$; the choice of these signs assures that the internal faces cancel. In particular, $\Delta^n = \bigcup_\alpha \sigma_\alpha(\Delta^n)$ and the interiors (in $\Delta^n$) of the simplices $\sigma_\alpha(\Delta^n)$ are pairwise disjoint. The chain homotopy $T$ is determined by the formula

$$T(\mathrm{id}_n) = p_\# \left( \tau + \sum_\alpha \varepsilon_\alpha \tau_\alpha \right)$$

where $p : \Delta^n \times \mathbb{I} \to \Delta^n$ is projection on the first factor and $\tau : \Delta^{n+1} \to \Delta^n \times \mathbb{I}$ are linear simplex whose vertices are $(b, 1)$ [here $b$ is the barycenter of $\Delta^n$] and $(v_i, 0)$ for $i = 0, \ldots, n$ and where $\tau_\alpha : \Delta^{n+1} \to \Delta^n \times \mathbb{I}$ is the linear simplex whose vertices are $(v_{\alpha(0)}, 0)$ and $(v_{\alpha(i)}, 1)$ for $i = 0, \ldots, n$. See [3] page 122.

*Proof of Theorem 5.2.* Let $X$ be any topological space. Define a chain map $S : C_n(X) \to C_n(X)$ and a chain homotopy $T : C_n(X) \to C_{n+1}(X)$ by

$$S(\sigma) = \sigma_\#(S(\mathrm{id}_n)), \qquad T(\sigma) = \sigma_\#(T(\mathrm{id}_n)).$$

The formula $\partial T + T\partial = \mathrm{id} - S$ holds so $S$ is chain homotopic to the identity id. By induction and the fact that compositions of chain homotopic maps are chain homotopic we have that the $m$th iterate $S^m$ of $S$ is chain homotopic to the identity, in fact

$$\partial D_m + D_m \partial = \mathrm{id} - S^m$$

where $D_m = \sum_{i<m} TS^i$. Etc. □

**Corollary 5.6 (Excision).** *Let $X$ be any topological space and $Z \subset A \subset X$ be such that the closure of $Z$ is a subset of the interior of $A$. Then the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces an isomorphism*

$$H(X \setminus Z, A \setminus Z) \approx H(X, A)$$

*of the relative homology groups.*

*Proof.* Let $\mathcal{U} = \{X \setminus Z, A\}$. The map $C(X \setminus Z)/C(A \setminus Z) \to C^{\mathcal{U}}(X)/C(A)$ is an isomorphism and the map $C^{\mathcal{U}}(X)/C(A) \to C(X)/C(A)$ induces an isomorphisms in homology by Theorem 5.2. Hence

$$H(X \setminus Z, A \setminus Z) \approx H^{\mathcal{U}}(X, A) \approx H(X, A)$$

where $H^{\mathcal{U}}(X, A)$ denotes the homology of $C^{\mathcal{U}}(X)/C(A)$. See page 124 of [3]. □

**Corollary 5.7.** *Assume that $(X, A)$ is a* **nice pair***, i.e. that $A$ has a neighborhood $V$ in $X$ which deformation retracts onto $A$. Then the projection*

$$(X, A) \to (X/A, A/A)$$

*induces an isomorphism is relative homology and hence*

$$H(X, A) \approx H(X/A, A/A) \approx \tilde{H}(X/A).$$

*Proof.* In the commutative diagram

$$\begin{tikzcd}
H(X, A) \arrow[r] \arrow[d] & H(X, V) \arrow[d] & H(X \setminus A, V \setminus A) \arrow[l] \arrow[d] \\
H(X/A, A/A) \arrow[r] & H(X/A, V/A) & H(X/A \setminus A/A, V/A \setminus A/A) \arrow[l]
\end{tikzcd}$$

the horizontal maps on the left are isomorphisms by homotopy, the horizontal maps on the right are isomorphisms by excision, the vertical map on the right is an isomorphism as it is induced by a homeomorphism, and hence the other two vertical maps are isomorphisms. See Proposition 2.22 page 124 of [3]. □

**Corollary 5.8 (Mayer Vietoris).** *Let $X$ be any topological space and $A, B \subset X$ be two subsets whose interiors cover $X$. Then there is an exact sequence*

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \xrightarrow{\partial_*} H_{n-1}(A \cap B) \to \cdots$$

*where the map $H(A \cap B) \to H(A) \oplus H(B)$ is the direct sum of the maps induced by the inclusions $A \cap B \to A$ and $A \cap B \to A$, the map $H(A) \oplus H(B) \to H(X)$ is the difference of the maps induced by the inclusions $A \to X$ and $B \to X$, and the boundary operator $\partial_*$ is defined in the proof.*

*Proof.* The collection $\mathcal{U} = \{A, B\}$ satisfies the hypothesis of Theorem 5.2, and

$$0 \to C(A \cap B) \to C(A) \oplus C(B) \to C^{\mathcal{U}}(X) \to 0$$

is an exact sequence of chain complexes. □

# 6 Cellular Homology

**Definition 6.1.** By Proposition 4.7 and Theorem 4.21 we have $H_n(\mathbb{S}^n) = \mathbb{Z}$. Hence each continuous map $f : \mathbb{S}^n \to \mathbb{S}^n$ determines an integer $\deg(f)$ called the **degree** of $f$ via the equation

$$f_*[\mathbb{S}^n] = \deg(f)[\mathbb{S}^n]$$

where $[\mathbb{S}^n]$ is a generator of $H_n(\mathbb{S}^n)$.

**6.2.** Now let $\Phi = \{\Phi_\alpha : D_\alpha \to X\}_{\alpha \in \Lambda}$ be a cell complex, $n_\alpha$ be the dimension of the disk $D_\alpha$, Let

$$C_n(\Phi) := \bigoplus_{n_\alpha = n} \mathbb{Z}\Phi_\alpha$$

be the free Abelian group generated by the $n$-dimensional cells of the complex. For each pair of indices $\alpha, \beta \in \Lambda$ with $n_\beta = n_\alpha - 1 = n - 1$ define

$$\phi_{\alpha\beta} : \partial D_\alpha = \mathbb{S}^{n-1} \to D_\beta / \partial D_\beta = \mathbb{D}^{n-1}/\mathbb{S}^{n-2} \approx \mathbb{S}^{n-1}$$

by $\phi_{\alpha\beta} = \Phi_\beta^{-1} \circ q_\beta \circ \phi_\alpha$ where $\phi_\alpha = \Phi_\alpha | \partial D_\alpha$ is the attaching map, $q_\beta : X^{(n-1)}/X^{(n-2)} \to \bar{e}_\beta / X^{(n-2)}$ is the identity on $e_\beta$ and sends the complement of $e_\beta$ to the wedge point, and $\Phi_\beta^{-1}$ is induced by the inverse of the characteristic map. Define $\partial : C_n(\Phi) \to C_{n-1}(\Phi)$ by

$$\partial \Phi_\alpha = \sum_\beta \deg(\phi_{\alpha\beta}) \Phi_\beta$$

**Theorem 6.3.** *The homomorphism $\partial : C(\Phi) \to C(\Phi)$ is a chain complex, i.e. $\partial^2 = 0$. The homology $H(\Phi)$ of this chain complex is isomorphic to the singular homology $H(X)$ of the space $X$.*

*Proof.* See [3] pages 137-141. An important point is that for each $n$ characteristic map $\Phi$ induces a homeomorphism

$$\bigvee_{n_\alpha=n} S_\alpha \approx X^{(n)}/X^{(n-1)}, \qquad S_\alpha := D_\alpha/\partial D_\alpha$$

from a wedge of $n$-spheres to the quotient of the $n$ skeleton by the $(n-1)$-skeleton. and that homeomorphism in turn induces an isomorphism

$$C_n(\Phi) \simeq H_n(X^{(n)}/X^{(n-1)})$$

of Abelian groups. $\square$

**Example 6.4.** The standard representation $P/\sim$ of the compact orientable surface $M_g$ of genus $g$ (connected sum of $g$ copies of the torus) is obtained from a $4g$-gon $P$ with boundary

$$\partial P = \alpha_1 \cup \alpha_2 \cup \beta_1 \cup \beta_2 \cup \cdots \cup \alpha_{2g-1} \cup \alpha_{2g} \cup \beta_{2g-1} \cup \beta_{2g},$$

where the sides are enumerated and oriented in the clockwise direction and where $\alpha_i$ and $\beta_i$ are identified by an orientation reversing homeomorphism. This gives a cell complex structure $\Phi$ whose cellular chain complex is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

so the homology is

$$H_0(M_g) = \mathbb{Z}, \qquad H_1(M_g) = \mathbb{Z}^{2g}, \qquad H_1(M_g) = \mathbb{Z}.$$

See Example 2.36 page 141 of [3].

**Example 6.5.** The standard representation $P/\sim$ of the compact nonorientable surface $N_k$ of genus $k$ (connected sum of $k$ copies of the projective plane) is obtained from a $2k$-gon $P$ with boundary

$$\partial P = \alpha_1 \cup \beta_1 \cup \cdots \cup \alpha_k \cup \beta_k,$$

where the sides are enumerated and oriented in the clockwise direction and where $\alpha_i$ and $\beta_i$ are identified by an orientation preserving homeomorphism. This gives a cell complex structure $\Phi$ whose cellular chain complex is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\partial} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

where $\partial(1) = (2, 2, \ldots, 2)$ so the homology is

$$H_0(N_g) = \mathbb{Z}, \qquad H_1(N_g) = \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2, \qquad H_2(N_g) = 0.$$

See Example 2.37 page 141 of [3].

**Example 6.6.** The 3-torus $T^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ may be viewed as a cube with opposite faces identified. This gives a cell complex structure $\Phi$ whose cellular chain complex is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

so the homology is

$$H_0(T^3) = \mathbb{Z}, \quad H_1(T^3) = \mathbb{Z}^3, \quad H_3(T^3) = \mathbb{Z}^3, \quad H_3(T^3) = \mathbb{Z}.$$

See Example 2.39 page 142 of [3].

**Example 6.7.** The **Moore Space** See Example 2.40 page 143 of [3].

**Example 6.8.** The **real projective space** $RP^n$ See Example 2.41 page 144 of [3].

**Example 6.9.** The **complex projective space** $CP^n$

**Example 6.10.** The **lens space** $L_m(p, q)$ See Example 2.42 page 144 of [3].

# References


[1] Charles Conley: *Isolated Inveariant Sets and the Morse Index*, CBMS Conferences in Math **38**, AMS 1976.

[2] William Fulton: *Algebraic Topology, A First Course* Springer GTM **153**, 1995.

[3] Allen Hatcher: *Algebraic Topology*, Cambridge U. Press, 2002.

[4] John G. Hocking & Gail S. Young: *Topology*, Addisow Wesley, 1961.

[5] Morris W. Hirsch: *Differential Topology*, Springer GTM **33**, 1976.

[6] William S. Massey: *A Basic Course in Algebraic Topology*, Springer GTM **127**, 1991.

[7] James R. Munkres: *Elements of Algebraic Topology*, Addison Wesley, 1984.

[8] S. Shelah: Can the fundamental group of a space be the rationals? Proc. Amer. Math. Soc. **103** (1988) 627–632.

[9] Stephen Smale: A Vietoris mapping theorem for homotopy. Proc. Amer. Math. Soc. 8 (1957), 604–610

[10] J. W. Vick: *Homology Theory* (2nd edition), Graduate Tests in Math **145**, Springer 1994.

[11] J.H.C. Whitehead: Combinatorial homotopy II, Bull. A.M.S **55** (1949) 453-496.

[12] George W. Whitehead: *Elements of Homotopy Theory* Graduate Texts in Math **61**, Springer 1978.