Sample Exam Questions

Math 542

Monday May 6 2001

Notation. Throughout $\mathbb{F}$ is either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers, V, U, W, are finite dimensional vector spaces over $\mathbb{F}$, $L(V, W)$ denotes the vector space of linear maps from V to W, $V^* = L(V, \mathbb{F})$, $L(V) = L(V, V)$, and

$$\mathrm{GL}(V) = \{P \in L(V) : P \text{ is invertible}\}.$$

(1) Complete the following definitions. Write complete sentences. Make sure you state any theorem which is required to make your definition valid (i.e. independent of any choices used in your definition).

(i) A *subspace* of a vector space V is a ...

(ii) The *range* $\mathcal{R}(T)$ of the linear map $T \in (V, W)$ is ...

(iii) The *null space* $\mathcal{N}(T)$ of the linear map $T \in (V, W)$ is ...

(iv) A sequence $v_1, v_2, \ldots, v_n$ of elements of a vector space V is *linearly independent* iff ...

(v) A sequence $v_1, v_2, \ldots, v_n$ of elements of a vector space V is *spans* V iff ...

(vi) A *basis* for a vector space V is ...

(vii) The *dimension* of a vector space V is ...

(viii) The *rank* of the linear map $T \in (V, W)$ is ...

(ix) The *nullity* of the linear map $T \in (V, W)$ is ...

(x) The *eigenspace* $\mathcal{E}_\lambda(T)$ of the linear map $T \in L(V)$ for $\lambda \in \mathbb{F}$ is ...

(xi) The *generalized eigenspace* $\mathcal{G}_\lambda(T)$ of the linear map $T \in L(V)$ for $\lambda \in \mathbb{F}$ is ...

(xii) A linear map $N \in L(V)$ is *nilpotent* iff ...

(xiii) A vector space V is the *direct sum* of subspaces U and W (notation $V = W \oplus U$) iff ...

(xiv) A subspace U of V is a *complement* to the subspace W of V iff ...

(xv) A subspace W of V is *invariant* under the linear map $T \in L(V)$ iff ...

(xvi) The *eigen ranks* of a matrix $A \in \mathbb{F}^{n\times n}$ are the numbers $\rho_{\lambda,k}(A)$ defined by ...

(xv) The orthogonal group O_n is defined by ...

(xvi) The special orthogonal group SO_n is defined by ...

(xvii) The Euclidean group E_n is defined by ...

(2) Proof or counterexample: If V is a finite dimensional vector space and $N \in L(V)$ satisfies

$$\forall v \in V \exists p \in \mathbb{Z}^+ \text{ such that } N^p v = 0,$$

then

$$\exists p \in \mathbb{Z}^+ \text{ such that } \forall v \in V \text{ we have } N^p v = 0.$$

What if the hypothesis that V is finite dimensional is dropped?

(3) A 12×12 matrix N satisfies $\mathrm{rank}(N) = 8$, $\mathrm{rank}(N^2) = 4$, $\mathrm{rank}(N^3) = 1$, and $N^4 = 0$. Find its Jordan Normal Form. To make your answer intelligible you should indicate the block structure: do not write 144 zeros and ones. (The question on the test may involve different numbers.)

(4) Let $A \in \mathbb{F}^{n\times n}$ and $\mathbb{F}[t]$ denote the ring of polynomials in the indeterminate t with coefficients from $\mathbb{F}$. Show that the set

$$I_A := \{f \in \mathbb{F}[t] : f(A) = 0\}$$

is an ideal in the ring $\mathbb{F}[t]$. The **minimal polynomial** $m_A(t)$ of a matrix $A \in \mathbb{F}^{n\times n}$ is the unique monic polynomial in $\mathbb{F}[t]$ which generates the ideal I_A, i.e. $I_A = (m_A)$ where

$$(f) = \{gf : g \in \mathbb{F}[t]\}.$$

The **characteristic polynomial** of A is the polynomial

$$c_A(t) = \det(tI - A).$$

(5) What is the characteristic polynomial of a nilpotent matrix? What is the minimal polynomial of a nilpotent matrix? What is the characteristic polynomial of a matrix in Jordan Normal Form? What is the minimal polynomial of a matrix in Jordan Normal Form? Conclude that the minimal polynomial of a matrix divides its characteristic polynomial.

(6) Give an example of a matrix whose minimal polynomial has lower degree than its characteristic polynomial.

(7) Proof or counter example: Two square matrices of the same size are similar if and only if they have the same eigenvalues each with the same algebraic multiplicity. (The algebraic multiplicity of an eigenvalue is the dimension of the corresponding generalized eigenspace.)

(8) Proof or counter example: Two square matrices of the same size are similar if and only if they have the same eigenvalues each with the same geometric multiplicity. (The geometric multiplicity of an eigenvalue is the dimension of the corresponding eigenspace.)

(9) Show that for a matrix $A \in \mathbb{R}^{n\times n}$ the following are equivalent:

(i) $A^{-1} = A^*$ (A^* is the transpose of A);

(ii) $(Av)\cdot(Aw) = v \cdot w$ for all $v, w \in \mathbb{R}^n$ ($v \cdot w = \sum_i v_i w_i$);

(iii) $|Ax - Ay| = |x - y|$ for all $x, y \in \mathbb{R}^n$ ($|v| = \sqrt{v \cdot v}$).

(10) Show that for any matrix $A \in \mathrm{SO}_3$ there is a matrix $Q \in \mathrm{SO}_3$ such that

$$QAQ^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$