Sample Exam Questions

Math 542

Monday April 16 2001

Notation. Throughout $\mathbb{F}$ is either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers, V, U, W, are finite dimensional vector spaces over $\mathbb{F}$, $L(V, W)$ denotes the vector space of linear maps from V to W, $V^* = L(V, \mathbb{F})$, $L(V) = L(V, V)$, and

$$\mathrm{GL}(V) = \{P \in L(V) : P \text{ is invertible}\}.$$

(1) Complete the following definitions. Write complete sentences. Make sure you state any theorem which is required to make your definition valid (i.e. independent of any choices used in your definition).

(i) A *subspace* of a vector space V is a ...

(ii) The *range* $\mathcal{R}(T)$ of the linear map $T \in (V, W)$ is ...

(iii) The *null space* $\mathcal{N}(T)$ of the linear map $T \in (V, W)$ is ...

(iv) A sequence $v_1, v_2, \ldots, v_n$ of elements of a vector space V is *linearly independent* iff ...

(v) A sequence $v_1, v_2, \ldots, v_n$ of elements of a vector space V is *spans* V iff ...

(vi) A *basis* for a vector space V is ...

(vii) The *dimension* of a vector space V is ...

(viii) The *rank* of the linear map $T \in (V, W)$ is ...

(ix) The *nullity* of the linear map $T \in (V, W)$ is ...

(x) The *eigenspace* $\mathcal{E}_\lambda(T)$ of the linear map $T \in (V)$ for $\lambda \in \mathbb{F}$ is ...

(xi) The *generalized eigenspace* $\mathcal{G}_\lambda(T)$ of the linear map $T \in L(V)$ for $\lambda \in \mathbb{F}$ is ...

(xii) A linear map $N \in L(V)$ is *nilpotent* iff ...

(xiii) A vector space V is the *direct sum* of subspaces U and W (notation $V = W \oplus U$) iff ...

(xiv) A subspace U of V is a *complement* to the subspace W of V iff ...

(xv) A subspace W of V is *invariant* under the linear map $T \in L(V)$ iff ...

(xvi) The *eigen ranks* of a matrix $A \in \mathbb{F}^{n\times n}$ are the numbers $\rho_{\lambda,k}(A)$ defined by ...

(2) State and prove the Rank Nullity Theorem. (You may use without proof the theorem that a linearly independent sequence extends to a basis.)

(3) Let

$$A = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 1 & 5 & 1 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ -1 & 2 \\ 0 & 0 \end{bmatrix}$$

so that $AB = I_2$. Show that $\mathcal{N}(BA) = \mathcal{N}(A)$. Find a 4×4 matrix C whose null space is a complement to the null space of BA

$$\mathbb{F}^{4\times 1} = \mathcal{N}(BA) \oplus \mathcal{N}(K).$$

(4) Let

$$A = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 1 & 5 & 1 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ -1 & 2 \\ 0 & 0 \end{bmatrix}$$

so that $AB = I_2$. Find a 4×2 matrix J whose range is a complement to the null space of BA

$$\mathbb{F}^{4\times 1} = \mathcal{N}(A) \oplus \mathcal{R}(J).$$

(5) Let $B \in \mathbb{F}^{n\times m}$ be a right inverse to $A \in F^{m\times n}$. i.e. $AB = I_m$. Find an $n \times m$ matrix J whose range is a complement to the null space of A

$$\mathbb{F}^{n\times 1} = \mathcal{N}(A) \oplus \mathcal{R}(J).$$

(6) Proof or counterexample: An invariant subspace has an invariant complement.

(7) Prove that the only eigenvalue of a nilpotent linear map is zero.

(8) Prove that the only eigenvalue of the restriction of T to $\mathcal{G}_\lambda(T)$ is λ.

(9) Suppose that $T \in L(V)$ and that $V = W \oplus U$ where W is invariant under the linear map T, say

$$T(w) = Aw, \qquad T(u) = Bu + Cu$$

for $w \in W$ and $u \in U$ where $A \in L(W)$, $B \in L(U, W)$, $C \in L(U)$. Show that if $\mathcal{N}(C) \neq \{0\}$ then $\mathcal{N}(T) \neq \{0\}$. (If you can't do this, prove that it under the hypothesis that A is invertible.)

(10) Suppose that $T \in \mathbb{F}^{n\times n}$ has form

$$T = \begin{bmatrix} A & B \\ 0_{p\times(n-p)} & C \end{bmatrix}$$

where $A \in \mathbb{F}^{p\times p}$, $B \in \mathbb{F}^{p\times(n-p)}$, $C \in \mathbb{F}^{(n-p)\times(n-p)}$. Show that if $\mathcal{N}(C) \neq \{0\}$ then $\mathcal{N}(T) \neq \{0\}$.

(11) Suppose that $T \in L(V)$ and that $V = W \oplus U$ where W is invariant under the linear map T, say

$$T(w) = Aw, \qquad T(u) = Bu + Cu$$

for $w \in W$ and $u \in U$ where $A \in L(W)$, $B \in L(U, W)$, $C \in L(U)$. Show that if λ is an eigenvalue of C, then λ is an eigenvalue of T. (If you can't do this, prove that it under the hypothesis that λ is not an eigenvalue of A.)

(12) Let $n = \dim(V)$ and $N \in L(V)$. Show that the following are equivalent.

(i) There exists an integer $p > 0$ such that $N^p(v) = 0$ for all $v \in V$;

(ii) For every $v \in V$ there exists an integer $p > 0$ with $N^p(v) = 0$;

(iii) $N^n v = 0$ for all $v \in V$.

(13) Let X be the vector space of all infinite sequences $x = (x_0, x_1, x_2, \ldots)$ with entries from $\mathbb{F}$; the vector space operations are defined by

$$(x_0, x_1, x_2, \ldots) + (y_0, y_1, y_2, \ldots) = (x_0 + y_0, x_1 + y_1, x_2 + y_2, \ldots)$$

$$a(x_0, x_1, x_2, \ldots) = (ax_0, ax_1, ax_2, \ldots).$$

Define $N : X \to X$ by

$$N(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots).$$

Show that the subset

$$X_0 = \{x \in X : \exists p \text{ such that } x_k = 0 \text{ for } k > p\}$$

is a subspace of X and that it is invariant under N. Show that

$$\forall x \in X_0 \exists p \in \mathbb{Z}^+ \text{ such that } N^p x = 0$$

but

$$\forall p \in \mathbb{Z}^+ \exists x \in X_0 \text{ such that } N^p x \neq 0.$$

(Here $\mathbb{Z}^+$ denotes the set of positive integers.)

(14) Proof or counterexample: If V is a finite dimensional vector space and $N \in L(V)$ satisfies

$$\forall v \in V \exists p \in \mathbb{Z}^+ \text{ such that } N^p v = 0,$$

then

$$\exists p \in \mathbb{Z}^+ \text{ such that } \forall v \in V \text{ we have } N^p v \neq 0.$$

What if the hypothesis that V is finite dimensional is dropped?

(15) A 12×12 matrix N satisfies $\operatorname{rank}(N) = 8$, $\operatorname{rank}(N^2) = 4$, $\operatorname{rank}(N^3) = 1$, and $N^4 = 0$. Find its Jordan Normal Form. To make your answer intelligible you should indicate the block structure: do not write 144 zeros and ones. (The question on the test may involve different numbers.)

(16) Proof or counter example: Two square matrices of the same size are similar if and only if they have the same eigenvalues each with the same multiplicity. (The multiplicity of an eigenvalue is the dimension of the corresponding generalized eigenspace.)