Math 541 Quiz V October 23, 2000

Theorem. Suppose that $\phi : G \to G'$ and $\psi : G \to G''$ are homomorphisms of groups. Assume that both are onto. Show that

(a) There is a homomorphism $f : G' \to G''$ such that $\psi = f \circ \phi$ if and only if $\mathrm{Ker}(\phi) \subset \mathrm{Ker}(\psi)$.

(b) The homomorphism f is unique (when it exists), i.e. if $f_1 : G' \to G''$ and $f_2 : G' \to G''$ are homomorphisms and $f_1 \circ \phi = f_2 \circ \phi$, then $f_1 = f_2$.

(c) Assume that $\psi = f \circ \phi$ as in (a). Then the homomorphism f is an isomorphism if and only if $\mathrm{Ker}(\phi) = \mathrm{Ker}(\psi)$

Answer: The theorem is closely related to the various homomorphism theorems in the text but is (strictly speaking) different: The groups G' and G'' are not necessarily quotient groups. We prove the theorem as a consequence of the following

Lemma. *Suppose that $\phi : S \to S'$ and $\psi : S \to S''$ are maps of sets. Assume that both are onto. Then*

(a') *There is a map $f : S' \to S''$ such that $\psi = f \circ \phi$ if and only if $\phi^{-1}(\phi(x)) \subset \psi^{-1}(\psi(x))$ for all $x \in S$.*

(b') *The map f is unique (when it exists), i.e. if $f_1 : S' \to S''$ and $f_2 : S' \to S''$ are maps and $f_1 \circ \phi = f_2 \circ \phi$, then $f_1 = f_2$.*

(c') *Assume that $\psi = f \circ \phi$ as in (a'). Then f is onto. Moreover f is one-one if and only if $\phi^{-1}(\phi(x)) = \psi^{-1}(\psi(x))$ for all $x \in S$.*

Proof of (a'$\Rightarrow$). Assume $\psi = f \circ \phi$. Choose $x \in S$. Choose $y \in \phi^{-1}(\phi(x))$. Then $\phi(y) = \phi(x)$ so $\psi(y) = f(\phi(y)) = f(\phi(x)) = \psi(x)$ so $y \in \psi^{-1}(\psi(x))$.

Proof of (b'). Assume $f_1 \circ \phi = f_2 \circ \phi$. Choose $x' \in S'$. As ϕ is onto there exists $x \in S$ with $x' = \phi(x)$. Then $f_1(y) = f_1(\phi(x)) = f_2(\phi(x)) = f_2(y)$. Hence $f_1 = f_2$.

Proof of (a'$\Leftarrow$). Assume $\phi^{-1}(\phi(x)) \subset \psi^{-1}(\psi(x))$ for all $x \in S$. By part (b') the only possible definition for f is $f(x') = \psi(x)$ where $x \in \phi^{-1}(x')$. There is such an x (as ϕ is onto so $\phi^{-1}(x') \neq \emptyset$) and $f(x')$ is independent of the choice of x (by the hypothesis that $\phi^{-1}(\phi(x)) \subset \psi^{-1}(\psi(x))$).

Proof of (c'-onto). We prove that f is onto. Choose $x'' \in S''$. As ψ is onto there exists $x \in S$ with $\psi(x) = x''$. Let $x' = \phi(x)$. Then $f(x') = f(\phi(x)) = \psi(x) = x''$.

Proof of (c'$\Leftarrow$). Assume that f is one-one. We show $\psi^{-1}(\psi(x)) \subset \phi^{-1}(\phi(x))$ for all $x \in S$ Choose $x \in S$. Choose $y \in \psi^{-1}(\psi(x))$. Then $f(\phi(y)) = \psi(y) = \psi(x) - f(\phi(x))$ so (as f is one-one) $\phi(y) = \phi(x)$. Hence $y \in \phi^{-1}(\phi(x)$.

Proof of (c'$\Rightarrow$). Assume that $\psi^{-1}(\psi(x)) \subset \phi^{-1}(\phi(x))$ for all $x \in S$. We show that f is one-one. Choose $x'_1, x'_2 \in S'$. Assume that $f(x'_1) = f(x'_2)$. As ϕ is onto there exist $x_1, x_2 \in S$ such that $x'_1 = \phi(x_1)$ and $x'_2 = \phi(x_2)$. Then $\psi(x_1) = f(\phi(x_1)) = f(x'_1) = f(x'_2) = f(\phi(x_2)) = \psi(x_2)$. Hence $x_2 \in \psi^{-1}(\psi(x_1))$. Hence $x_2 \in \psi^{-1}(\psi(x_1))$ by our hypothesis. Hence $x'_1 = \phi(x_1) = \phi(x_2) = x'_2$.

The theorem is almost the same as the lemma. In fact, since

$$\mathrm{Ker}(\phi) = \phi^{-1}(\phi(e)) \qquad \text{and} \qquad \mathrm{Ker}(\phi) = \phi^{-1}(\phi(e)),$$

the only difference is that the condition

$$\phi^{-1}(\phi(x)) \subset \psi^{-1}(\psi(x)) \qquad \forall x \tag{$*$}$$

is replaced by the apparently weaker condition

$$\phi^{-1}(\phi(e)) \subset \psi^{-1}(\psi(e)). \tag{$**$}$$

But for homomorphisms

$$\phi^{-1}(\phi(x)) = x\phi^{-1}(\phi(e)) \qquad \text{and} \qquad \psi^{-1}(\psi(x)) = x\psi^{-1}(\psi(e))$$

so conditions (*) and (**) are equivalent.

Corollary (First Homomorphism Theorem) *Suppose that $\phi : G \to G'$ is homomorphism, that ϕ is onto and let $K = \mathrm{Ker}(\phi)$. Then G' and G/K are isomorphic.*

Proof: Take $G'' = G/K$ and $\psi : G \to G/K$ the homomorphism defined by $\psi(x) = xK$ (the "projection to the quotient").

Corollary (Third Homomorphism Theorem) *If $K \triangleright N \triangleright G$ then the groups G/N and $(G/K)/(N/K)$ are isomorphic.*

Proof: Take $G' = G/N$ and $G'' = (G/K)/(N/K)$, the homomorphism $\phi : G \to G/N = G'$ is the projection and the homomorphism $\psi : G \to G''$ is the composition of the projection $G \to G/K$ with the projection $(G/K) \to (G/K)/(N/K)$.