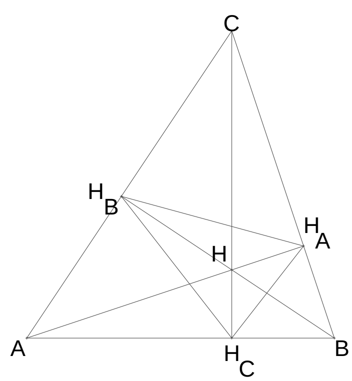
Orthocenter and Incenter

JWR

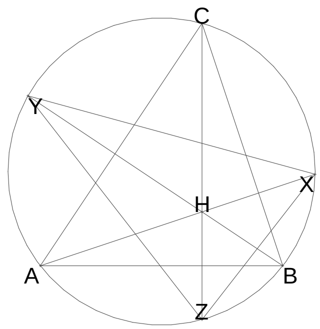
November 3, 2003

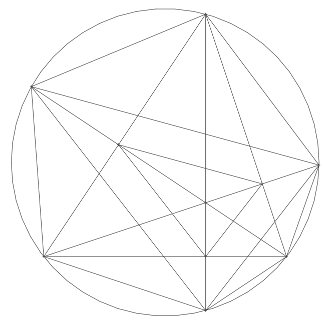


Let $\triangle ABC$ be a triangle and H_A, H_B, H_C be the feet of the altitudes from A, B, C respectively. The triangle $\triangle H_A H_B H_C$ is called the **orthic triangle** (some authors call it the *pedal triangle*) of $\triangle ABC$. We denote the **orthocenter** by H; it is the point of concurrence of the three altitudes. The **incenter** of a triangle is the center of its inscribed triangle. It is equidistant from the three sides and is the point of concurrence of the angle bisectors.

Theorem. *The orthocenter H of $\triangle ABC$ is the incenter of the orthic triangle $\triangle H_A H_B H_C$.*

Proof. Because $\angle AH_AC = 90°$, $\angle CAH = \angle CAH_A$, $\angle ACB = \angle ACH_A$, we have that $\angle CAH = 90° - \angle ACB$. Because the quadrilateral $H_B AH_C H$ has right angles at H_A and H_B it is cyclic, in fact H_B and H_C lie on the circle with diameter AH. Hence $\angle H_B AH = \angle H_B H_C H$ as they subtend the same are on this circle. But $\angle H_B AH = \angle CAH$ so $\angle H_B H_C H = 90° - \angle ACB$. The same argument (reading A for B) shows that also $\angle H_A H_C H = 90° - \angle ACB$. Hence $\angle H_B H_C H = \angle H_A H_C H$, i.e. the line HH_C bisects $\angle H_B H_C H_A$. By the same reasoning HH_A bisects $\angle H_B H_A H_C$ and HH_B bisects $\angle H_C H_B H_A$. $\square$





Theorem. *Let the altitudes AH, BH, CH meet the circumcircle of triangle $\triangle ABC$ respectively in X, Y, Z. Then H is the incenter of $\triangle XYZ$.*

Proof. As in the previous proof $\angle CAX$ (i.e. $\angle CAH_A$) is is the complement of $\angle ACB$. But $\angle CAX = \angle CZX$ as they subtend the same arc on the circumcircle. Hence $\angle CZX$ is the complement of $\angle ACB$. The same argument (reversing the roles of A and B) shows that $\angle CZY$ is the complement of $\angle ACB$. Hence $\angle CZX = \angle CZY$, i.e. CZ bisects $\angle XZY$. The same argument applies at the other vertices X and Y. By hypothesis the lines AX, BY, CZ are the altitudes of $\triangle ABC$ and we have just shown that they are they are the angle bisectors of $\triangle XYZ$. Hence their common point H is both the orthocenter of $\triangle ABC$ and the incenter of $\triangle XYZ$. □

Corollary. $H_CH = H_CZ$, $H_BH = H_BY$, $H_AH = H_AX$.

Proof. By what we have already proved and the principle that equal angles subtend equal arcs we have that $\angle XAB = \angle XYB = \angle ZYB = \angle ZAB$ so the right triangles $\triangle AH_CH$ and $\triangle AH_CZ$ are congruent. Hence $H_CH = H_CZ$. The other equalities follow in the same way. □

Remark. In other words X, Y, Z are the reflections of H in the sides BC, CA, AB respectively. Note also that the triangle $\triangle XYZ$ is similar to the orthic triangle $\triangle H_AH_BH_C$ as corresponding sides make equal angles with the altitudes (e.g. $\angle HH_CH_B = 90° - \angle ACB = \angle HZY$), so they are parallel.