Coordinate Geometry

JWR

Tuesday September 6, 2005

Contents

1 Introduction

These are notes to Math 461, a course in plane geometry I sometimes teach at the University of Wisconsin. Students who take this course have completed the calculus sequence and have thus seen a certain amount of analytic geometry. Many have taken (or take concurrently) the first course in linear algebra. To make the course accessible to those not familiar with linear algebra, there are three appendices explaining matrix notation, determinants, and the language of sets and transformations.

My object is to explain that classical plane geometry is really a subset of algebra, i.e. every theorem in plane geometry can be formulated as a theorem which says that the solutions of one system of polynomial equations satisfy another system of polynomial equations. The upside of this is that the criteria for the correctness of proofs become clearer and less reliant on pictures.

The downside is evident: algebra, especially complicated but elementary algebra, is not nearly so beautiful and compelling as geometry. Even the weakest students can appreciate geometric arguments and prove beautiful theorems on their own. For this reason the course also includes synthetic arguments as well. I have not reproduced these here but instead refer to the excellent texts of Isaacs [4] and Coxeter & Greitzer [3] as needed. It is my hope that the course as a whole conveys the fact that the foundations of geometry can be based on algebra, but that it is not always desirable to replace traditional (synthetic) forms of argument by algebraic arguments. The following quote of a quote which I got from page 31 of [3] should serve as a warning.

> The following anecdote was related by E.T. Bell [1] page 48. Young Princess Elisabeth had successfully attacked a problem in elementary geometry using coordinates. As Bell states it, "The problem is a fine specimen of the sort that are *not* adapted to the crude brute force of elementary Cartesian geometry." Her teacher René Descartes (who invented the coordinate method) said that "he would not undertake to carry out her solution … in a month."

The reduction of geometry to algebra requires the notion of a transformation group. The transformation group supplies two essential ingredients. First it is used to define the notion of equivalence in the geometry in question. For example, in Euclidean geometry, two triangles are congruent iff there is distance preserving transformation carrying one to the other and they are similar iff there is a similarity transformation carrying one to the other. Secondly, in each kind of geometry there are *normal form theorems* which can be used to simplify coordinate proofs. For example, in affine geometry every triangle is equivalent to the triangle whose vertices are $A_0 = (0, 0)$, $B_0 = (1, 0)$, $C_0 = (0, 1)$ (see Theorem 3.13) and in Euclidean geometry every triangle is congruent to the triangle whose vertices are of form $A = (a, 0)$, $B = (b, 0)$, $C = (0, c)$ (see Corollary 4.14).

This semester the official text is [3]. In past semesters I have used [4] and many of the exercises and some of the proofs in these notes have been taken from that source.

---

2 Some Fallacies

Pictures sometimes lead to erroneous reasoning, especially if they are not carefully drawn. The three examples in this chapter illustrate this. I got them from [6]. See if you can find the mistakes. Usually the mistake is a kind of sign error resulting from the fact that some point is drawn on the wrong side of some line.

2.1 Every Angle is a Right Angle!?

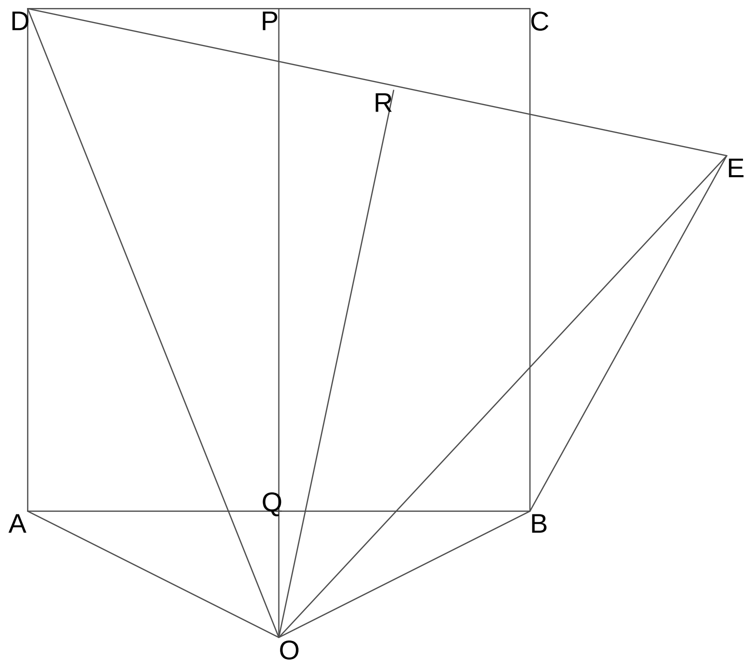


Figure 1: Every Angle is a Right Angle!?

Let $ABCD$ be a square and E be a point with $BC = BE$. We will show that $\angle ABE$ is a right angle. Take R to be the midpoint of DE, P to be the midpoint of DC, Q to be the midpoint of AB, and O to be the point where the lines PQ and the perpendicular bisector of DE intersect. (See Figure 2.1.) The triangles AQO and BQO are congruent since OQ is the perpendicular bisector of AB; it follows that $AO = BO$. The triangles DRO and ERO are congruent since RO is the perpendicular bisector of DE; it follows that $DO = EO$. Now $DA = BE$ as $ABCD$ is a square and E is a point with $BC = BE$. Hence the triangles OAD and OBE are congruent because the corresponding sides are equal. It follows that $\angle ABE = \angle OBE - \angle ABO = \angle OAD - \angle BAO = \angle BAD$.

2.2 Every Triangle is Isosceles!?

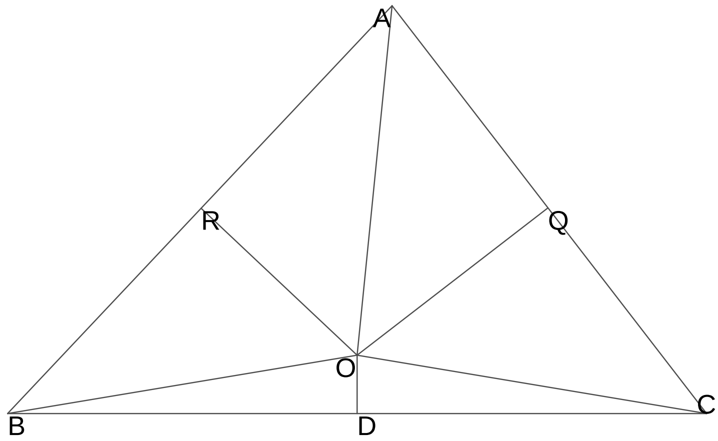


Figure 2: An Isosceles Triangle!?

Let ABC be a triangle; we will prove that $AB = AC$. Let O be the point where the perpendicular bisector of BC and the angle bisector at A intersect, D be the midpoint of BC, and R and Q be the feet of the perpendiculars from O to AB and AC respectively (see Figure 2.2.) The right triangles ODB and ODC are congruent since $OD = OD$ and $DB = DC$. Hence $OB = OC$. Also the right triangles AOR and AOQ are congruent since $\angle RAO = \angle QAO$ (AO is the angle bisector) and $\angle AOR = \angle AOQ$ (the angles of a triangle sum to 180 degrees) and AO is a common side. Hence $OR = OQ$. The right triangles BOR and COQ are congruent since we have proved $OB = OC$ and $OR = OQ$. Hence $RB = QC$. Now $AR = AQ$ (as AOR and AOQ are congruent) and $RB = QC$ (as BOR and COQ are congruent) so $AB = AR + RB = AQ + QC = AC$ as claimed.

2.3 Every Triangle is Isosceles!? -II

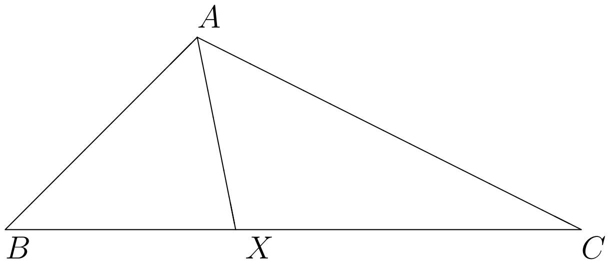


Figure 3: AX bisects $\angle BAC$

In a triangle ABC, let X be the point at which the angle bisector of the angle at A meets the segment BC. By Exercise 2.2 below we have

$$\frac{XB}{AB} = \frac{XC}{AC}. \tag{1}$$

Now $\angle AXB = \angle ACX + \angle CAX = \angle C + \frac{1}{2}\angle A$ since the angles of a triangle sum to 180 degrees. By the Law of Sines (Exercise 2.1 below) applied to triangle AXB we have

$$\frac{XB}{AB} = \frac{\sin \angle BAX}{\sin \angle AXB} = \frac{\sin \frac{1}{2}\angle A}{\sin(\angle C + \frac{1}{2}\angle A)} \tag{2}$$

Similarly $\angle AXC = \angle ABX + \angle BAX = \angle B + \frac{1}{2}\angle A$ so

$$\frac{XC}{AC} = \frac{\sin \frac{1}{2}\angle A}{\sin(\angle B + \frac{1}{2}\angle A)}. \tag{3}$$

From (1-3) we get $\sin(\angle C + \frac{1}{2}\angle A) = \sin(\angle B + \frac{1}{2}\angle A)$ so $\angle C + \frac{1}{2}\angle A = \angle B + \frac{1}{2}\angle A$ so $\angle C = \angle B$ so $AB = AC$ so ABC is isosceles.

Exercise 2.1. The law of sines asserts that for any triangle ABC we have

$$\frac{\sin \angle A}{BC} = \frac{\sin \angle B}{CA} = \frac{\sin \angle C}{AB}$$

Prove this by computing the area of ABC in three ways. Does the argument work for an obtuse triangle? What is the sign of the sine?

Exercise 2.2. Prove (1). Hint: Compute the ratio of the area of ABX to the area of ACX in two different ways.

3 Affine Geometry

3.1 Lines

3.1. Throughout $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^2$ denotes the set of pairs of real numbers. Thus a **point** of $P \in \mathbb{R}^2$ is an ordered pair $P = (x, y)$ of real numbers.

Definition 3.2. A **line** in $\mathbb{R}^2$ is a set of form

$$\ell = \{(x, y) \in \mathbb{R}^2 : ax + by + c = 0\}$$

where $a, b, c \in \mathbb{R}$ and either $a \neq 0$ or $b \neq 0$ (or both). Three or more points are called **collinear** iff there is a line ℓ which contains them all. Three or more lines are called **concurrent** iff they have a common point. Two lines are said to be **parallel** iff they do not intersect.

3.3. The two most fundamental axioms of plane geometry are

Axiom (1) Two (distinct nonparallel) lines intersect in a (unique) point.

Axiom (2) Two (distinct) points determine a line.

Axiom (1) says that two equations

$$a_1x + b_1y + c_1 = 0, \qquad a_2x + b_2y + c_2 = 0$$

for lines have a unique common solution (the usual case), no common solution (this means that the lines are parallel), or else define the same line (which is case if and only if the equations are nonzero multiples of one another). The latter two cases are characterized by the condition $a_1b_2 - a_2b_1 = 0$ and in the first case the intersection point is

$$x = -\frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \qquad y = -\frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

Axiom (2) says that for any two distinct points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ there is a unique line

$$\ell = \{(x, y) : ax + by + c = 0\}$$

containing both. Remark 3.5 below gives a formula for this line.

Theorem 3.4. **(I)** *Three points $P_i = (x_i, y_i)$ are collinear if and only if*

$$\det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0.$$

(II) *Three distinct lines $\ell_i = \{(x, y) : a_i x + b_i y + c_i = 0\}$ are concurrent or parallel if and only if*

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = 0.$$

Proof. A determinant is unchanged if one row is subtracted from another. Hence

$$\det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = \det \begin{bmatrix} x_1 - x_3 & y_1 - y_3 & 0 \\ x_2 - x_3 & y_2 - y_3 & 0 \\ x_3 & y_3 & 1 \end{bmatrix}.$$

Evaluating the determinant on the right gives

$$\det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = (x_1 - x_3)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_3).$$

Dividing by $(x_1 - x_3)(x_2 - x_3)$ shows that the determinant vanishes if and only if

$$\frac{y_1 - y_3}{x_1 - x_3} = \frac{y_2 - y_3}{x_2 - x_3}.$$

This last equation asserts that the slope of the line P_1P_3 equals the slope of the line P_2P_3. Since P_3 lies on both lines, this occurs if and only if the lines are the same, i.e. if and only if the points P_1, P_2, P_3 are collinear.

The above proof assumes that $x_1, x_2 \neq x_3$; a special argument is required in the contrary case. We give another proof which handles both cases at the same time. The matrix equation

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

says that the points P_i lie on the line $ax + by + c = 0$. Any nonzero solution (a, b, c) of this equation must have either $a \neq 0$ or $b \neq 0$ or both. Hence the three points P_i are collinear if and only if this matrix equation (viewed as a system of three homogeneous linear equations in three unknowns (a, b, c)) has a nonzero solution. Part (I) thus follows from the following

> **Key Fact.** *A homogeneous system of n linear equations in n unknowns has a nonzero solution if and only if the matrix of coefficients has determinant zero.*

Part (II) is similar, but there are several cases. The matrix equation

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

says that the point (x_0, y_0) lies on each of the three lines $a_i x + b_i y + c_i = 0$. The three lines are parallel (and not vertical) if and only if they have the same slope, i.e. if and only if $-a_1/b_1 = -a_2/b_2 = -a_3/b_3$. This happens if and only if the matrix equation

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\ m \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

has a solution m. The lines are vertical (and hence parallel) if and only if $b_1 = b_2 = b_3 = 0$. This happens if and only if the matrix equation

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3}$$

holds. This (and the above Key Fact) proves "only if". For "if" assume that the matrix equation

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has a nonzero solution (u, v, w). If $w \neq 0$, then $x_0 = u/w$, $y_0 = v/w$ satisfies (1). If $w = 0$ and $u \neq 0$, then $m = v/u$ satisfies (2). If $w = u = 0$, then $v \neq 0$ so (1) holds. □

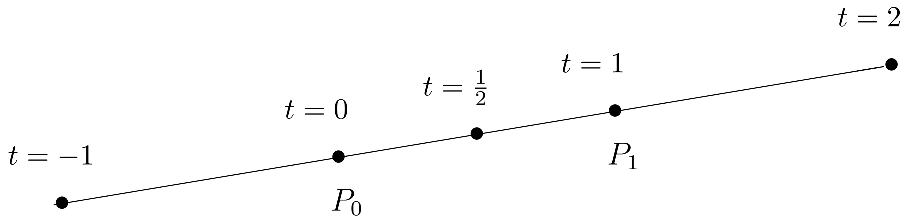


Figure 4: $P = tP_1 + (1-t)P_0$

Remark 3.5. The point $P = (x, y)$ lies on the line joining the distinct points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ if and only if the points P_1,, P_2, P are collinear. Thus Theorem 3.4 implies that an equation for this line is

$$\det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x & y & 1 \end{bmatrix} = 0.$$

It has form $ax + by + c = 0$ where

$$a = y_1 - y_2, \qquad b = x_2 - x_1, \qquad c = x_1y_2 - x_2y_1.$$

The points P_1 and P_2 satisfy this equation since a determinant vanishes if two of its rows are the same.

Theorem 3.6. *The line connecting the two distinct points $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$ is given by*

$$\ell = \{tP_1 + (1-t)P_0 : t \in \mathbb{R}\},$$

i.e. a point $P = (x, y)$ lies on ℓ if and only if

$$x = tx_1 + (1-t)x_0, \qquad y = ty_1 + (1-t)y_0$$

for some $t \in \mathbb{R}$. (See Figure 4.)

Proof. These are the parametric equations for the line as taught in Math 222. The formula

$$\det \begin{bmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ x & y & 1 \end{bmatrix} = t \det \begin{bmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ x_1 & y_1 & 1 \end{bmatrix} + (1-t) \det \begin{bmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ x_0 & y_0 & 1 \end{bmatrix} = 0.$$

shows that any point P of form $P = tP_1+(1-t)P_0$ lies on the line. Conversely in P lies on the line, choose t to satisfy one of the two parametric equations and then the equation for the line in the form $ax + by + c = 0$ shows that the other parametric equation holds as well. $\square$

Definition 3.7. The **line segment** connecting points P_0 and P_1 is the set

$$[P_0, P_1] = \{tP_1 + (1-t)P_0 : 0 \le t \le 1\}.$$

We say that a point P on the line joining P_0 and P_1 lies **between** P_0 and P_1 iff it lies in the segment $[P_0, P_1]$. We call P_0 and P_1 the **end points** of the line segment. The **ray** emanating from P_0 in the direction of P_1 is the set $\{tP_1 + (1-t)P_0 : t \ge 0\}$. We call P_0 the **initial point** of the ray.

Definition 3.8. *(Some general terminology.)* An ordered sequence

$$(P_1, P_2, \ldots, P_n)$$

of n distinct points no three of which are collinear is called a **polygon** or an n**-gon**. The points are called the **vertices** of the polygon. Two consecutive vertices in the list are said to be **adjacent** and also the vertices P_n and P_1 are adjacent. The lines and line segments joining adjacent vertices are called the **sides** of the polygon and the other lines and line segments joining vertices are called **diagonals**. The term **extended side** is employed if we want to emphasize that the line and not the line segment is intended. for emphasis. but the adjective is often omitted. A 3-gon is also called a **triangle**, a 4-gon is also called a **quadrangle**, a 5-gon is also called a **pentagon**, a 6-gon is also called a **hexagon**, etc. For triangles the two notations (A, B, C) and $\triangle ABC$ are synonymous; the latter is more common.

3.2 Affine Transformations

Definition 3.9. A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is called **affine** iff it has the form

$$(x', y') = T(x, y) \iff \begin{array}{l} x' = ax + by + p \\ y' = cx + dy + q \end{array}.$$

where $ad - bc \neq 0$.

3.10. It is convenient to use matrix notation to deal with affine transformations. To facilitate this we will we not distinguish between points and column vectors, i.e. we write both

$$P = (x, y) \qquad \text{and} \qquad P = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then, in matrix notation, an affine transformation takes the form

$$T(P) = MP + V,$$

where

$$T(P) = \begin{bmatrix} x' \\ y' \end{bmatrix}, \quad M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad P = \begin{bmatrix} x \\ y \end{bmatrix}, \quad V = \begin{bmatrix} p \\ q \end{bmatrix},$$

and $\det(M) \neq 0$.

Theorem 3.11. *The set of all affine transformations is a group, i.e.*

(1) *the identity transformation $I(P) = P$ is affine,*

(2) *the composition $T_1 \circ T_2$ of two affine transformations T_1 and T_2 is affine, and*

(3) *the inverse T^{-1} of an affine transformation T is affine.*

Proof. The identity transformation I has the requisite form with $a = d = 1$ and $b = c = p = q = 0$. Suppose T_1 and T_2 are affine, say $T_1(P) = M_1P + V_1$ and $T_2(P) = M_2P + V_2$. Then $(T_1 \circ T_2)(P) = T_1(T_2(P)) = M_1(M_2P + V_2) + V_1 = MP + V$ where $M = M_1M_2$ and $V = M_1V + V_2$. Since $\det(M_1M_2) = \det(M_1)\det(M_2) \neq 0$, this shows that $(T_1 \circ T_2)$ is affine. To compute the inverse transformation we solve the equation $P' = T(P)$ for P; we get $P' = MP + V \iff MP = P' - V \iff P = M^{-1}P' + M^{-1}V$. In other words,

$$T^{-1}(P') = M'P' + V'$$

where

$$M' = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \qquad V' = \frac{1}{ad - bc} \begin{bmatrix} dp - bq \\ -cp + aq \end{bmatrix}.$$

This shows that the inverse T^{-1} of the affine transformation T is itself an affine transformation. $\square$

Theorem 3.12. *An affine transformation maps lines onto lines, line segments onto line segments, and rays onto rays.*

Proof. Let ℓ be a line and T be an affine transformation; the theorem asserts that the image

$$T(\ell) = \{T(P) : P \in \ell\}$$

is again a line. Fix two distinct points $P_0, P_1 \in \ell$. Let M and V be the matrices which define T, i.e. $T(P) = MP + V$. Choose $P \in \ell$. Then $P = (1-t)P_0 + tP_1$ for some $t \in \mathbb{R}$. Hence

$$\begin{aligned} T(P) &= M\big((1-t)P_0 + tP_1\big) + V \\ &= (1-t)\big(MP_0 + V\big) + t\big(MP_1 + V\big) \\ &= (1-t)T(P_0) + tT(P_1) \end{aligned}$$

which shows that $T(P)$ lies on the line ℓ' connecting $T(P_0)$ and $T(P_1)$. The same argument (reading T^{-1} for T) shows that if $P' \in \ell'$ then $T^{-1}(P') \in \ell$. Hence $T(\ell) = \ell'$ as claimed. Reading $0 \le t \le 1$ for $t \in \mathbb{R}$ proves the theorem for line segments. Reading $t \ge 0$ for $t \in \mathbb{R}$ proves the theorem for rays. $\square$

Theorem 3.13. *For any two triangles $\triangle ABC$ and $\triangle A'B'C'$ there is a unique affine transformation T such that $T(\triangle ABC) = \triangle A'B'C'$, i.e. $T(A) = A'$, $T(B) = B'$, and $T(C) = C'$.*

Proof. Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$. Define T_0 by $T_0(x, y) = (x', y')$ where

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 - c_1 & b_1 - c_1 \\ a_2 - c_2 & b_2 - c_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Then $T_0(A_0) = A$, $T_0(B_0) = B$, $T_0(C_0) = C$ where $A_0 = (1, 0)$, $B_0 = (0, 1)$, $C_0 = (0, 0)$. As in the proof of Theorem 3.4 we have

$$\det \begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_1 & 1 \\ c_1 & c_2 & 1 \end{bmatrix} = \det \begin{bmatrix} a_1 - c_1 & b_1 - c_1 \\ a_2 - c_2 & b_2 - c_2 \end{bmatrix}$$

and this is nonzero since A, B, C are not collinear. Hence T_0 is an affine transformation. Similarly there is an affine transformation T_1 such that $T_1(A_0) = A$, $T_1(B_0) = B'$, $T_1(C_0) = C'$ By Theorem 3.11 $T := T_1 \circ T_0^{-1}$

is an affine transformation. It satisfies the conclusion of the theorem. For example, $T(A) = T_2(T_1^{-1}(A)) = T_2(A_0) = A'$.

To prove uniqueness let T' be another affine transformation such that $T'(A) = A'$, $T'(B) = B'$, and $T'(C) = C'$. Let $I = T_2^{-1} \circ T' \circ T_1$. By Theorem 3.11 I is affine so $I(P) = MP + V$ for some 2×2 matrix M and 2×1 matrix V. Also $I(C_0) = C_0$, $I(A_0) = A_0$, and $I(B_0) = B_0$. From $V = MC_0 + V = I(C_0) = C_0 = 0$ it follows that V is the zero matrix and from $MA_0 = A_0$ and $MB_0 = B_0$ it follows that M is the identity matrix. Hence I is the identity transformation. Thus

$$T = T_2 \circ T_1^{-1} = T_2 \circ I \circ T_1^{-1} = T_2 \circ T_2^{-1} \circ T' \circ T_1 \circ T_1^{-1} = T'$$

as required. □

Remark 3.14. Often it is possible to find a coordinate proof of a theorem which is both straight forward and uncomplicated by using affine transformations. For example, imagine a theorem involving five points A, B, C, D, and three lines a, b, c, and suppose the hypothesis includes the condition that A, B, C are the vertices of a triangle, i.e. they are not collinear. We can the prove the theorem by arguing as follows: Choose an affine transformation T with $T(A_0) = A$, $T(B_0) = B$, $T(C_0) = C$. Let $D_0 = T^{-1}(D)$, $a_0 = T^{-1}(a)$, $b_0 = T^{-1}(b)$, $c_0 = T^{-1}(c)$. Prove the theorem for A_0, B_0, C_0, D_0, a_0, b_0, c_0. Check that the hypotheses and conclusion are preserved by affine transformations. This will be true if the hypotheses and conclusion involve only assertions about lines (see Theorem 3.12, ratios of collinear distances (see Theorem 3.25), and ratios of areas (see Theorem 3.32). For example, if the lines a and b are parallel, then so are a_0 and b_0. We can now conclude that the theorem holds for A, B, C, D, a, b, c. We will often signal this kind of proof by saying *Choose affine coordinates* (x, y) *so that* .... The following theorem illustrates this technique.

Theorem 3.15 (Parallel Pappus' Theorem)**.** *Assume that the three points* A, B, C *are collinear and that the three points* $A', B,' C'$ *are collinear. Let the lines joining them in pairs intersect as follows:*

$$X = BC' \cap B'C, \qquad Y = CA' \cap C'A, \quad Z = AB' \cap A'B.$$

(See Figure 5.) If the lines ABC *and* $A'B'C'$ *are parallel, then the points* X,Y, Z *are collinear. (This theorem is a special case of Theorem 6.19 below.)*

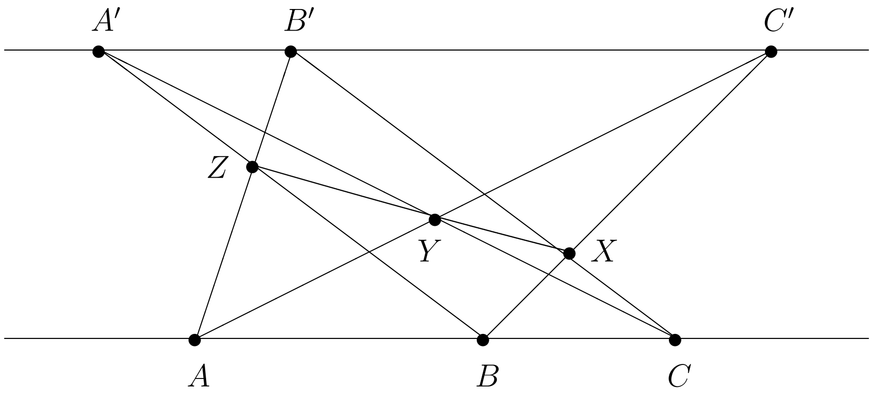


Figure 5: Parallel Pappus' Theorem

Proof. Choose affine coordinates (x, y) so that the line ABC has equation $y = 0$ and the line $A'B'C'$ has equation $y = 1$. Then $A = (a, 0)$, $B = (b, 0)$, $C = (c, 0)$, $A' = (a', 1)$, $B' = (b', 1)$, $C' = (c', 1)$. The equation of the line AB' is

$$0 = \begin{vmatrix} a & 0 & 1 \\ b' & 1 & 1 \\ x & y & 1 \end{vmatrix} = -x + (b' - a)y + a$$

and similarly the equation of the line $A'B$ is $x + (b - a')y - b = 0$. To find the intersection point $Z = (z_1, z_2)$ we solve these two equations. The result is

$$z_2 = \frac{a - b}{a - b + a' - b'}, \qquad z_1 = \frac{aa' - bb'}{a - b + a' - b'}.$$

Now calculate the coordinates of $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ by cyclically permuting the symbols and then use Theorem 3.4. (Note that the columns of the resulting matrix sum to zero.) $\square$

Exercise 3.16. Do the calculations required to complete the proof of Theorem 3.15.

Theorem 3.17 (Parallel Desargues Theorem)**.** *Let the corresponding sides of two triangles $\triangle ABC$ and $\triangle A'B'C'$ intersect in*

$$X = BC \cap B'C', \qquad Y = CA \cap C'A', \quad Z = AB \cap A'B'.$$

(See Figure 6.) If the lines AA', BB', CC' are parallel, then the points X, Y, Z are collinear. (This theorem is a special case of Theorem 6.16 below.)

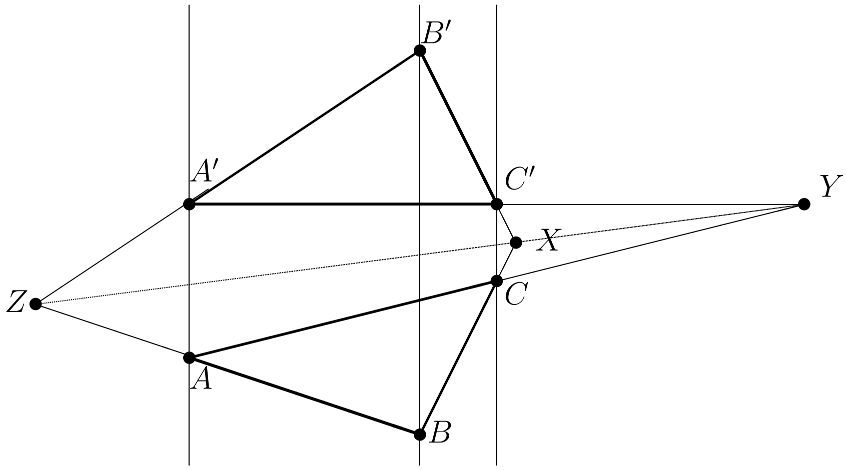


Figure 6: Parallel Desargues Theorem

Proof. Choose coordinates (x, y) so that the lines AA', BB', CC' are the vertical lines $x = a$, $x = b$, $x = c$. Then $A = (a, p)$, $A' = (a, p')$, $B = (b, q)$, $B' = (b, q')$, $C = (c, r)$, $C' = (c, r')$. The line AB has equation

$$\begin{vmatrix} a & p & 1 \\ b & q & 1 \\ x & y & 1 \end{vmatrix} = 0$$

i.e. $(p - q)x + (b - a)y + aq - bp = 0$. Similarly, the equation of line $A'B'$ is $(p' - q')x + (b - a)y + aq' - bp' = 0$. The two lines intersect in the solution of the matrix equation

$$\begin{bmatrix} q - p & a - b \\ q' - p' & a - b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} aq - bp \\ aq' - bp' \end{bmatrix}$$

so the intersection is $Z = (z_1, z_2)$ where

$$z_1 = \frac{bp - aq + aq' - bp'}{p - q - p' + q'}, \qquad z_2 = \frac{pq' - p'q}{p - q - p' + q'}$$

Similarly $X = (x_1, x_2)$ where

$$x_1 = \frac{cq - br + br' - cq'}{q - r - q' + r'}, \qquad x_2 = \frac{qr' - q'r}{q - r - q' + r'}$$

and $Y = (y_1, y_2)$ where

$$y_1 = \frac{ar - cp + cp' - ar'}{r - p - r' + p'}, \qquad y_2 = \frac{rp' - r'p}{r - p - r' + p'}$$

At this point we could complete the proof using Theorem 3.4 (see Remark 3.18 below), but here is a trick which finishes the proof more easily. The proof uses the following two assertions:

(I) An affine transformation of form

$$T(x, y) = (x_0 + x, y_0 + wx + y)$$

transforms each vertical line $x = k$ to another vertical line.

(II) Given any line $\ell = \{(x, y) : y = mx + y_0\}$ there is an affine transformation T as in part (I) such that $T(\ell)$ is the x-axis.

Using these two facts we may suppose w.l.o.g. that the line XY is the x-axes, i.e. that $x_2 = y_2 = 0$. From the above formulas it follows that $qr' = q'r$ and $rp' = r'p$. Multiplying these two equations and dividing by rr' gives $qp' = q'p$, i.e. $z_2 = 0$. Hence Z lies on the x-axis as well, i.e. the points X, Y, Z are collinear as required. □

Remark 3.18. To show that X, Y, Z in Theorem 3.17 are collinear we could show that the determinant

$$\begin{vmatrix} x_1 & x_2 & 1 \\ y_1 & y_2 & 1 \\ z_1 & z_2 & 1 \end{vmatrix} = \frac{1}{m} \begin{vmatrix} cq - br + br' - cq' & qr' - q'r & q - r - q' + r' \\ ar - cp + cp' - ar' & rp' - r'p & r - p - r' + p' \\ bp - aq + aq' - bp' & pq' - p'q & p - q - p' + q' \end{vmatrix}$$

vanishes. Here $m = (q - r - q' + r')(r - p - r' + p')(p - q - p' + q')$. A three by three determinant has six terms each of which has three factors: one from the first column, one from the second, and one from the third. In the case at hand the first and third factors have four terms each and the second has two. Thus the fully expanded determinant has $6 \times 4 \times 2 \times 4 = 192$ terms. I evaluated it using a computer program (Maple) which does symbolic calculation and all the terms cancel leaving zero. If you believe in computers, this is an alternative proof.

Exercise 3.19. Complete the proof of Theorem 3.17 by proving (I) and (II) in the proof.

Exercise 3.20. The proof of Theorem 3.17 assumes that $m \neq 0$, i.e. that the three numbers $p - p'$, $q - q'$, $r - r'$ are distinct. What if this is false?

Exercise 3.21. The last step in the proof of Theorem 3.17 assumes that $rr' \neq 0$. What if this is false? Hint: If $rr' = 0$ then either $r = 0$ or $r' = 0$. There are three cases: $r = r' = 0$, $r \neq r' = 0$, $r' \neq r = 0$, and the last two are treated the same way.

3.3 Directed Distance

Theorem 3.22. *Let $P = (1-t)P_0 + tP_1$ and $Q = (1-s)P_0 + sP_1$ be two points on the line P_0P_1. Then the distance*[1] *$|PQ|$ between P and Q is given by*

$$\bigl|PQ\bigr| = |s-t|\,\bigl|P_0P_1\bigr|$$

Proof. $P - Q = (s-t)\bigl(P_0 - P_1\bigr)$. □

Definition 3.23. The **directed distance** (PQ) from P to Q in the direction from P_0 to P_1 is defined by

$$(PQ) = (s-t)\,\bigl|P_0P_1\bigr|.$$

3.24. Distances are always nonnegative. However, the directed distance can be negative. For example, this is the case if the points appear on the line in the order P_0, P_1, Q, P, i.e. if P_1 is between P_0 and Q and Q is between P_1 and P. (See Definition 3.7.) Interchanging P_0 and P_1 reverses the sign of the directed distance and hence leaves a ratio of directed distances unchanged. Most affine transformations do not preserve distances; those which do preserve distance are called *Euclidean transformations* and will be studied in Section 4. However, affine transformations preserve ratios of collinear distances. In fact,

Theorem 3.25. *Affine transformations preserve ratios of collinear directed distances.*

Proof. As in the proof of Theorem 3.12, $T(P) = (1-t)T(P_0) + tT(P_1)$. Hence $T(P) - T(Q) = (s-t)\bigl(T(P_0) - T(P_1)\bigr)$ so

$$\frac{(P'Q')}{(P_0'P_1')} = s - t = \frac{(PQ)}{(P_0P_1)}$$

for $P_0' = T(P_0)$, $P_1' = T(P_1)$, $P' = T(P)$, $Q' = T(Q)$. □

Corollary 3.26. *Affine transformations preserve midpoints of segments.*

Proof. The **midpoint** of the segment $[A, B]$ is the unique point M on the line AB which is equidistant from A and B. It is given by

$$M = \tfrac{1}{2}(A + B)$$

(Read $A = P_0$, $B = P_1$, $M = P$ and $t = 1/2$ in the parametric equation $P = (1-t)P_0 + tP$ for a line.) □

[1] See Definition 4.5 below.

3.4 Points and Vectors

Definition 3.27. The difference $W = P_1 - P_0$ between two points P_0 and P_1 is called the **vector** from P_0 to P_1.

Remark 3.28. An affine transformation $T(P) = MP$ where $V = 0$ is called a **linear transformation**.[2] These are studied in the first course in linear algebra. An affine transformation T is a linear transformation if and only if it fixes the origin , i.e. if and only if $T(0) = 0$. When points undergo an affine transformation, the corresponding vectors undergo a linear transformation. This means the following. If $T(P) = MP + V$ is an affine transformation, and $P_0' = T(P_0)$, $P_1' = T(P_1)$ are the images of points P_0, P_1 under T, then the vectors $W = P_1 - P_0$ and $W' = P_1' - P_0'$ are related by the formula

$$W' = MW.$$

Contrast this with the formula

$$P' = MP + V$$

for $P' = T(P)$. The set of all linear transformations form a group: the fact that the composition of two linear transformations is again linear follows from the associative law for matrix multiplication, i.e. $M_2(M_1W) = (M_2M_1)W$.

Remark 3.29. An affine transformation $T(P) = P + V$ where M is the identity matrix is called a **translation**. The set of all translations forms a group: the identity transformation is the translation with $V = 0$, the composition of two translations $T_1(P) = P + V_1$ and $T_2(P) = P + V_2$ is

$$T_2 \circ T_1(P) = P + (V_1 + V_2)$$

and the inverse transformation of the translation $T(P) = P + V$ is

$$T^{-1}(P) = P - V.$$

3.5 Area

In Math 222 you learn how to compute the area of a parallelogram using determinants (cross products). We'll take this as the definition of area. The calculations in this section are easy if you are familiar with matrix algebra. The neophyte can skip the proofs in this section.

[2] The reader is cautioned that some authors use the term *linear transformation* for what we call *affine transformation*.

Definition 3.30. The **oriented area** of a triangle $\triangle ABC$ is half the determinant

$$(ABC) := \tfrac{1}{2}\det[B - A \quad C - A]$$

of the 2×2 matrix whose columns are the edge vectors $B - A$ and $C - A$ from A. Thus if $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$, then

$$(ABC) = \tfrac{1}{2}\det\begin{bmatrix} b_1 - a_1 & c_1 - a_1 \\ b_2 - a_2 & c_2 - a_2 \end{bmatrix}$$

The **area** is the absolute value of the oriented area.

Remark 3.31. The oriented area of $\triangle ABC$ is positive (and hence equal to the area) when the points A, B, C occur in counter clockwise order.

Theorem 3.32. *An affine transformation preserves ratios of oriented areas, i.e. it changes all oriented areas by the same factor.*

Proof. Suppose that $T(P) = MP + V$. Then $T(B) - T(A) = M(B - A)$ and $T(C) - T(A) = M(C - A)$. Hence we have an equality of 2×2 matrices

$$\Big[T(B) - T(A) \quad T(C) - T(A)\Big] = M\Big[B - A \quad C - A\Big].$$

Taking the determinant of both sides and using the fact that the determinant of a product is the product of the determinants shows that the oriented area of $T(\triangle ABC)$ is the determinant of M times the oriented area of $\triangle ABC$. □

Theorem 3.33. *The oriented areas of the four triangles obtained by deleting a vertex of a quadrangle satisfy*

$$(ABD) + (DBC) = (ABC) + (CDA).$$

Proof. (See Figure 7.) The determinant is linear in its columns, reverses sign if the columns are interchanged, and vanishes if two of the columns are the same. From this we see that

$$(ABC) = \det[A\,B] + \det[B\,C] + \det[C\,A].$$

Combining this with the corresponding formulas for the other three triangles in the quadrangle gives the result. □

Figure 7: Two ways to compute the area of a quadrangle

Definition 3.34. The oriented area of the convex quadrangle (A, B, C, D) is the sum of the oriented areas of the two triangles $\triangle ABC$ and $\triangle CDA$, i.e. by definition it is the sum of the oriented areas of the two triangles with common side AC. By Theorems 3.33 this is the same as the sum of the oriented areas the two triangles with common side DB. (See Figure 7.) In other words *the oriented area of a quadrangle is independent of the choice of the diagonal used to compute it.* Figure 8 shows that this might not work for areas (as opposed to oriented areas) because one of the terms in the formula can have the wrong sign.

Definition 3.35. Two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are said to be **separated** by the line $\ell := \{(x, y) : ax + by + c = 0\}$ if $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite signs. This gives a precise meaning to the phrase "P_1 and P_2 lie on opposite sides of ℓ". A polygon is called **convex** iff no edge separates any two of the remaining vertices. For example, the quadrangle on the right in Figure 8 is not convex because C and D lie on opposite sides of the edge AB.

Figure 8: Convex and not convex

Theorem 3.36. *If the quadrangle (A, B, C, D) is convex, then all four terms in the formula in Theorem 3.33 have the same sign. Hence Theorem 3.33 holds for convex quadrangles and area (instead of oriented area).*

Proof. By Theorems 3.13 and 3.32 and Exercise 3.39, we may assume w.l.o.g. that $A = (0, 0)$, $B = (1, 0)$ and $C = (0, 1)$. Now use Exercise 3.40. $\square$

Remark 3.37. One could can carry out the theory of this section for any polygon and prove algebraically that the oriented area of any polygon may be defined by breaking it up triangles and that this area is independent of how the polygon is thus broken up. Similarly for areas and convex polygons.

Exercise 3.38. Show that

$$(ABC) = (BCA) = (CAB) = -(BAC) = -(ACB) = -(CBA).$$

Exercise 3.39. Let T be an affine transformation, ℓ be a line, and A and B be points not on ℓ. Show that ℓ separates A and B if and only if $T(\ell)$ separates $T(A)$ and $T(B)$. Hint: Given a, b, c there are numbers a', b', c' such that

$$(x', y') = T(x, y) \implies ac + by + c = a'x' + b'y' + c'.$$

If $\ell = \{(x, y) : ax + by + c = 0\}$, then $T(\ell) = \{(x', y') : a'x' + b'y' + c = 0\}$.

Exercise 3.40. Let $A = (1, 0)$, $B = (0, 0)$, $C = (0, 1)$, and $D = (d_1, d_2)$. Calculate the four areas (ABD), (DBC), (ABC), (CDA) and verify the formula in Theorem 3.33. Use the definition to show that the quadrangle (A, B, C, D) is convex if and only if $d_1 > 0$, $d_2 > 0$, and $d_1 + d_2 > 1$ and that in this case all four oriented areas are positive.

3.6 Parallelograms

Definition 3.41. A **parallelogram** is a quadrangle (A, B, C, D) such that the lines AB and CD are parallel and the lines AD and BC are parallel.

Theorem 3.42. *A quadrangle (A, B, C, D) is a parallelogram if and only if $B - A = C - D$.*

Proof. Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$, $D = (d_1, d_2)$. Two lines are parallel if and only if they have the same slope. Hence (A, B, C, D) is a parallelogram if and only if

$$\frac{b_2 - a_2}{b_1 - a_1} = \frac{c_2 - d_2}{c_1 - d_1} \qquad \text{and} \qquad \frac{d_2 - a_2}{d_1 - a_1} = \frac{c_2 - b_2}{c_1 - b_1}.$$

The first equation clearly holds if $B - A = C - D$ since the numerators and denominators are equal. But $B - A = C - D \implies D - A = C - B$ so the second equation holds as well. Conversely assume that (A, B, C, D) is a parallelogram and let m and n be the slopes of the sides. Then

$$\begin{aligned} b_2 - a_2 &= m(b_1 - a_1), & c_2 - b_2 &= n(c_1 - b_1), \\ c_2 - d_2 &= m(c_1 - d_1), & d_2 - a_2 &= n(d_1 - a_1). \end{aligned}$$

Subtracting the two equations on the left and subtracting the two on the right gives two equations $v = mu$ and $v = nu$ where $v = (b_2 - a_2) - (c_2 - d_2)$ and $u = (b_1 - a_1) - (c_1 - d_1)$. Since $m \neq n$ (else the points would be collinear) we conclude $u = v = 0$ so $B - A = C - D$ as required. $\square$

Exercise 3.43. *(Addition in Affine Geometry)* Let $O = (0, 0)$, $A = (a, 0)$, $B = (b, 0)$, $C = (c, 0)$ be four points on the x-axis and (O, A', C', B') be a parallelogram (i.e. the lines OA' and $B'C'$ are parallel and the lines OB' and $A'C'$ are parallel) such that the lines AA', BB', CC' are parallel. Show that $c = a + b$

Exercise 3.44. *(Subtraction in Affine Geometry)* Let $O = (0, 0)$, $A = (a, 0)$, $B = (b, 0)$ be three points on the x-axis and (O, A', B', O') be a parallelogram such that the lines OO', AA', BB' are parallel, i.e. the lines OA' and $B'O'$ are parallel and the lines OB' and $A'O'$ are parallel. Show that $b = -a$.

Exercise 3.45. *(Multiplication in Affine Geometry)* Let $O = (0, 0)$, $I = (1, 0)$, $A = (a, 0)$, $B = (b, 0)$, $C = (c, 0)$ be five points on the x-axis, and O', I', B' be three points such that (a) the lines OO', II', BB' are parallel, (b) the lines IB and $I'B'$ are parallel, (c) the points O', I', A are collinear, and (d) the points O', B', C are collinear. Show that $c = ab$.

Exercise 3.46. *(Division in Affine Geometry)* Let $O = (0, 0)$, $I = (1, 0)$, $A = (a, 0)$, $B = (b, 0)$ be four points on the x-axis, and O', I', B' be three points such that (a) the lines OO',BB', II' are parallel, (b) the lines IB and $I'B'$ are parallel, (c) the points O', I', A are collinear, and (d) the points O', I, B' are collinear. Show that $b = 1/a$.

3.7 Menelaus and Ceva

3.47. Let P, Q, R be points on (extended) sides BC, CA, AB of triangle $\triangle ABC$. Thus there are numbers p, q, r with

$$P = pB + (1 - p)C, \qquad Q = qC + (1 - q)A, \qquad R = rA + (1 - r)B.$$

The six distances (BP), (PC), (CQ), (QA), (AR), (RB) in Theorems 3.48 and 3.49 below are directed , i.e. $(XY) = -(YX)$. The sign convention is such that the signs of (BP), (PC), (CQ), (QA), (AR), (RB) are the same as the signs of $1-p$, p, $1-q$, q, $1-r, r$ respectively.

Theorem 3.48 (Menelaus)**.** *The points P, Q, and R are collinear if and only if*

$$\frac{(BP)}{(PC)} \cdot \frac{(CQ)}{(QA)} \cdot \frac{(AR)}{(RB)} = -1$$

Proof. (See also [4] page 146 and [3] page 66.) By Theorem 3.12 the condition that the lines AP, BQ, and CR are concurrent is preserved by affine transformations. By Theorem 3.25 the ratios (and hence product of the ratios) is preserved by affine transformations. Hence by Corollary 4.14 we may assume that $A = (1, 0)$, $B = (0, 1)$ and $C = (0, 0)$ so $P = (0, p)$, $Q = (1-q, 0)$ and $R = (r, 1-r)$. By Theorem 3.4 the points P, Q, R are collinear if and only if

$$\det \begin{bmatrix} 0 & p & 1 \\ 1-q & 0 & 1 \\ r & 1-r & 1 \end{bmatrix} = 0.$$

The condition on the ratios is

$$\frac{1-p}{p-0} \cdot \frac{1-q}{q-0} \cdot \frac{\sqrt{2}(1-r)}{\sqrt{2}(r-0)} = -1.$$

Both conditions simplify to $1-p-q-r+pq+qr+rp = 0$. □

Theorem 3.49 (Ceva)**.** *The lines (AP), (BQ), and (CR) are concurrent if and only if*

$$\frac{(BP)}{(PC)} \cdot \frac{(CQ)}{(QA)} \cdot \frac{(AR)}{(RB)} = 1.$$

Proof. (See also [4] page 126 and [3] page 4.) As in Theorem 3.48 we may assume that $A = (1, 0)$, $B = (0, 1)$, $C = (0, 0)$, $P = (0, p)$, $Q = (1-q, 0)$, $R = (r, 1-r)$. The lines AP, BQ, CR have equations

$$px + y = p, \qquad x + (1-q)y = 1-q, \qquad (r-1)x + ry = 0.$$

By Theorem 3.4 these lines are concurrent if and only if

$$\det \begin{bmatrix} p & 1 & -p \\ 1 & 1-q & q-1 \\ r-1 & r & 0 \end{bmatrix} = 0.$$

The condition on the ratios is

$$\frac{1-p}{p-0} \cdot \frac{1-q}{q-0} \cdot \frac{\sqrt{2}(1-r)}{\sqrt{2}(r-0)} = 1.$$

Both conditions simplify to $1 - p - q - r + pq + qr + rp = 2pqr$. $\square$

Remark 3.50. Ceva's Theorem has the following six corollaries.

1. The medians of a triangle are concurrent. The common point is called the **centroid**. (See 3.8.)
2. The altitudes of a triangle are concurrent. The common point is called the **orthocenter**. (See 5.3.)
3. The perpendicular bisectors of a triangle are concurrent. The common point is called the **circumcenter**. (See 5.2.)
4. The angle bisectors of a triangle are concurrent. The common point is called the **incenter**. (See 5.5.)
5. The lines connecting each vertex of a triangle to the opposite point of tangency of the inscribed circle are concurrent. The common point is called the **Gergonne point**. (See 5.5.)
6. The lines connecting each vertex of a triangle to the point of tangency of the opposite exscribed circle are concurrent. The common point is called the **Nagel point**. (See 5.5.)

3.8 The Medians and the Centroid

Definition 3.51. The **medians** of a triangle are the lines connecting the vertices to the midpoints of the opposite sides. The triangle formed by joining these midpoints is called the **medial triangle**.

Theorem 3.52. *The medians of a triangle $\triangle ABC$ are concurrent, The common point G is called the* **centroid** *and is given by*

$$G = \tfrac{1}{3}(A + B + C).$$

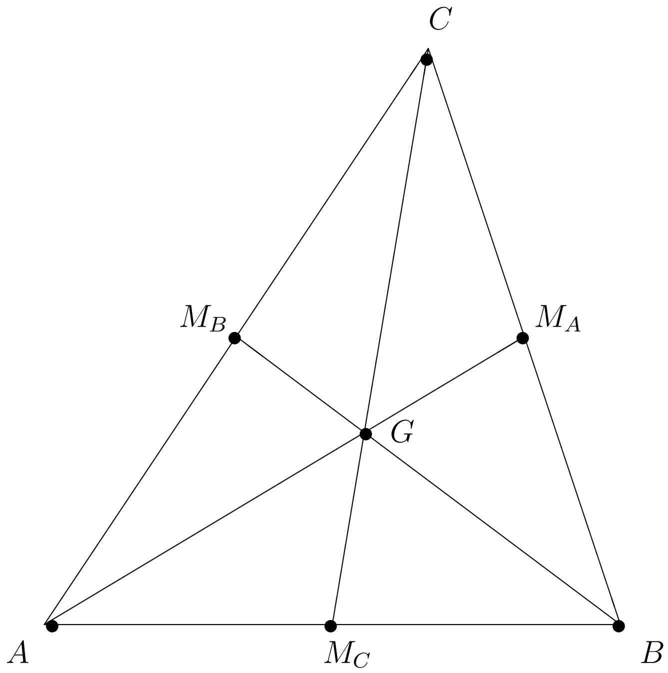


Figure 9: The Centroid.

Proof. The three ratios in Ceva's Theorem are all one. To check the formula for G let M_A, M_B, M_C be the midpoints of the sides BC, CA, AB respectively. Then as in the proof of Corollary 3.26

$$M_A = \tfrac{1}{2}(B + C), \qquad M_B = \tfrac{1}{2}(C + A), \qquad M_C = \tfrac{1}{2}(A + B).$$

so

$$G = \tfrac{1}{3}A + \tfrac{2}{3}M_A = \tfrac{1}{3}B + \tfrac{2}{3}M_B = \tfrac{1}{3}C + \tfrac{2}{3}M_C.$$

In other words G lies on each median two thirds of the way from the vertex to the opposite midpoint. □

Remark 3.53. In Math 221 one learns that the **center of mass** of a collection of n point masses $m_1, m_2, \ldots, m_n$ located at point $P_1, P_2, \ldots, P_n$ is the weighted average

$$\bar{P} = \frac{\sum_i m_i P_i}{\sum_i m_i}.$$

An analogous formula

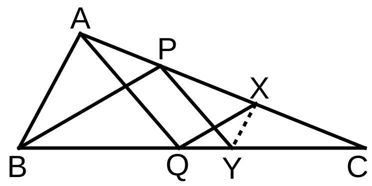
$$\bar{P} = \frac{\int P\, dm}{\int dm}$$

holds for continuous mass distributions. The centroid of a triangle is both the center of mass of three equal mass points at its vertices and also the center of mass of a uniform mass distribution spread over its area. (See [4] page 58.) Somewhat surprisingly, the center of mass of a triangle made from three uniform rods is the incenter, not the centroid. (See [4] page 59.) As explained in [4] on page 129, the point of concurrency in Ceva's theorem can also be viewed as the center of mass of three (unequal) point masses.

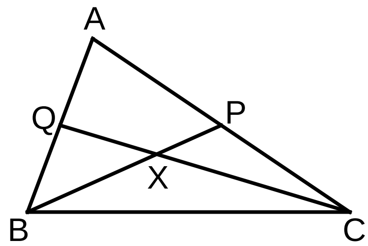
Exercise 3.54. Show that the medians of a triangle divide it into six triangles of equal area. Hint: It is enough to prove this for an equilateral triangle.

Exercise 3.55. Show that the centroid of a triangle divides each median into two segments one of which is twice as long as the other.

Exercise 3.56. Points P and Q are selected on two sides of $\triangle ABC$, as shown, and segments AQ and BP are drawn. Then QX PY are drawn parallel to BP and AQ, respectively. Show that $XY \| AB$.

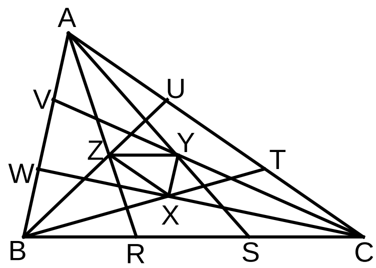


Exercise 3.57. In the figure, vertices B and C of $\triangle ABC$ are joined to points P and Q on the opposite sides, and lines BP and CQ meet at point X. Suppose that $BX = (2/3)BP$ and $CX = (2/3)CQ$. Prove that BP and CQ are medians of $\triangle ABC$.

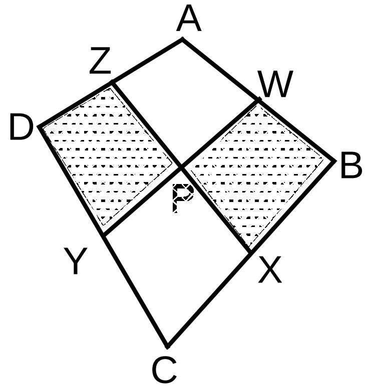


Exercise 3.58. Show that there is no point P inside $\triangle ABC$ such that *every* line through P cuts the triangle into two pieces of equal area. Hint: Show that if there were such a point, it would have to lie on each median of the triangle.

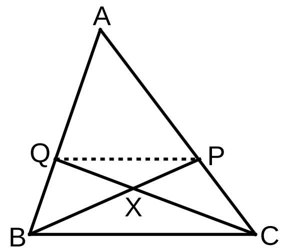
Exercise 3.59. In the figure, the side BC of $\triangle ABC$ is trisected by points R and S. Similarly, T and U trisect side AC and V and W trisect side AB. Each vertex of $\triangle ABC$ is joined to the two trisection points on the opposite side, and the intersections of these trisecting lines determine $\triangle XYZ$, as shown. Prove that the sides of $\triangle XYZ$ are parallel to the sides of $\triangle ABC$.



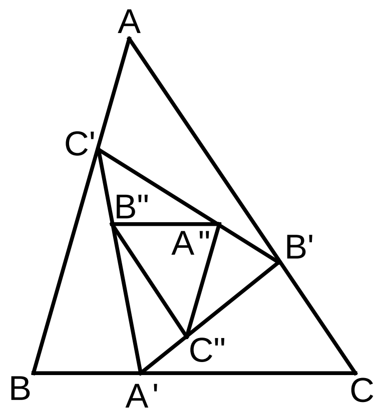
Exercise 3.60. (**Varignon**'s *Theorem.)* Points W, X, Y and Z are the midpoints of the sides of quadrangle $ABCD$ as shown, and P is the intersection of WY with XZ. Two of the four small quadrangles are shaded. Show that P is the midpoint of both WY and XZ and that the shaded area is exactly half of the area of quadrangle $ABCD$. Hint: For the second part, decompose the whole area into four triangles so that exactly half the area of each triangle is shaded.



Exercise 3.61. Points P and Q are chosen on two sides of $\triangle ABC$, as shown, and lines BP and QC meet at X. Show that X lies on the median from vertex A if and only if $QP \| BC$.



Exercise 3.62. Given $\triangle ABC$, let A' be the point 1/3 of the way from B to C, as shown. Similarly, B' is the point 1/3 of the way from C to A and C' lies 1/3 of the way from A to B. In this way, we have constructed a new triangle, $\triangle A'B'C'$ starting with an arbitrary triangle. Now apply the same procedure to $\triangle A'B'C'$, thereby creating $\triangle A''B''C''$. Show that the sides of $\triangle A''B''C''$ are parallel to the (appropriate) sides of $\triangle ABC$. What fraction of the area of $\triangle ABC$ is the area of $\triangle A''B''C''$?



Exercise 3.63. If we draw two medians of a triangle, we see that the interior of the triangle is divided into four pieces: three triangles and a quadrilateral. Prove that two of these small triangles have equal areas, and show that the other small triangle has the same area as the quadrilateral.

4 Euclidean Geometry

4.1 Orthogonal Matrices

Definition 4.1. A square matrix M is said to be **orthogonal** iff its transpose M^* is it inverse.

Theorem 4.2. *A 2×2 matrix M is orthogonal if and only if it has one of the two forms*

$$M = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \qquad \text{or} \qquad M = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

where $a^2 + b^2 = 1$.

Proof. Let a, b, c, d be the entries of M so that M and M^* are given by

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad M^* = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

Then $M^{-1} = M^*$ if and only if $M^*M = MM^* = I :=$ the 2×2 identity matrix, i.e.

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + c^2 & ab + cd \\ ba + dc & b^2 + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ca + db & c^2 + d^2 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This happens if and only if $a^2 + c^2 = b^2 + d^2 = a^2 + b^2 = c^2 + d^2 = 1$ and $ab + cd = ac + bd = 0$. From the former we get $a^2 = d^2$, $b^2 = c^2$ so $d = \pm a$ and $c = \pm b$. The additional equation $ab + cd = 0$ shows that if $d = a \neq 0$ then $b = -c$ and if $d = -a \neq 0$ then $b = c$. $\square$

Remark 4.3. The two forms in Theorem 4.2 are distinguished by their determinants: $\det(M) = a^2 + b^2 = 1$ for the first and $\det(M) = -a^2 - b^2 = -1$ for the second.

Remark 4.4. Each of the equations $M^*M = I$ and $MM^* = I$ implies the other. More generally, if M and N are square matrices and $NM = I$, then $N = M^{-1}$. (See paragraph A.5.) Hence to show that a 2×2 matrix M is orthogonal it is enough to verify one of the two equations $M^*M = I$ and $MM^* = I$.

4.2 Euclidean Transformations

Definition 4.5. The **distance** between the two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ is defined by the Pythogorean formula

$$|AB| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}.$$

(In elementary geometry it is customary to we AB instead of $|AB|$.) We also write

$$|W| = \sqrt{u^2 + v^2}$$

for the distance between the origin $(0, 0)$ and the point $W = (u, v)$.

Definition 4.6. A **Euclidean transformation** is an affine transformation T which preserves distance, i.e.

$$|A'B'| = |AB|$$

whenever $A' = T(A)$ and $B' = T(B)$. Euclidean transformations are called **isometries** in [3]. The term **rigid motion** is also commonly used.

Remark 4.7. The Euclidean transformations form a group. This is obvious and has nothing to do with distance. The set of (invertible) transformations which preserve any function contains the identity and is closed under composition and inverses.

Theorem 4.8. *An affine transformation $T(P) = MP + V$ is Euclidean if and only if the matrix M is orthogonal.*

Proof. Given points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ let $W = B - A$ denote the vector from A to B, let $A' = T(A)$, $B' = T(B)$ be the images of A and B under T, and let $W' = B' - A'$. Then $W' = MW$ (see Remark 3.28). Thus the theorem asserts that $|W'| = |W|$ for all W if and only if M is orthogonal. If $W = (u, v)$ then $W' = (au + bv, cu + dv)$ and

$$|W|^2 = u^2 + v^2, \qquad |W'|^2 = (a^2 + c^2)u^2 + 2(ab + cd)uv + (b^2 + d^2)v^2.$$

If M is orthogonal, then $a^2 + c^2 = b^2 + d^2 = 1$ and $ab + cd = 0$ (see the proof of Theorem 4.2) so $|W'|^2 = |W|^2$. Conversely, assume that $|W'|^2 = |W|^2$ for all W. Taking $W = (1, 0)$ and $W = (0, 1)$ gives $a^2 + c^2 = b^2 + d^2 = 1$. $ab + cd = 0$. These equations say $M^*M = I$. (It follows that $MM^* = I$. See Remark 4.4.) Hence M is orthogonal. $\square$

4.3 Congruence

Definition 4.9. Two figures in the plane are said to be **congruent** iff there is a Euclidean transformation carrying one onto the other. In particular two triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent iff there is a Euclidean transformation T such that $T(A) = A'$, $T(B) = B'$, and $T(C) = C'$. We write

$$F \cong F'$$

as an abbreviation for the sentence F *is congruent to* F'

Remark 4.10. We use the word *figure* rather than the word *set* because the order of the points is important. The triangles $\triangle ABC$ and $\triangle CBA$ are different, and usually there is no Euclidean transformation T such that $T(A) = C$, $T(B) = B$, and $T(C) = A$.

Remark 4.11. Congruence is an equivalence relation, i.e. for any figures

1. $F \cong F$;
2. if $F \cong F'$ and $F' \cong F''$, the $F \cong F''$;
3. if $F \cong F'$, then $F' \cong F$.

This is an immediate consequence of the fact that the Euclidean transformations form a group.

4.12. Any two points are congruent. In fact, for any two points A and B there is a unique translation $T(P) = P + (B - A)$ such that $T(A) = B$. Two directed line segments AB and $A'B'$ are congruent if and only if they have the same length, i.e. $|AB| = |A'B'|$. This is an immediate consequence of the following

Theorem 4.13. *Let $b = |AB|$ be the distance between distinct points A and B. Then there are exactly two Euclidean transformations T such that $T(A) = (0, 0)$ and $T(B) = (b, 0)$.*

Proof. Using a translation we may assume w.l.o.g. that $A = (0, 0)$. Let $B = (b_1, b_2)$ and define a, b, and M by

$$a = \frac{b_1}{\sqrt{b_1^2 + b_2^2}}, \qquad b = \frac{b_2}{\sqrt{b_1^2 + b_2^2}}, \qquad M = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

Then M is orthogonal and $MB = (b, 0)$.

Now suppose that $T_1(A) = T_2(A) = (0,0)$ and $T_1(B) = T_2(B) = (b, 0)$, Then $T(0,0) = (0,0)$ and $T(b,0) = (b,0)$ where $T = T_1 \circ T_2^{-1}$. As T is an Euclidean transformation fixing the origin Theorem 4.2 says that it has form $T(x, y) = (\alpha x + \beta y, \mp\beta x \pm \alpha y)$ where $\alpha^2 + \beta^2 = 1$. From $T(b, 0) = (b, 0)$ we conclude that $\beta = 0$ and $\alpha = 1$. We conclude that there are two choices for T, namely $T = I$ the identity or $T = S$ where $S(x, y) = (x, -y)$ is reflection in the x-axis. Hence either $T_1 = T_2$ or $T_1 = S \circ T_2$. □

Corollary 4.14. *Every triangle is congruent to a triangle $\triangle ABC$ where $A = (a, 0)$, $B = (b, 0)$, and $C = (0, 0)$.*

Proof. By Theorem 4.13 We may assume that the triangle has vertices $A_1 = (0, 0)$, $B_1 = (b_1, 0)$, $C_1 = (c_1, c_2)$. Apply the translation $T(P) = P - V$ where $V = (c_1, 0)$. □

Theorem 4.15 (SSS)**.** *Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent if and only if the corresponding sides are equal:*

$$\triangle ABC \cong \triangle A'B'C' \iff |AB| = |A'B'|,\ |BC| = |B'C'|,\ |CA| = |C'A'|.$$

Proof. "Only if" is immediate since Euclidean transformations preserve distance. Hence assume that the corresponding sides are equal. By Theorem 4.13 we may assume that $A = A' = (0,0)$ and $B = B' = (c, 0)$. Let $a = |BC|$ and $b = |AC|$ then both C and C' lie on the intersection of the two circles

$$x^2 + y^2 = b^2, \qquad (x - c)^2 + y^2 = a^2.$$

But these two circles intersect in the two points $(x_0, \pm y_0)$. (Specifically, $x_0 = (c^2 - a^2)/2c$ and $y_0 = \sqrt{b^2 - x_0^2}$.) Thus either $C = C'$ or the reflection $S(x, y) = (x, -y)$ carries C to C'. Either way $\triangle ABC \cong \triangle A'B'C'$. □

4.4 Similarity Transformations

Definition 4.16. A **similarity transformation** is an affine transformation T which preserves ratios of distances, i.e. there is a positive constant μ such that

$$|A'B'| = \mu|AB|$$

whenever $A' = T(A)$ and $B' = T(B)$.

Definition 4.17. Two figures in the plane are said to be **similar** iff there is a similarity transformation carrying one onto the other. In particular two triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar iff there is a Euclidean transformation T such that $T(A) = A'$, $T(B) = B'$, and $T(C) = C'$.

Theorem 4.18. *The similarity transformations form a group and similarity is an equivalence relation.*

Proof. As for Euclidean transformations and congruence. □

Theorem 4.19. *An affine transformation $T(P) = MP + V$ is a similarity transformation if and only M has one of the two forms*

$$M = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \qquad \textit{or} \qquad M = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

where $a^2 + b^2 > 0$.

Proof. If $T(P) = MP + V$ is a similarity transformation, then $\mu^{-1}T(P) = \mu^{-1}MP + \mu^{-1}V$ is Euclidean. □

4.20. We summarize Definitions 4.6 and 4.16 and Theorem 3.25:

- Euclidean transformations preserve distance.
- Similarity transformations preserve ratios of distances.
- Affine transformations preserve ratios of collinear distances.

4.5 Rotations

Definition 4.21. An affine transformation $T(P) = MP+V$ is called **orientation preserving** iff $\det(M) > 0$ and **orientation reversing** iff $\det(M) < 0$.

Remark 4.22. A Euclidean transformation has form $T(P) = MP + V$ where M has one of the two forms in Theorem 4.2. The first is orientation preserving and the second is orientation reversing.

Definition 4.23. A **fixed point** of a transformation T is a point P which is not moved by T, i.e. $T(P) = P$.

Theorem 4.24. *An orientation preserving Euclidean transformation which is not a translation has a unique fixed point O; such a transformation is called a* **rotation** *about O.*

Proof. Assume that T is an orientation preserving Euclidean Transformation which is not a translation. Then $T(P) = MP + V$ where

$$M = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \qquad V - \begin{bmatrix} p \\ q \end{bmatrix}.$$

and $a^2 + b^2 = 1$, $a < 1$. To find the fixed point O we solve $MO + V = O$ for $O = (x, y)$. These equations are

$$(1 - a)x - by = p, \qquad by + (1 - a)y = q.$$

The determinant of the matrix of coefficients is $(1 - a)^2 + b^2 = 2(1 - a) > 0$ so there is a unique solution. $\square$

4.25. Recall from Math 222 that rotation through angle θ is described by the orthogonal matrix

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

The rotation R_θ carries the point $(1, 0)$ to the point $(\cos\theta, \sin\theta)$. The matrices R_θ satisfy the identities

$$R_0 = I, \qquad R_\alpha R_\beta = R_{\alpha+\beta}, \qquad R_\alpha^{-1} = R_{-\alpha}.$$

The second identity is an immediate consequences of the trigonometric addition formulas:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta, \qquad \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Definition 4.26. A matrix of form

$$R = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}, \qquad c^2 + s^2 = 1$$

rotation matrix; the corresponding Euclidean transformation is then a rotation about the origin $(0, 0)$.

4.6 Review

Next we review how angles are treated in Math 222. We use calculus notation. For the proofs see any calculus textbook. We will not use the material in this section in the formal development, but it motivates the definitions.

Definition 4.27. The **dot product** of two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = \cos\theta |\mathbf{u}|\, |\mathbf{v}|$$

where $|\mathbf{u}|$ is the length of the vector $\mathbf{u}$ and θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$.

Theorem 4.28. *The dot product between $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ is given by*

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

Definition 4.29. The **cross product** of two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \times \mathbf{v} = \sin\theta |\mathbf{u}|\, |\mathbf{v}| \mathbf{w}$$

where $\mathbf{w}$ is the unit vector normal to the plane of $\mathbf{u}$ and $\mathbf{v}$ and θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. (There are two unit vectors perpendicular to a plane: the choice of $\mathbf{w}$ is determined by the condition that $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ forms a right hand frame like $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ and not a left hand frame like $(\mathbf{i}, \mathbf{j}, -\mathbf{k})$.)

Theorem 4.30. *The cross product between $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ is given by*

$$\mathbf{u} \times \mathbf{v} = (u_1 v_2 - u_1 v_2)\mathbf{k}.$$

Corollary 4.31. *Let θ be the angle from the vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ to the vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$. Then*

$$\cos\theta = \frac{u_1 v_1 + u_2 v_2}{\sqrt{u_1^2 + u_2^2}\sqrt{v_1^2 + v_2^2}} \qquad \textit{and} \qquad \sin\theta = \frac{u_1 v_2 - u_2 v_1}{\sqrt{u_1^2 + u_2^2}\sqrt{v_1^2 + v_2^2}}.$$

4.32. In the sequel we will write $U = (u_1, u_2)$ instead of $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $U = A - O$ for the vector from O to A instead of the notation $\mathbf{u} = \overrightarrow{OA}$ used in Math 222.

4.7 Angles

Definition 4.33. An **angle** is a pair (ρ_1, ρ_2) of rays having a common initial point. The common initial point of ρ_1 and ρ_2 is called the **vertex** of the angle and the rays ρ_1 and ρ_2 are called the **arms** of the angle. Given three distinct points O, P, Q the notation $\angle POQ$ denotes the angle (ρ_1, ρ_2) where ρ_1 is the ray with initial point O through P and ρ_2 is the ray with initial point O through Q.

Definition 4.34. The **trigonometric measure** of an angle (ρ_1, ρ_2) is the pair (c, s) defined by

$$c = \frac{u_1 v_1 + u_2 v_2}{\sqrt{u_1^2 + u_2^2}\sqrt{v_1^2 + v_2^2}}, \qquad s = \frac{u_1 v_2 - u_2 v_1}{\sqrt{u_1^2 + u_2^2}\sqrt{v_1^2 + v_2^2}},$$

where O is the vertex of the angle (ρ_1, ρ_2), and A, B are points on ρ_1, ρ_2 distinct from O, and

$$(u_1, u_2) := U := A - O, \qquad (v_1, v_2) := V := B - O.$$

(The definition is independent of the choice of A and B as follows. If A', B' are other points on ρ_1, ρ_2 distinct from O and $U' = A' - O$, $V' = B' - O$, then $U' = \mu U$ and $V' = \nu V$ where $\mu, \nu > 0$. The numbers μ and ν factor out in the above expressions for c and s.)

Exercise 4.35. Show that $c^2 + s^2 = 1$.

Theorem 4.36. *Let (c, s) and (c', s') be the trigonometric measures of angles (ρ_1, ρ_2) and (ρ_1', ρ_2') respectively. Then*

(1) *There is an orientation preserving Euclidean transformation T such that $T(\rho_1) = \rho_1'$ and $T(\rho_2) = \rho_2'$ if and only if $(c', s') = (c, s)$.*

(2) *There is an orientation reversing Euclidean transformation T such that $T(\rho_1) = \rho_1'$ and $T(\rho_2) = \rho_2'$ if and only if $(c', s') = (c, -s)$.*

Proof. □

Definition 4.37. For three points $A = (a_1, a_2)$, $O = (o_1, o_2)$, $B = (b_1, b_1)$ of $\mathbb{R}^2$ with $A \neq O$ and $B \neq O$ define

$$\cos \angle AOB := c, \qquad \sin \angle AOB := s$$

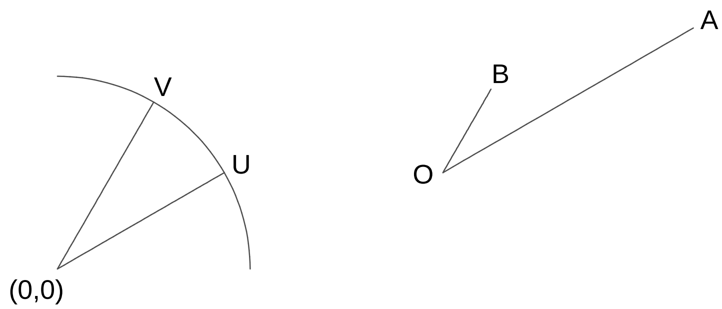


Figure 10: An Angle

as in Theorem 4.36 with $u_i = a_i - o_i$, $v_i = b_i - o_i$. In other words, the pair $(\cos\angle AOB, \sin\angle AOB)$ is the trigonometric measure of the (ρ_1, ρ_2) where ρ_1 is the ray $)A$ and ρ_2 is the ray OB. *In these notes the notation $\angle AOB = \angle A'O'B'$ shall henceforth be considered as an abbreviation for the two equations $\cos\angle AOB = \cos\angle A'O'B'$ and $\sin\angle AOB = \sin\angle A'O'B'$.* Thus

$$\angle AOB = \angle A'O'B' \iff \begin{cases} \cos\angle AOB = \cos\angle A'O'B' \\ \qquad\text{and} \\ \sin\angle AOB = \sin\angle A'O'B'. \end{cases}$$

4.38. In elementary trigonometry the **radian measure** of an angle is defined as follows. Slide the angle in the plane so that the vertex is at the origin and the first ray ρ_1 is the positive x-axis. Then the first ray ρ_1 intersects the unit circle $x^2 + y^2 = 1$ in the point $(1, 0)$ and the second ray ρ_2 intersects the unit circle a point (a, b). The radian measure θ of the angle is then the length of the arc of the unit circle traced out by a point moving counter clockwise from the point $(1, 0)$ to the point (c, s); moreover, the sine and cosine of the angle are given by

$$c = \cos\theta, \qquad s = \sin\theta.$$

In coordinate calculations we generally trigonometric measure rather than radian measure. A careful treatment of radian measure requires calculus.

Remark 4.39. It is an immediate consequence of Definitions 4.33 and 3.30 that the oriented area of $\triangle AOB$ is given by

$$(AOB) = \tfrac{1}{2}|AO|\,|BO|\,\sin\angle AOB.$$

Remark 4.40. The reader is cautioned that in most elementary books no distinction is made between $\angle AOB$ and $\angle BOA$, but with the definitions presented here

$$\sin \angle BOA = -\sin \angle AOB,$$

i.e. reversing the order of the rays in an angle reverses the sign of the sine. With our definitions

$$\cos \angle BOA = \cos \angle AOB,$$

and in elementary texts the condition that two angles are equal can usually be replaced by the condition that they have the same cosine. When a diagram is present $\angle AOB$ is often denoted by $\angle O$, especially when the sign of the sine not important.

Corollary 4.41. *Orientation preserving similarity transformations preserve trigonometric measures of angles. Orientation reversing similarity transformations preserve cosines of angles.*

Corollary 4.42. *Vertical angles are equal, i.e. if A_1, O, A_2 are collinear with O between A_1 and A_2 and B_1, O, B_2 are collinear with O between B_1 and B_2, then $\angle A_1OB_1 = \angle A_2OB_2$.*

Corollary 4.43. *Corresponding angles determined by parallel lines are equal, i.e. if lines A_1B_1 and A_2B_2 are parallel and a point C is on the line A_1A_2 is such that C is between A_1 and A_2, then $\angle B_1A_1C = \angle B_2A_2C$.*

Proof. The translation $T(P) = P + (A_2 - A_1)$ sends the ray A_1B_1 onto the ray A_2B_2 and the ray A_1C onto the ray A_2C. □

4.8 Addition of Angles

Theorem 4.44. *Let (ρ_1, ρ_2), O, A, B, c, s be as in Definitions 4.33, and 4.34, and $T(P) = R(P - O) + O$ where*

$$R = \begin{bmatrix} c & -s \\ s & c \end{bmatrix},$$

Then T is the unique orientation preserving Euclidean transformation such that $T(\rho_1) = \rho_2$.

Proof. The transformation T is orientation preserving and Euclidean: the condition $c^2 + s^2 = 1$ is Exercise 4.35. It follows from the identity

$$(u_1v_1 + u_2v_2)^2 + (u_1v_2 - u_2v_1)^2 = (u_1^2 + u_2^2)(v_1^2 + v_2^2).$$

Clearly $T(O) = O$. Let$(u_1, u_2) := U := A - O$ and $(v_1, v_2) := V := B - O$ as in Definition 4.34. The equations

$$\begin{aligned}(u_1v_1 + u_2v_2)u_1 - (u_1v_2 - u_2v_1)u_2 &= (u_1^2 + u_2^2)v_1, \\ (u_1v_2 - u_2v_1)u_1 + (u_1v_1 + u_2v_2)u_2 &= (u_1^2 + u_2^2)v_2.\end{aligned}$$

show that RU is a positive multiple of V (i.e. $T(A) - O$ is a positive multiple of $B - O$) and hence that T sends ρ_1 to ρ_2. The uniqueness of T is a consequence of the uniqueness in Theorem 4.13. $\square$

Definition 4.45. Let α, β, γ be angles. Then the notation

$$\gamma = \alpha + \beta$$

is understood to be an abbreviation for the assertion that there exist four distinct points O, A, B, C such that

$$\gamma = \angle AOC, \qquad \alpha = \angle AOB, \qquad \beta = \angle BOC.$$

If $\alpha = \angle AOB$, the notation $\alpha = 180^\circ$ means A, O, B are collinear with O between A and B; the notation $\beta = -\alpha$ means $\beta = \angle BOA$; the notation $\alpha = 90^\circ$ means $\cos\angle AOB = 0$ and $\sin\angle AOB = 1$; the notation $\alpha = 45^\circ$ means $\alpha+\alpha = 90^\circ$; etc. We also say that lines OA and OB are **perpendicular** (the word *orthogonal* is used in some textbooks) when $\angle AOB = \pm 90^\circ$. Angles which sum to 90° are called **complementary**; angles which sum to 180° are called **supplementary**.

Theorem 4.46. *Let R_α, R_β, R_γ be the rotation matrices corresponding to the angles α, β, γ as in Theorem 4.44. Then*

$$\gamma = \alpha + \beta \iff R_\gamma = R_\alpha R_\beta.$$

Proof. Let ρ_1, ρ_2, ρ_3 be three rays through the origin. If $R_\alpha(\rho_1) = \rho_2$, $R_\beta(\rho_2) = \rho_3$ and then $R_\beta R_\alpha(\rho_1) = \rho_3$. $\square$

Theorem 4.47. *The angles of a triangle sum to* 180°.

Proof. The theory presented thus far justifies the usual proof: draw a parallel to a side through the third vertex. $\square$

Theorem 4.48 (SSS,SAS,ASA)**.** *For triangles $\triangle ABC$ and $\triangle A'B'C'$ the following conditions are equivalent:*

(i) *the triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent;*

(ii) $|AB| = |A'B'|$, $|BC| = |B'C'|$, *and* $|CA| = |C'A'|$*;*

(iii) $|AB| = |A'B'|$, $|BC| = |B'C'|$ *and* $\cos\angle B = \cos\angle B'$*;*

(iv) $\cos\angle A = \cos\angle A'$, $\cos\angle B = \cos\angle B'$, *and* $|AB| = |A'B'|$.

Proof. The assertion (i) $\iff$ (ii) is Theorem 4.15. The other assertions are similar. $\square$

Corollary 4.49 (Pons Asinorum)**.** *The base angles of an isosceles triangle are equal.*

Proof. If $|AB| = |CB|$, Then $\triangle ABC \cong \triangle CBA$ by ASA, so $\cos\angle A = \cos\angle C$ since Euclidean transformations preserve cosines of angles. $\square$

Remark 4.50. Strictly speaking, the correct conclusion of Pons Asinorum is

$$\cos\angle BAC = \cos\angle BCA, \qquad \sin\angle BAC = -\sin\angle BCA.$$

See Remark 4.40.

Theorem 4.51 (AAA,S:S:S,S:A:S)**.** *For triangles $\triangle ABC$ and $\triangle A'B'C'$ the following conditions are equivalent:*

(i) *the triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar;*

(ii) $\cos\angle A = \cos\angle A'$, $\cos\angle B = \cos\angle B'$, *and* $\cos\angle C = \cos\angle C'$*;*

(iii) *the sides are proportional, i.e.* $\dfrac{A'B'}{AB} = \dfrac{B'C'}{BC} = \dfrac{C'A'}{CA}$;

(iv) $\cos\angle A = \cos\angle A'$ *and* $\dfrac{A'B'}{AB} = \dfrac{C'A'}{CA}$.

Proof. Two triangles are similar if and only if there is a number μ such that the affine transformation $T(P) = \mu P$ carries one of them to a triangle congruent to the other. Apply Theorem 4.48 $\square$

Exercise 4.52. Let $A = (0,0)$, $B = (1,0)$, $C = (0.8, 1.2)$, $A' = (2,3)$, $B' = (2.4, 3.6)$. Then $|AB| = |A'B'| = 1$. There are two points C' such that $\triangle ABC$ and $\triangle A'B'C'$ are congruent. Find them and for each one find the Euclidean transformation T such that $T(A) = A'$, $T(B) = B'$, $T(C) = C'$. For which one(s) is T orientation preserving?

5 More Euclidean Geometry

5.1 Circles

Definition 5.1. The locus of all points P at a fixed distance r from a given point O is called the **circle** of **radius** r and **center** (or *centered at*) O. If $O = (a, b)$ circle is the set of all points $P = (x, y)$ satisfying the equation

$$(x - a)^2 + (y - b)^2 = r^2. \tag{$*$}$$

Any line segment $[O, P]$ where P lies on the circle is called a **radius** of the circle.

Theorem 5.2. *Though any point P_0 on the circle $(*)$ there is a unique line ℓ intersecting the circle at the single point P_0. This line is called the* **tangent line** *to the circle at P_0. The radius OP_0 is perpendicular to the line tangent line ℓ through P_0.*

Proof. The line through $P_0 = (x_0, y_0)$ has an equation of form

$$v(x - x_0) - u(y - y_0) = 0$$

(v/u is the slope) and hence parametric equations

$$x = x_0 + tu, \qquad y = y_0 + tv$$

where $u^2 + v^2 = 1$. Inserting this into $(*)$ and gives

$$(x_0 - a)^2 + 2(x_0 - a)tu + t^2u^2 + (y_0 - b)^2 + 2(y_0 - b)tv + t^2v^2 = r^2.$$

But P_0 lies on the circle, i.e $(x_0 - a)^2 + (y_0 - b)^2 = r^2$, so the equation simplifies to

$$2(x_0 - a)tu + 2(y_0 - b)tv + t^2 = 0.$$

There are two distinct solutions (for t) unless $(x_0 - a)u + (y_0 - b)v = 0$, in which case the line has equation

$$(x_0 - a)(x - x_0) + (y_0 - b)(y - y_0) = 0.$$

Note that this equation says that the vector $\overrightarrow{OP_0}$ from the center O of the circle to a point P_0 on its circumference is perpendicular to vector $\overrightarrow{P_0P}$ from P_0 to a point P on the tangent line at P_0. This is in agreement with the method used in Math 222. □

Theorem 5.3. *The locus of all points P equidistant from two given points A and B is a straight line perpendicular to line AB and passing through the midpoint M of the segment $[A, B]$. It is called the* **perpendicular bisector** *of the segment $[A, B]$.*

Proof. Exercise. $\square$

5.2 The Circumcircle and the Circumcenter

Theorem 5.4. *For any triangle $\triangle ABC$ there is a unique circle containing the three points A, B, C. This circle is called the* **circumcircle** *(short for* **circumscribed circle***) of $\triangle ABC$ and its center O is called the* **circumcenter***. The circumcenter is the intersection of the perpendicular bisectors of the sides.*

Proof. Let O be the intersection of the perpendicular bisectors of $[A, B]$ and $[B, C]$. Then $|OA| = |OB|$ and $|OB| = |OC|$ so$|OA| = |OC|$ O lies on the perpendicular bisector of $[A, C]$ and is equidistant from the vertices. $\square$

Remark 5.5. In high school algebra you learned that equation of form

$$x^2 + y^2 - 2ax - 2by - c = 0$$

determines a circle, a point, or the empty set; you determine which by completing the square, and that is also how you find the center of the circle. Three non collinear points (x_1, y_1), (x_2, y_2), (x_3, y_3) determine (a multiple of) an equation of this form, namely

$$\det \begin{bmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{bmatrix} = 0.$$

Since each of the three points $(x, y) = (x_i, y_i)$ satisfies this equation (because a determinant with two identical rows vanishes), this must be the equation of the circumcircle.

5.3 The Altitudes and the Orthocenter

Theorem 5.6. *Let ℓ be a line and P be a point. The there is a unique line through P perpendicular to ℓ. The point H on ℓ where the perpendicular*

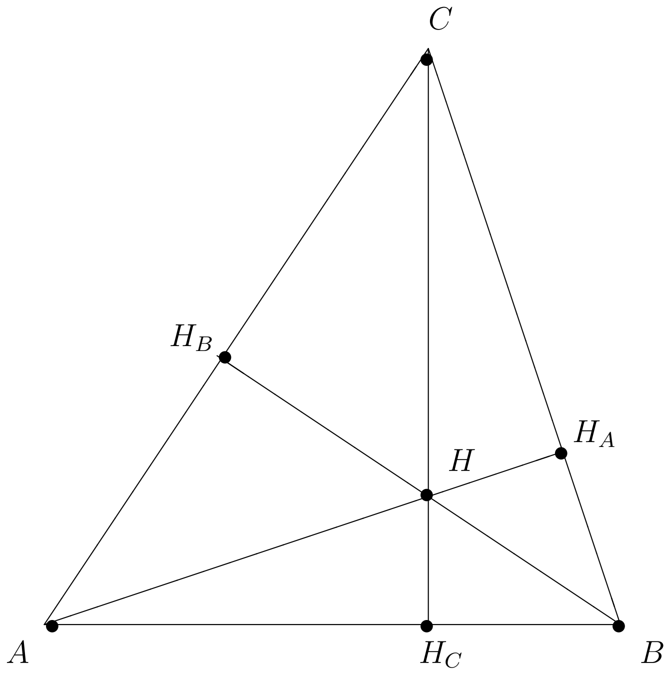


Figure 11: The Othocenter.

through P to ℓ intersects ℓ is called the **foot** *of the perpendicular yo ℓ from P.*

Exercise 5.7. The foot H of the perpendicular to ℓ from P is the point on ℓ closest to P, i.e. the distance from P to H is less than the distance from P to any other point of ℓ. The distance $|PH|$ is called the **distance** from the point P to the line ℓ.

Definition 5.8. The **altitudes** of a triangle are the lines through the vertices perpendicular the opposite sides.

Theorem 5.9. *The altitudes of a triangle are concurrent. The point of concurrency is called the* **orthocenter** *of the triangle.*

Proof. Let $\triangle ABC$ be a triangle and H_A, H_B, H_C be the feet of the altitudes through A, B, C respectively. The lengths in Ceva's Theorem are $BH_A = AB\cos\angle B$, $H_AC = CA\cos\angle C$, $CH_B = BC\cos\angle C$, $H_BC = AB\cos\angle A$, $AH_C = CA\cos\angle A$, $H_CB = BC\cos\angle B$. The corresponding product or ratios is one so the altitudes are concurrent. $\square$

Remark 5.10. Another proof can be based on the fact that circumcenter of a triangle is the orthocenter of its medial triangle.

Definition 5.11. The triangle $\triangle H_A H_B H_C$ formed by joining the feet of the altitudes of $\triangle ABC$ is called the **orthic triangle**[3] of $\triangle ABC$.

5.4 Angle Bisectors

Definition 5.12. Let A, O, B be distinct points. Then the locus of all points P such that either $P = O$ or $\angle AOP = \angle POB$ is called the **angle bisector** of the angle AOB.

Theorem 5.13. *Every point on the angle bisector of the angle AOB is equidistant from the lines AO and BO.*

Proof. Drop perpendiculars from P to OA and OB and use ASA. □

5.5 The Incircle and the Incenter

Theorem 5.14. *The angle bisectors of a triangle are concurrent. The point of concurrency is equidistant from the sides of the triangle and is therefore the center of a circle tangent to the three sides of the triangle. This circle is called the* **incircle** *(short for* **inscribed circle***) of the triangle and its center is called the* **incenter***.*

Proof. The bisector of an angle is the locus of all points equidistant from the arms of the angle. Hence the point of intersection of two angle bisectors of $\triangle ABC$ is equidistant from all thee sides and hence must lie on the third angle bisector □

Remark 5.15. One can give a proof using Ceva's Theorem. See [4] page 21 Theorem 1.12 and [3] page 10 Theorem 1.34.

Theorem 5.16. *Let X on BC, Y on CA, Z on AB be the points of tangency of the inscribed circle of triangle $\triangle ABC$. Then the lines AX, BY, CZ are concurrent. Their common intersection is called the* **Gergonne point** *of $\triangle ABC$.*

[3] Some authors call it the *pedal triangle*.

Proof. Ceva's Theorem plus the fact that the two tangents to a circle from a point are equal. ☐

Theorem 5.17. *Let X on BC, Y on CA, Z on AB be the points of tangency of the exscribed circles of triangle $\triangle ABC$. Then the lines AX, BY, CZ are concurrent. Their common intersection is called the* **Nagel point** *of $\triangle ABC$.*

Proof. The line joining two center of excircle tangent at X and the center of the excircle tangent at Z passes through the corresponding vertex B. The ratio of BX to BZ is the same as the ratio of the radii of these two excircles. Now use Ceva. ☐

5.6 The Euler Line

Theorem 5.18. *Let $\triangle ABC$ be a triangle, G be its centroid, O be its circumcenter, and H be its orthocenter. Then H, G, O are collinear and*

$$G = \tfrac{2}{3}O + \tfrac{1}{3}H.$$

The line OGH is called the **Euler line**.

Proof. Consider the similarity transformation

$$T(P) = G - \tfrac{1}{2}(P - G)$$

This transformation satisfies $T(G) = G$, $T(A) = M_A$, $T(B) = M_B$, $T(C) = M_C$ where M_A, M_B, M_C are the midpoints of $[B, C]$, $[C, A]$, $[A, B]$. It carries the orthocenter H to the circumcenter O. As its shrinks distances by a factor of $\frac{1}{2}$ we conclude that $|OG| = \frac{1}{2}|HG|$. The transformation rotates through 180° about G so H, G, and $O = T(G)$ are collinear. ☐

5.7 The Nine Point Circle

This section uses material not covered above. See [3] page 20.

Theorem 5.19. *The medial triangle and the orthic triangle of a triangle have the same circumcircle. This circumcircle is called the* **nine point circle**[4] *of the original triangle. It also contains the three midpoints, called the* **Euler points**, *of the line segments joining the orthocenter to the vertices.*

[4] Sometimes called the **Euler circle** or the **Feuerbach circle**.

Proof. Denote the midpoints of the sides of $\triangle ABC$, the feet of the altitudes, and the Euler points by M_A, M_B, M_C, H_A, H_B, H_C, E_A, E_B, E_C, with the subscripts chosen so that the medians are AM_A, BM_B, CM_C, the altitudes are AH_A, BH_B, CH_C, and the points E_A, E_B, E_C lie on these altitudes respectively.

We claim that $M_AM_BE_BE_A$ is a rectangle, i.e. adjacent sides are perpendicular. This is because

- $\triangle M_AM_BC$ is similar to $\triangle ABC$ by SAS so M_AM_B is parallel to AB;
- $\triangle E_AE_BH$ is similar to $\triangle ABH$ by SAS so E_AE_B is parallel to AB;
- $\triangle AM_BE_A$ is similar to $\triangle ACH_C$ by SAS so M_BE_A is parallel to CH_C and hence perpendicular to the parallel lines AB and M_AM_B;
- $\triangle BM_AE_B$ is similar to $\triangle BCH_C$ by SAS so M_AE_B is parallel to CH_C and hence perpendicular to the parallel lines AB and M_AM_B.

The rectangle $M_AM_BE_BE_A$ is inscribed in a circle with diameters M_AE_A and M_BE_B. Reading B and C for A and B we obtain that the rectangle $M_BM_CE_CE_B$ is inscribed in a circle with diameters M_BE_B and M_CE_EC. These two circles share a common diameter M_BE_B and so must be the same circle. As this circle contains M_A, M_B, M_C it is the circumcircle of the medial triangle, i.e. the nine point circle. The line E_AH_A is the altitude AH_A and the line $H)AM_A$ is the side BC opposite A. Hence $\angle E_AH_AM_A$ is a right angle so the point H_A (and similarly H_B and H_C) lies on the nine point circle. □

Theorem 5.20. *The orthocenter H of an acute angled triangle $\triangle ABC$ is the incenter of its orthic triangle $\triangle H_AH_BH_C$.*

Proof. We must show $\angle H_AH_CH = \angle H_BH_AH$. (The same argument shows $\angle H_CH_BH = \angle H_CH_BH$ and $\angle H_CH_BH = \angle H_AH_CH$.) Now the quadrangle HH_CBH_A is cyclic (i.e. is inscribed in a circle) since $HH_CB = HH_AB = 90°$. Hence $\angle HH_AH_C = \angle HBH_C$. But $\angle HBH_C = \angle H_BBA$ is the complement α of $\angle CBA$ so $HH_AH_C = \alpha$. The same argument (reverse the roles of B and C) shows that $HH_AH_B = \alpha$ as required. □

Remark 5.21. For an obtuse triangle the argument shows that H is the center of one of the exscribed circles, not the center of the inscribed circle.

Corollary 5.22. *The triangle of least perimeter inscribed in a given triangle $\triangle ABC$ is the orthic triangle of $\triangle ABC$.*

Proof. For three points X, Y, Z, on lines BC, CA, AB respectively let $f = f(X, Y, Z) = XY + YZ + ZX$. This function is continuous as a function of three variables. (Each of X, Y, Z is constrained to a line.) When one or more of the points X, Y, Z are far from A, B, and C the function f is large so the minimum occurs either a point where the partial derivatives of f are zero, or else at a point where the function f is not differentiable. The function f is differentiable except where one of the three lengths is zero, i.e. when $X = Y = C$ or $Y = Z = A$ or $Z = X = B$. □

5.8 A Coordinate Proof

Here are some coordinate calculations which could be used to prove Theorems 5.18 and 5.19. I used a Maple program to do these calculations and verify that the nine points of the nine point circle are equidistant from N, the midpoint of the segment OH.

5.23. By Corollary 4.14 we may assume that the vertices of the triangle are

$$A = (a, 0), \qquad B = (b, 0), \qquad C = (0, c).$$

The midpoints of the sides are given by

$$M_A = \left(\frac{a+b}{2}, 0\right), \qquad M_B = \left(\frac{a}{2}, \frac{c}{2}\right), \qquad M_C = \left(\frac{b}{b}, \frac{c}{2}\right),$$

and the centroid is

$$G = \tfrac{1}{3}(A + B + C) = \left(\frac{a+b}{3}, \frac{c}{3}\right).$$

The slopes of the lines AB, AC, BC are given by

$$m_{AB} = 0, \qquad m_{AC} = -\frac{c}{a}, \qquad m_{BC} = -\frac{c}{b};$$

The equations for these lines are respectively $y = 0$ and

$$cx + ay = ac, \qquad cx + by = bc.$$

The orthocenter H lies on the y-axis, say $H = (0, h)$. The slopes of the altitudes AH and BH are

$$m_{AH} = -\frac{h}{a} = -\frac{1}{m_{BC}} = \frac{b}{c}, \qquad m_{BH} = -\frac{h}{a} = -\frac{1}{m_{AC}} = \frac{a}{c},$$

and solving for h gives

$$H = \left(0, -\frac{ab}{c}\right).$$

Equations for the altitudes CH, AH, BH are $x = 0$ and

$$bx - cy = ab, \qquad ax - cy = ab$$

The feet of these altitudes are respectively $H_C = (0, 0)$ and

$$H_A = \left(\frac{ab^2 + bc^2}{b^2 + c^2}, \frac{b^2c - abc}{b^2 + c^2}\right), \qquad H_B = \left(\frac{ba^2 + ac^2}{a^2 + c^2}, \frac{a^2c - abc}{a^2 + c^2}\right).$$

The Euler points $E_C = \frac{1}{2}(C + H)$, $E_A = \frac{1}{2}(A + H)$, $E_B = \frac{1}{2}(B + H)$ are

$$E_C = \left(0, \frac{c^2 - ab}{2c}\right), \qquad E_A = \left(\frac{a}{2}, -\frac{ab}{2c}\right), \qquad E_B = \left(\frac{b}{2}, -\frac{ab}{2c}\right).$$

The perpendicular bisectors of the sides AB, AC, BC have equations

$$x = \frac{a + b}{2}, \qquad y - \frac{c}{2} = \frac{a}{c}\left(x - \frac{a}{2}\right), \qquad y - \frac{c}{2} = \frac{b}{c}\left(x - \frac{b}{2}\right)$$

The common point is the circumcenter

$$O = \left(\frac{a + b}{2}, \frac{c^2 + ab}{2c}\right).$$

5.9 Simson's Theorem

See [3] page 40.

5.24. Let ABC be a triangle. The lines AB, BC, AC divide the plane into seven regions; let the point P lie in the unbounded region containing the edge AC in its boundary (see diagram). Let X, Y, Z be the feet of the perpendiculars from P to the lines BC, AC, AB respectively. See Figure 5.9.

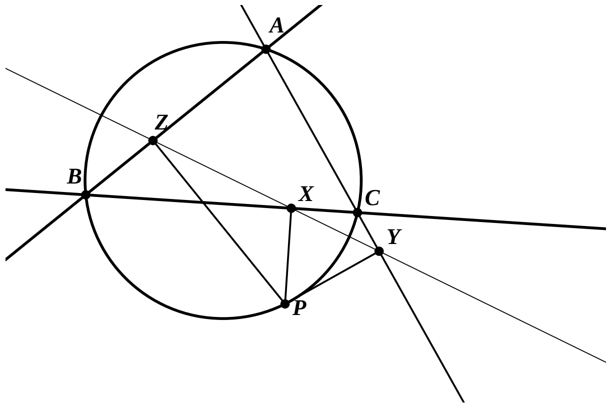


Figure 12: Simson's Theorem

Theorem 5.25 (Simson's Theorem)**.** *The points X, Y, Z are collinear if and only if P lies on the circumcircle of ABC.*

Proof. The point P lies on the circumcircle of ABC if and only if

$$\angle APC = 180^o - \angle B. \tag{1}$$

Because the opposite angles at X and Z in the quadrangle $BXPZ$ are right angles we have

$$180^o - \angle B = \angle ZPX$$

so condition (1) is equivalent to

$$\angle APC = \angle ZPX \tag{2}$$

and on subtracting $\angle APX$ we see that (2) is equivalent to

$$\angle XPC = \angle ZPA. \tag{3}$$

A quadrilateral containing a pair of opposite right angles is *cyclic*, i.e. its vertices lie on a circle; in fact, the other two vertices are the endpoints of a diameter of this circle. Hence each of the quadrilaterals $AYPZ$, $BXPZ$, $CXPY$ is cyclic. Since the quadrangle $CXPY$ is cyclic we have

$$\angle XYC = \angle XPC. \tag{4}$$

Since the quadrangle $AYPZ$ is cyclic we have

$$\angle ZYA = \angle ZPA. \tag{5}$$

From (4) and (5) we conclude that (3) is equivalent to

$$\angle XYC = \angle ZYA. \tag{6}$$

But clearly (6) holds if and only if the points X, Y, Z are collinear. $\square$

Alternate Proof. Let O be the circumcenter of $\triangle ABC$. By trigonometry

$$\begin{aligned} BX &= BP\cos\angle PBC, & XC &= PC\cos\angle PCB, \\ CY &= CP\cos\angle PCA, & YA &= PB\cos\angle PAC, \\ AZ &= AP\cos\angle PAB, & ZB &= PA\cos\angle PBA. \end{aligned}$$

Using the fact that the inscribed angle is half the subtended arc

$$\begin{aligned} BX &= BP\cos\tfrac{1}{2}\angle POC, & XC &= PC\cos\tfrac{1}{2}\angle POB, \\ CY &= CP\cos\tfrac{1}{2}\angle POA, & YA &= PB\cos\tfrac{1}{2}\angle POC, \\ AZ &= AP\cos\tfrac{1}{2}\angle POB, & ZB &= PA\cos\tfrac{1}{2}\angle POA. \end{aligned}$$

Now use Menalaus. $\square$

5.26. Here is a computer assisted coordinate calculation which proves Simson's Theorem. It shouldn't be to difficult to do by hand, especially if we take $\gamma = -\alpha$ below to simplify the formulas. We begin by loading the Maple package for doing linear algebra.

```
with(LinearAlgebra);
```

To calculate the foot Z of the perpendicular from the point P to the line AB we use the formulas

$$Z = A + t(B - A), \qquad PZ \perp AB,$$

solve for t, and plug back in to get Z. Here is a Maple procedure to compute this.

```
foot:=proc(A,B,P) local t;
   t:=((B[1]-A[1])*(P[1]-A[1])+(B[2]-A[2])*(P[2]-A[2]))/
         ((B[1]-A[1])^2+(B[2]-A[2])^2);
   [A[1]+t*(B[1]-A[1]),A[2]+t*(B[2]-A[2])]
end proc;
```

We choose A, B, C, on the unit circle and P arbitrarily.

```
A:=[cos(alpha),sin(alpha)];
B:=[cos(beta),sin(beta)];
C:=[cos(gamma),sin(gamma)];
P:=[x,y];
```

We use the procedure to calculate X, Y,Z:

```
X:= foot(B,C,P); Y:=foot(C,A,P); Z:=foot(A,B,P);
```

We define the matrix whose determinant vanishes when X, Y, Z are collinear.

```
M:=Matrix([
                    [X[1], X[2], 1],
                    [Y[1], Y[2], 1],
                    [Z[1], Z[2] ,1]
                ]);
```

We compute its determinant.

```
W:=Determinant(M);
```

The determinant W vanishes exactly when X, Y, Z are collinear. The following commands show that x^2+y^2-1 divides W and that the quotient m is independent of $P=(x,y)$.

```
m:=simplify(W/(x^2+y^2-1));
simplify(W-m*(x^2+y^2-1));
```

The last command evaluates to 0 and proves that $W=m(x^2+y^2-1)$. Thus X, Y, Z are collinear if and only if $x^2+y^2=1$, i.e. if and only if P lies on the circumcircle of $\triangle ABC$. The commands

```
mm:=expand(sin(alpha-beta)+sin(beta-gamma)+sin(gamma-alpha))/4;
simplify(m-mm);
```

produce an output of zero which shows that

$$m=\frac{\sin(\alpha-\beta)+\sin(\beta-\gamma)+\sin(\gamma-\alpha)}{4}$$

We have proved the following

Theorem 5.27 (Algebraic form of Simson's Theorem). *Let*

$$A = (\cos\alpha, \sin\alpha), \qquad B = (\cos\beta, \sin\beta), \qquad C = (\cos\gamma, \sin\gamma)$$

be three points on the unit circle $x^2 + y^2 = 1$, let $P = (x, y)$ be an arbitrary point, and

$$X = (x_1, x_2), \qquad Y = (y_1, y_2), \qquad Z = (z_1, z_2)$$

be the feet of the perpendiculars from P to the lines BC, CA, AB respectively. Then

$$\begin{vmatrix} x_1 & x_2 & 1 \\ y_1 & y_2 & 1 \\ z_1 & z_2 & 1 \end{vmatrix} = m(x^2 + y^2 - 1)$$

where

$$m = \frac{\sin(\alpha - \beta) + \sin(\beta - \gamma) + \sin(\gamma - \alpha)}{4}.$$

5.10 The Butterfly

See [3] page 45.

5.11 Morley's Theorem

See [3] page 47.

5.12 Bramagupta and Heron

See [3] page 56.

5.13 Napoleon's Theorem

See [3] page 63.

5.14 The Fermat Point

See [3] page 83.

6 Projective Geometry

Projective geometry developed historically at the same time artists learned to draw in perspective, i.e. to draw pictures on a flat canvas which look three dimensional. See for example Figure 13 where a two dimensional theorem is illustrated by a three dimensional picture. A line in space will be drawn as a line in the picture , but a circle in space will be drawn as an ellipse in the picture. Parallel lines in space will (if extended) meet in the picture. Look down State street from Bascom hill: the sidewalks and roofs of the buildings all aim at a point behind the Capitol.

6.1 Homogeneous coordinates

6.1. When (x, y, z) is a point of $\mathbb{R}^3$ distinct from the origin $(0, 0, 0)$ we denote the line connecting the origin and this point by $(x:y:z)$. Thus

$$(x:y:z) = \{(tx, ty, tz) : t \in \mathbb{R}\}.$$

Note that $(x':y':z') = (x:y:z)$ if and only if $x' = \mu x$, $y' = \mu y$, $z' = \mu z$ for some non zero number μ.

Definition 6.2. The **projective plane** is the set $\mathbb{P}^2$ of all lines through the origin in $\mathbb{R}^3$. We say that (x, y, z) are **homogeneous coordinates** for the point $(x:y:z)$.

6.3. A point $(x:y:z) \in \mathbb{P}^2$ is a line in $\mathbb{R}^3$ and, if $z \neq 0$, intersects the plane $z = 1$ in the unique point $(xz^{-1}, yz^{-1}, 1)$, i.e.

$$z \neq 0 \implies (x:y:z) = (xz^{-1}:yz^{-1}:1).$$

The points (i.e. lines) of form $(x:y:0)$ do not intersect the plane $z = 1$; the set of these points is called the **line at infinity**.

Exercise 6.4. Find z if $(1 : 2 : 3) = (2 : 4 : z)$.

Remark 6.5. Imagine an object in three dimensional space $\mathbb{R}^3$ above the plane $z = 1$. Place your eye at the origin $(0, 0, 0)$ view the object through a transparent canvas lying on the plane $z = 1$. Each point on the object determines a line though your eye. For each point of the object paint a dot on the point of intersection of this line with the plane $z = 1$ and you have a portrait of the object. If you have another object and a correspondence between the points of the objects such that each pair of corresponding points is collinear with the origin, then the two portraits will be indistinguishable. This is why two points, (x, y, z) and $(\mu x, \mu y, \mu z)$, on the same line through the origin determine the same point $(x:y:z)$ in projective space.

6.6. A plane through the origin in $\mathbb{R}^3$ has an equation of form $ax+by+cz=0$ where $(a, b, c) \neq (0, 0, 0)$. We denote by

$$[a:b:c] := \{(x:y:z) \in P^2 : ax + by + cz = 0\}$$

the set of lines through the origin which lie in this plane. If $\mu \neq 0$, the equations $\mu ax + \mu by + \mu cz = 0$ and $ax + by + cz = 0$ define the same plane. Hence

$$[\mu a:\mu b:\mu c] = [a:b:c].$$

We call $[a:b:c]$ a **projective line**. Several points $(x_1:y_1:z_i1)$, $(x_2:y_2:z_2)$, ... are said to be **collinear** if there is a line $[a:b:c]$ which contains all of them; several lines $[a_1:b_1:c_1]$, $[a_2:b_2:c_2]$, ... are said to be **concurrent** if there is a point $(x:y:z)$ in their intersection.

Theorem 6.7. (i) *Two distinct points in $\mathbb{P}^2$ lie in a unique projective line.* **(ii)** *Two distinct projective lines intersect in a unique point.*

Proof. (i) Two lines in $\mathbb{R}^3$ through the origin lie in a unique plane necessarily containing the origin. (ii) Two distinct planes in $\mathbb{R}^3$ which pass through the origin intersect in a line which passes through the origin. □

Exercise 6.8. Find the intersection of the two lines $[1:2:3]$ and $[3:2:1]$.

Remark 6.9. Parallel lines in the affine plane have equations $ax+by+c=0$ and $ax + by + d = 0$ where $c \neq d$. The corresponding projective lines $[a:b:c]$ and $[a:b:d]$ intersect in the point $(-b:a:0)$. This point lies in the line at infinity. Hence *parallel lines intersect at infinity.*

In affine geometry two distinct points determine a unique line, but distinct lines need not determine a point: they might be parallel. Because of this asymmetry the statement proof of part (II) of Theorem 3.4 was more complicated then that of part (I). In projective geometry this asymmetry disappears: two distinct points determine a unique line and two distinct lines intersect in a unique point. We recover affine geometry by specifying a "line at infinity" where parallel lines meet.

Theorem 6.10. **(I)** *Three points* $(x_i\!:\!y_i\!:\!z_i)$ *are collinear if and only if*

$$\det\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_3 & z_3 & z_3 \end{bmatrix} = 0.$$

(II) *Three lines* $[a_i\!:\!b_i\!:\!c_i]$ *are concurrent if and only if*

$$\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = 0.$$

Proof. As in Theorem 3.4 but without the special cases. Note that in part (I) the determinant is zero if and only if

$$\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_3 & z_3 & z_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

for some $\begin{bmatrix} a & b & c \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$, and in part (II) the determinant is nonzero if and only if

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

for some $(x, y, z) \neq (0, 0, 0)$. □

6.2 Projective Transformations

Definition 6.11. Let (t_{ij}) be a 3×3 matrix with nonzero determinant. Then there is a unique transformation $T : \mathbb{P}^2 \to \mathbb{P}^2$ satisfying the condition

$$\begin{aligned} x' &= t_{11}x + t_{12}y + t_{13}z \\ y' &= t_{21}x + t_{22}y + t_{23}z \\ z' &= t_{31}x + t_{32}y + t_{33}z \end{aligned} \quad \Longrightarrow \quad (x'\!:\!y'\!:\!z') = T(x\!:\!y\!:\!z).$$

(The definition is legal since matrix multiplication sends lines through the origin to lines through the origin.) A transformation of this form is called a **projective transformation**. Note that multiplying the matrix (t_{ij}) by a nonzero number does not change the transformation T (but it does change the matrix).

6.12. As in affine geometry, the formulas defining a projective transformation can be written using matrix notation:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \qquad (*)$$

A matrix transformation preserves the plane $z = 1$ if and only if $t_{31} = t_{32} = 0$ and $t_{33} = 1$. For such a transformation we may rewrite $(*)$ in the form

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & p \\ c & d & q \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

with $a = t_{11}$, $b = t_{12}$, $c = t_{21}$, $d = t_{22}$, $p = t_{13}$, $q = t_{23}$. This is equivalent to $x' = ax+by+p$ and $y' = cx+dy+q$ in agreement with Definition 3.9. In this way every affine transformation is a projective transformation. The power of the theory will become evident when we use projective transformations which are not affine.

6.13. We say that the triple (x, y, z) **represents** the point $(x:y:z)$. Thus if $\mu \neq 0$ the triples $(\mu x, \mu y, \mu z)$ and (x, y, z) represent the same point. It is convenient to think of (x, y, z) as a column matrix:

$$(x, y, z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

We also say that the row matrix $[a, b, c]$ represents the line $[a:b:c]$ and the matrix (t_{ij}) represents the projective transformation T.

Theorem 6.14. *A projective transformation maps lines to lines.*

Proof. If $L = [a, b, c]$ represents the line $[a:b:c]$ and $X = (x, y, z)$ represents the point $(x:y:z)$, then $(x:y:z)$ lies on $[a:b:c]$ if and only if $LX = 0$. But $LX = LT^{-1}TX$ so

$$LX = 0 \iff (LT^{-1})(TX) = 0$$

so the line represented by LT^{-1} is the image under T of the line $[a:b:c]$. $\square$

Theorem 6.15. *Given four points A, B, C, D no three of which are collinear, there is a unique projective transformation, T such that*

$$T(1{:}0{:}0) = A, \quad T(0{:}1{:}0) = B, \quad T(0{:}0{:}1) = C, \quad T(1{:}1{:}1) = D.$$

(Compare Theorem 3.13 and Corollary 4.14.)

Proof. Let $A = (a_1{:}a_2{:}a_3)$, $B = (b_1{:}b_2{:}b_3)$, $C = (c_1{:}c_2{:}c_3)$. Then the projective transformation S represented by the matrix

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

satisfies the first three of the four conditions, namely

$$S(1{:}0{:}0) = A, \qquad S(0{:}1{:}0) = B, \qquad S(0{:}0{:}1) = C.$$

Let $S(D) = (d_1{:}d_2{:}d_3)$ and

$$R = \begin{bmatrix} d_1^{-1} & 0 & 0 \\ 0 & d_2^{-1} & 0 \\ 0 & 0 & d_3^{-1} \end{bmatrix}.$$

(We have $d_1 \neq 0$ since otherwise $S(B)$, $S(C)$, $S(D)$ would be collinear contradicting the hypothesis that B, C, D are not collinear. Similarly $d_2 \neq 0$ and $d_3 \neq 0$.) Now $R(S(A)) = R(1{:}0{:}0) = (d_1^{-1}{:}0{:}0) = (1{:}0{:}0)$ and similarly $R(S(B) = (0{:}1{:}0)$ and $R((S(C)) = (0{:}0{:}1)$. Also $R(S(D)) = R(d_1{:}d_2{:}d_2) = (1 : 1 : 1)$. The projective transformation $T = R \circ S$ satisfies all four conditions.

To prove uniqueness suppose that T' is another projective transformation satisfying the four conditions. Then

$$T' \circ T^{-1}(1{:}0{:}0) = 1{:}0{:}0), \quad T' \circ T^{-1}(0{:}1{:}0) = (0{:}1{:}0),$$
$$T' \circ T^{-1}(0{:}0{:}1) = (0{:}0{:}1), \quad T' \circ T^{-1}(1{:}1{:}1) = (1{:}1{:}1).$$

The first three conditions imply that $T' \circ T^{-1}$ is represented by a diagonal matrix and then the fourth condition says that the three diagonal entries must be equal. This means that $T' \circ T^{-1}$ is the identity transformation of $\mathbb{P}^2$ so $T = T'$. $\square$

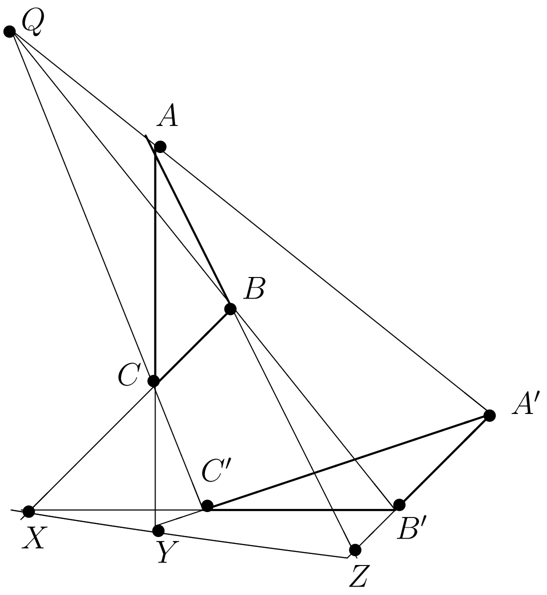


Figure 13: Desargues' Theorem

6.3 Desargues and Pappus

Theorem 6.16 (Desargues)**.** *Given triangles $\triangle ABC$ and $\triangle A'B'C'$ let X, Y, Z be the intersections of the corresponding sides of $\triangle ABC$ and $\triangle A'B'C'$, i.e.*

$$X := BC \cap B'C', \qquad Y := CA \cap C'A', \qquad Z := AB \cap A'B'.$$

Then the lines AA', BB', CC' are concurrent if and only if the points X, Y, Z are collinear. (See Figure 13.)

Proof. Assume the lines AA', BB', CC' are concurrent, i.e. there is a point Q such that each triple (Q, A, A'), (Q, B, B'), (Q, C, C') is collinear. Choose an affine transformation T which sends Q to the line at infinity. Then the lines $T(A)T(A')$, $T(B)T(B')$, $T(C)T(C')$ are parallel. By Exercise 3.17 the points $T(X)$, $T(Y)$, $T(Z)$ are collinear. Apply T^{-1} to conclude that the points X, Y, Z are collinear. The converse can be proved in the same way, but it also follows from the principle of duality as explained in 6.23 below. $\square$

Remark 6.17. Sometimes Desargues' Theorem is stated as follows: *Two triangles are perspective from a point if and only if they are perspective from a line.*

Remark 6.18. Think of Figure 13 as representing a three dimensional picture where Q is the apex of two tetrahedra, one with base $\triangle ABC$ and the other with base $\triangle A'B'C'$. The plane $A'B'C'$ intersects the plane ABC in a line containing X, Y, Z. Note that projecting the three dimensional figure to a plane either from a point (your eye) or by parallel lines gives the configuration in Desargues' theorem if no triangle projects to a line. This observation is undoubtedly how the theorem was discovered. We give a proof based on it.

Synthetic proof of Desargues' theorem. Denote by Π the plane containing the points in the statement of the theorem, i.e. the points A, B, C, A', B', C', X, Y, Z, and Q. Let E (the eye) be a point not on Π. Each point P in Π determines a line EP and each line L in Π determines a plane EL. In particular the points E, Q, A, A' are coplanar. Let $\tilde{A}$ be the intersection of the three planes EAB, EAC, and $EQAA'$. Similarly, let $\tilde{B}$ be the intersection of the three planes EBC, EBA, $EQBB'$, and $\tilde{C}$ be the intersection of the three planes ECA, ECB, $EQCC'$. The three planes $EQAA'\tilde{A}$, $EQBB'\tilde{B}$, $EQCC'\tilde{C}$ intersect in a point $\tilde{Q}$. (A diagram illustrating the situation can be obtained from Figure 13 by replacing Q, A, B, C, by $\tilde{Q}$, $\tilde{A}$, $\tilde{B}$, $\tilde{C}$. The original points Q, A, B, C, are hidden because they are on the line of sight with E.) The points

$$\begin{aligned} X &= \Pi \cap EQBB'\tilde{B} \cap EQCC'\tilde{C}, \\ Y &= \Pi \cap EQCC'\tilde{C} \cap EQAA'\tilde{A}, \\ Z &= \Pi \cap EQAA'\tilde{A} \cap EQBB'\tilde{B} \end{aligned}$$

lie in the intersection of the plane of $\tilde{A}\tilde{B}\tilde{C}$ and Π and are therefore collinear. □

Theorem 6.19 (Pappus)**.** *Assume that the three points A, B, C are collinear and that the three points $A', B,' C'$ are collinear. Let the lines joining them in pairs intersect as follows:*

$$X = BC' \cap B'C, \qquad Y = CA' \cap C'A, \quad Z = AB' \cap A'B.$$

Then the points X,Y, Z are collinear. (See Figure 14.)

Proof. As in the proof of Desargues' theorem we may assume w.l.o.g. that the lines ABC and $A'B'C'$ are parallel. Then the theorem follows from Theorem 3.15. □

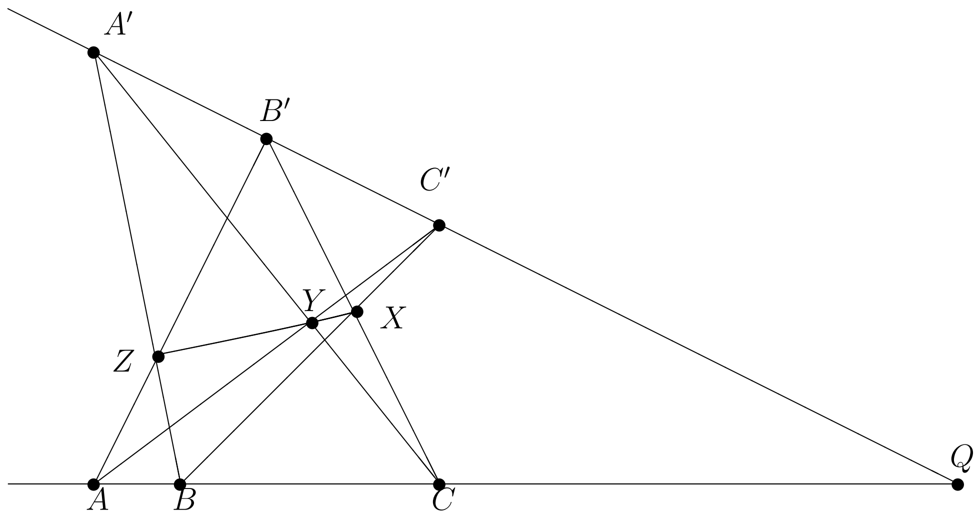


Figure 14: Pappus' Theorem

Remark 6.20. Any theorem involving only points and lines in the projective plane can be viewed as a theorem about determinants. For example, Desargues' theorem assumes we are given column matrices A, B, C, A', B', C', X, Y, Z, satisfying

$$\begin{aligned}&\det(ABZ) = \det(BCX) = \det(CAY) = \\ &\det(A'B'Z) = \det(B'C'X) = \det(C'A'Y) = 0.\end{aligned}$$

and concludes that

$$\exists Q \ \det(QAA') = \det(QBB') = \det(QCC') = 0 \iff \det(XYZ) = 0.$$

Pappus' theorem assumes we are given column matrices A, B, C, A', B', C', X, Y, Z, satisfying

$$\begin{aligned}&\det(AB'Z) = \det(BC'X) = \det(CA'Y) = \\ &\det(A'BZ) = \det(B'CX) = \det(C'AY) = 0.\end{aligned}$$

and concludes that

$$\det(ABC) = \det(A'B'C') = 0 \implies \det(XYZ) = 0.$$

Exercise 6.21. Let the lines joining three points A, B, C to three other points $A', B,' C'$ in pairs intersect as follows:

$$X = BC' \cap B'C, \qquad Y = CA' \cap C'A, \quad Z = AB' \cap A'B.$$

Show that if A, B, C are collinear and X, Y, Z are collinear, then A', B', C' are collinear. Hint: Don't work too hard.

6.4 Duality

6.22. Let the column matrix (x, y, z) represent the point $(x:y:z)$ and the row matrix $[a, b, z]$ represent the line $[a:b:c]$. Then the equation

$$ax + by + cz = 0$$

says that the point $(x:y:z)$ lies on the line $[a:b:c]$. But it also says that the point $(a:b:c)$ lies on the line $[x:y:z]$. In matrix theoretic terms, taking the transpose converts points to lines and line to points. If we have a theorem about matrices and replace every matrix in the statement by its transpose, we get another theorem about matrices. If we have a theorem in projective geometry and systematically replace points by lines and lines by points and phrases like "the point P lies on the line ℓ" by "the line p passes through the line L", then we get another theorem in projective geometry. This is called the principle of **duality**. We illustrate this using the principle of duality to complete the proof of Desargues' Theorem (Theorem 6.16).

6.23. Consider $\triangle ABC$ with opposite sides a, b, c. Then

$$\begin{array}{lll} a = BC, & b = CA, & c = AB, \\ A = b \cap c, & B = c \cap a, & C = a \cap b. \end{array}$$

Similarly for $\triangle A'B'C'$ we have

$$\begin{array}{lll} a' = B'C', & b' = C'A', & c' = A'B', \\ A' = b' \cap c', & B' = c' \cap a', & C' = a' \cap b'. \end{array}$$

Define

$$\begin{array}{lll} X = a \cap a', & Y = b \cap b', & Z = c \cap c', \\ x = AA', & y = BB', & z = CC'. \end{array}$$

Then Desargues' Theorem is

$$x,\ y,\ z \text{ are concurrent} \iff X,\ Y,\ Z \text{ are collinear.}$$

We proved $\Longrightarrow$ in our proof of Theorem 6.16 above. This, together with the principle of duality, proves $\Longleftarrow$.

Remark 6.24. The theorems of Menelaus and Ceva (Theorems 3.48 and 3.49 above) appear to be dual as they are usually stated. When they are stated with directed distances instead of distances as above it becomes clear that they are not dual. Menelaus has a minus one in the conclusion and a quadratic equation in the proof; Ceva has a plus one in the conclusion and a cubic equation in the proof. One cannot transform one to the other by taking transposes of matrices.

Exercise 6.25. State the dual of Pappus' theorem and draw a diagram illustrating it.

6.5 The Projective Line

6.26. The **projective line** $\mathbb{P}^1$ is the space of lines through the origin in $\mathbb{R}^2$. In analogy with the projective plane each point $(x, y) \in \mathbb{R}^2$ with $(x, y) \neq (0, 0)$ determines a point

$$(x : y) := \{(tx, ty) \in \mathbb{R}^2 : t \in \mathbb{R}\}$$

in the projective line $\mathbb{P}^2$ and $(\mu x : \mu y) = (x : y)$ for $\mu \neq 0$. A transformation $M : \mathbb{P}^1 \to \mathbb{P}^1$ satisfying

$$\begin{matrix} x' = ax + by \\ y' = cx + dy \end{matrix} \implies (x' : y') = M(x : y)$$

where $ad - bc \neq 0$ is called a **projective transformation** (of the line). These definitions exactly parallel Definition 6.2 of the projective plane and Definition 6.11 of projective plane transformation.

Theorem 6.27. *Given three distinct points A, B, C in $\mathbb{P}^1$ there is a unique projective transformation of the line M such that*

$$M(1:0) = A, \qquad M(0:1) = B, \qquad M(1:1) = C.$$

Proof. The same as Theorem 6.15 but easier. □

6.28. If $y \neq 0$ then $(x : y) = (z : 1)$ where $z = x/y$. This establishes a one-one correspondence between $\mathbb{P}^1$ and and the space $\mathbb{R} \cup \{\infty\}$ obtained from the real line $\mathbb{R}$ by adjoining a point which we denote by ∞. The real number z corresponds to the point $(z : 1)$ on the projective line $\mathbb{P}^1$ and the point ∞

corresponds to the point $(1{:}0)$. With this identification $\mathbb{P}^1$ and $\mathbb{R} \cup \{\infty\}$ a projective transformation takes the form

$$M(z) = \frac{az+b}{cz+d} \tag{$\dagger$}$$

where it is understood that $M(z) = \infty$ if $cz + d = 0$, $M(\infty) = a/c$ if $c \neq 0$, and $M(\infty) = \infty$ if $c = 0$. These conventions are consistent with the following formulas from Math 221:

$$\lim_{z\to\infty} \frac{az+b}{cz+d} = \lim_{z\to-\infty} \frac{az+b}{cz+d} = \frac{a}{c}$$

if $c \neq 0$ and, for $z_0 = -d/c$,

$$\lim_{z\to z_0+} \frac{az+b}{cz+d} = \pm\infty, \qquad \lim_{z\to z_0-} \frac{az+b}{cz+d} = \pm\infty,$$

where the signs on the two $\pm\infty$'s are opposite. (In calculus we usually think of adjoining two points, $+\infty$ and $-\infty$, to the real numbers, but here there is only one infinity.) A transformation of form ($\dagger$) is called a **fractional linear transformation** or a **Möbius transformation**.

6.29. Let $A = (x_1{:}y_1{:}z_1)$ and $B = (x_2{:}y_2{:}z_2)$ be two distinct points in the projective plane $\mathbb{P}^2$. Define $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ by

$$\phi(t{:}s) = (x{:}y{:}z)$$

where

$$x = tx_1 + sy_1, \qquad y = ty_1 + sy_2, \qquad z = tz_1 + sz_2.$$

This definition is legal since $\phi(\mu t{:}\mu s) = (\mu x{:}\mu y{:}\mu z) = (x{:}y{:}z)$ for $\mu \neq 0$. It is easy to see that ϕ is a one-one correspondence between the projective line $\mathbb{P}^1$ and the line AB in $\mathbb{P}^2$. We call ϕ a **projective parameterization** of the line AB. Note that ϕ depends not just on the points A and B but also on the representatives chosen.

Exercise 6.30. Let A, B, C, be distinct collinear points in P^2. Show that there is a unique projective parameterization of the common line such that $\phi(1{:}0) = A$, $\phi(0{:}1) = B$, and $\phi(1{:}1) = C$.

Exercise 6.31. Assume that ϕ and ψ are projective parameterizations of the same line in $\mathbb{P}^2$. Show that there is a unique prpjective transformation $M : \mathbb{P}^1 \to \mathbb{P}^1$ of the projective line such that $\psi = \phi \circ M$.

Exercise 6.32. Assume that ϕ is a projective parameterization of a line ℓ in $\mathbb{P}^2$ and that $T : \mathbb{P}^2 \to \mathbb{P}^2$ is a projective transformation of the projective plane. Show that $T \circ \phi$ is a projective parameterization of a line $T(\ell)$.

6.6 Cross Ratio

Definition 6.33. The **cross ratio** of four distinct points A, B, C, D on the projective line is defined by

$$\{AB, CD\} := \frac{\det(A\,C) \cdot \det(B\,D)}{\det(A\,D) \cdot \det(B\,C)}.$$

In the formula the column vectors on the right are representatives respectively of the points by the same name. It does not matter which representatives are used since the cross ratio is unchanged if any of the four column vectors is replaced by a nonzero scalar multiple of itself since the scalar factors out in the numerator and denominator and then cancels. In particular, if $A = (a{:}1)$, $B = (b{:}1)$, $C = (c{:}1)$, $D = (d{:}1)$, then

$$\{AB, CD\} := \frac{(a-c)(b-d)}{(a-d)(b-c)}.$$

Theorem 6.34. *A projective transformation of the projective line preserves cross ratio.*

Proof. This is an immediate consequence of the fact that the determinant of the product is the product of the determinants, i.e.

$$\det(TA \;\; TB) = \det(T)\det(A\,B)$$

where A, B are 2×1 column vectors and T is a 2×2 matrix . $\square$

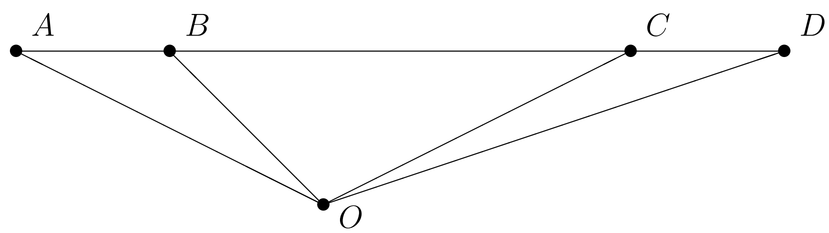


Figure 15: Cross Ratio

Exercise 6.35. Let $A, B, C, D \in \mathbb{R}^2 \setminus \{O\}$ represent distinct points in $\mathbb{P}^1$ and $O = (0,0)$ denote the origin in $\mathbb{R}^2$. Then

$$\begin{aligned}\{AB, CD\} &= \frac{(AOC) \cdot (BOD)}{(AOD) \cdot (BOC)} \\ &= \frac{\sin \angle AOC \cdot \sin \angle BOD}{\sin \angle AOD \cdot \sin \angle BOC} \\ &= \frac{(AC) \cdot (BD)}{(AD) \cdot (BC)}\end{aligned}$$

In the first formula (AOB) denotes the oriented area of $\triangle AOB$, and in the last formula it is assumed that the four points are collinear and (AB) denotes the directed distance along the common line. (The choice of direction doesn't matter since reversing the direction produces four sign reversals which cancel.) See Figure 6.6.

Remark 6.36. Recall the relation between projective geometry and drawing in perspective from Remark 6.5. From a perspective drawing of three points A, B, C, on a line in $\mathbb{R}^3$ we cannot conclude anything about the distances between them. If A', B', C' lie respectively on the line of sight from the origin $(0,0,0)$ with A, B, C then the points A', B', C' wll appear the same as the points A, B, C in the drawing although the ratios AB/AC and $A'B'/A'C'$ can be different. However in an accurate perspective drawing of four collinear points A, B, C, D, the cross ratio in the drawing will be the same as the cross ratio of the four points in space.

6.7 A Geometric Computer

In the following four exercises we will design a geometric computer. It works better than the one in Exercises 3.43-3.46 because that one required us to draw parallel lines whereas this one requires only a straightedge to connect two points with a line. Let O, I, Q, A, B, C be six collinear points such that O, I, Q are distinct and a, b, c denote the three cross ratios

$$a := \{AQ, IO\}, \qquad b := \{BQ, IO\}, \qquad c := \{CQ, IO\}.$$

Exercise 6.37. *(Addition in Projective Geometry.)* Assume given points M, N, P, A', B', C' satisfy the following conditions:

(a) the points M, N, P, Q are collinear,

(b) the points M, O, A' are collinear,

(c) the points M, B', C' are collinear,

(d) the points N, O, B' are collinear,

(e) the points N, A', C' are collinear,

(f) the points P, A', A are collinear,

(g) the points P, B', B are collinear, and

(i) the points P, C', C are collinear.

Draw a diagram illustrating this configuration and show that $c = a + b$. Hint: Take the common line of M, N, P, Q to be the line at infinity and use Exercise 3.43.

Exercise 6.38. *(Subtraction in Projective Geometry.)* State and prove a theorem analogous to Exercise 3.44 in the same way that Exercise 6.37 is analogous to Exercise 3.43. The conclusion should be that $b = -a$. Draw a diagram illustrating the configuration.

Exercise 6.39. *(Multiplication in Projective Geometry.)* State and prove a theorem analogous to Exercise 3.45 in the same way that Exercise 6.37 is analogous to Exercise 3.43. The conclusion should be that $c = ab$. Draw a diagram illustrating the configuration.

Exercise 6.40. *(Division in Projective Geometry.)* State and prove a theorem analogous to Exercise 3.46 in the same way that Exercise 6.37 is analogous to Exercise 3.43. The conclusion should be that $b = 1/a$. Draw a diagram illustrating the configuration.

7 Inversive Geometry

7.1 The complex projective line

7.2 Feuerbach's theorem

8 Klein's view of geometry

8.1 The elliptic plane

8.2 The hyperbolic plane

8.3 Special relativity

A Matrix Notation

Matrix notation is a handy way to describe messy calculations. The matrix operations obey most of the rules of ordinary algebra; the most important exception is the commutative law for multiplication. The material presented in this section contains all the matrix algebra we shall need. For more details consult any book in linear algebra.

A.1. Fix positive integers m and n. An $m \times n$ **matrix** is an array

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

where each entry a_{ij} is a number. There are m rows and n columns in an $m \times n$ matrix. We say that an $m \times n$ matrix is a matrix of **size** $m \times n$ when we want to call attention to the number of rows and columns. A **square matrix** is one having the same number of rows as columns; a **column vector**) is an $m \times 1$ matrix; a **row vector**) is an $1 \times n$ matrix.

A.2. To add or subtract matrices, add or subtract the corresponding entries as in

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{bmatrix}.$$

To multiply a matrix by a number, multiply each entry by the number as in

$$c \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \end{bmatrix}.$$

The zero matrix of any size has all its entries zero and is usually denoted by 0. Thus $A + 0 = A$.

A.3. If the number n of columns in A is the same as the number of rows in B the matrix product $C = AB$ is defined by the rule that the number c_{ij} in the ith row and jth column of C is given by

$$c_{ij} = \sum_k a_{ik} b_{kj}.$$

For example,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.$$

A.4. The $m \times m$ **identity matrix** I is the square matrix with 1's on the diagonal and 0's elsewhere. For example, if $m = 2$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The identity matrix behaves like the number 1: multiplication by I leaves a matrix unchanged.

A.5. A square matrix A is called **invertible** there is a (necessarily unique) matrix A^{-1} called the **inverse** of A which satisfies

$$AA^{-1} = A^{-1}A = I$$

where I is the identity matrix. The following calculation shows that if A is invertible, then the only matrix B satisfying $BA = I$ is $B = A^{-1}$:

$$B = BI = B(AA^{-1}) = (BA)A^{-1} = IA^{-1} = A^{-1}.$$

A square matrix is invertible if and only if its determinant is nonzero. For example, if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then $\det(A) = ad - bc$ and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

A.6. The **transpose** A^* of a matrix A is defined by the rule that the entry of A^* in the ith column and jth row is the same as the entry of A in the ith row and jth column. For example,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^* = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

and

$$\begin{bmatrix} x & y & z \end{bmatrix}^* = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

B Determinants

In elementary linear algebra (Math 340 at UW) one studies an operation which assigns to any square matrix A a number $\det(A)$ called the *determinant* of A. In this appendix we state its key properties. In the notes we only need determinants of 2×2 and 3×3 matrices; these are often taught in high school algebra. For these matrices one can check the properties using elementary algebra, although the 3×3 case is a bit hairy. For the proofs in the general case see any book on elementary linear algebra. (My favorite is [9].)

Theorem B.1. *There is a unique function called the* **determinant** *which assigns to each square matrix A a number* $\det(A)$ *satisfying the following properties:*

(Scale) *If B results from A by multiplying some row by a number c, then*

$$\det(B) = c \det(A).$$

(Swap) *If B results from A by interchanging two rows, then*

$$\det(B) = -\det(A).$$

(Shear) *If B results from A by adding a multiple of one row to another row, then*

$$\det(B) = \det(A).$$

(Identity) *The determinant of the identity matrix I is*

$$\det(I) = 1.$$

Exercise B.2. The determinant of a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

Verify the properties listed in Theorem B.1 using this formula.

Exercise B.3. The determinant of a 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is

$$\begin{aligned} \det(A) &= \quad a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad -a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33}. \end{aligned}$$

Verify the properties listed in Theorem B.1 using this formula.

Remark B.4. There is a general definition for the determinant of an $n \times n$ matrix as a sum

$$\det(A) = \sum_{\sigma} \pm a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

where the sum is over all $n!$ permutations of $\{1, 2, \ldots, n\}$. (The sign depends on the permutation σ in a subtle way.) We do not need this formula in these notes and will not explain it.

Theorem B.5. *A square matrix is invertible if and only if its determinant is not zero.*

Exercise B.6. Find AB and BA where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and

$$B = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Then show that A is invertible if and only if its determinant is not zero and give a formula for A^{-1}.

Theorem B.7. *The determinant of a matrix and its transpose are the same:* $\det(A) = \det(A^*)$. *Hence the properties of Theorem B.1 hold reading "column" for "row".*

Exercise B.8. Check this for 2×2 matrices.

Theorem B.9. *For a square matrix A, the homogeneous system $AX = 0$ has a nonzero solution X if and only if* $\det(A) = 0$*.*

Exercise B.10. Check this for 2×2 matrices.

Corollary B.11. *The determinant of a square matrix vanishes if and only if one of its columns is a linear combination of the others.*

Theorem B.12. *The determinant function satisfies the following:*

(1) $\det(I) = 1$*;*

(2) $\det(A^{-1}) = \det(A)^{-1}$*;*

(3) $\det(AB) = \det(A)\det(B)$*.*

Exercise B.13. Check this for 2×2 matrices.

C Sets and Transformations

C.1. A **set** X divides the mathematical universe into two parts: those objects x that **belong** to X and those that don't. The notation $x \in X$ means x belongs to X. The notation $x \notin X$ means that x does not belong to X. In geometry the word *locus* is often used as a synonym for *set* as in the sentence *The locus of all points P such that $\angle APB$ is a given constant is an arc of a circle.*

C.2. If X is a set and $P(x)$ is a property that either holds or fails for each element $x \in X$, then we may form a new set S consisting of all $x \in X$ for which $P(x)$ is true. This set S is denoted by

$$S = \{x \in X : P(x)\} \tag{1}$$

and called "the set of all $x \in V$ such that $P(x)$". The set S is a **subset** of X meaning that every element of S is an element of X. The notation used in equation (1) is called **set builder notation**. Having defined S by (1), we may assert that for all x

$$x \in S \iff x \in X \text{ and } P(x)$$

and that for all $x \in X$

$$x \in S \iff P(x).$$

Since the property $P(x)$ may be quite cumbersome to state, the notation $x \in W$ is both shorter and easier to understand. (The symbol $\iff$ is an abbreviation for *if and only if.*)

C.3. Let X and Y be sets. The notation $f : X \to Y$ means that f is a function which assigns to each point $x \in X$ and element $f(x) \in Y$. Mathematicians have more words for this concept than any other: calculus textbooks call f a **function** with domain X (or defined on X) taking values in Y, while textbooks on linear algebra or plane geometry call f a **transformation** from X to Y, and in more advanced mathematics one says that f is a **map** (or mapping) from X to Y. In these notes we use the word *transformation* and almost always we take $X = Y = \mathbb{R}^2$ = the set of pairs of real numbers. We will also use the letter T rather than f.

C.4. For any set X there is a transformation $I_X : X \to X$ called the **identity transformation** of X and defined by

$$I_X(x) = x$$

for $x \in X$. The composition $g \circ f : X \to Z$ of two transformations $f : X \to Y$ and $g : Y \to Z$ is defined by

$$(g \circ f)(x) = g(f(x))$$

for $x \in X$. A transformation $f : X \to Y$ is called **invertible** iff there is a (necessarily unique) transformation $f^{-1} : Y \to X$ such that

$$f^{-1}(f(x)) = x, \qquad \text{and} \qquad f(f^{-1}(y)) = y$$

for $x \in X$ and $y \in Y$. The transformation f^{-1} is called the **inverse transformation** to f. Clearly $(h \circ g) \circ f = h \circ (f \circ g)$, $f \circ I_X = f$, $I_Y \circ f = f$, and

$$f^{-1} \circ f = I_X, \qquad \text{and} \qquad f \circ f^{-1} = I_Y.$$

C.5. When $f : X \to Y$ and S is a subset of X we define the **image** of S by f to be the set of all points $f(x)$ as x ranges over S. It is denoted by $f(S)$. In set builder notation this is written as

$$f(S) := \{f(x) : x \in S\}. \tag{2}$$

(The notation := is often used to emphasize that the right hand side is the definition of the left hand side, so that no proof is required.) To prove that a point y lies in the set $f(X)$ one must show there is an $x \in S$ with $y = f(x)$.

Example C.6. In calculus one learns that

$$x = \cos\theta, \qquad y = \sin\theta \tag{3}$$

are parametric equations for the unit circle. In the notation introduced thus far this could be written

$$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} = \{(\cos\theta, \sin\theta) : \theta \in \mathbb{R}\}. \tag{4}$$

The left hand side uses the set builder notation of equation (1) while the right hand side uses the set builder notation of equation (2). It is also true that

$$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} = \{(\cos\theta, \sin\theta) : 0 \le \theta < 2\pi\}.$$

C.7. To prove that two sets are equal on must show that every element of one set is an element of the other set and vice versa. For example, to prove (4) we argue as follows. If (x,y) lies in the set defined on the right hand side of (4), then (3) holds for some number θ. Hence $x^2 + y^2 = \cos^2\theta + \sin^2\theta = 1$ by the Pythagorean Theorem so (x,y) lies in the set on the left hand side of (4). Conversely, if $x^2 + y^2 = 1$, then $(x,y) = (\cos\theta, \sin\theta)$ where $\theta = \tan^{-1}(y/x)$ if $x > 0$, $\theta = \tan^{-1}(y/x) + \pi$ if $x < 0$, $\theta = \pi/2$ if $(x,y) = (0,1)$, and $\theta = 3\pi/2$ if $(x,y) = (0,-1)$. The case analysis is necessary because the equivalence

$$m = \tan\theta \iff \theta = \tan^{-1}(m)$$

holds only if $-\pi/2 < \theta < \pi/2$.

References

[1] E. T. Bell, *Men of Mathematics*, Simon & Schuster, 1937.

[2] L. M. Blumenthal, *A Modern View of Geometry*, W. H. Freeman, 1961.

[3] H. S. M. Coxeter & S. L. Greitzer, *Geometry Revisited*, Random House, 1967, Reissued by the MAA (see `http://www.maa.org/pubs/books/nml19.html`) A classic written by a famous geometer.

[4] I. Martin Isaacs, *Geometry for College Students*, Brooks Coles, 2001. A beautiful book with lots of problems written by a professor at UW.

[5] H. Levy, *Projective and Related Geometries*, Macmillan, 1961. Professor Levy introduced me to the subject in a course at the University of Illinois in 1961-62.

[6] E. A. Maxwell, *Fallacies in Mathematics*, Cambridge University Press, 1959. (See `http://books.cambridge.org/0521057000.htm`)

[7] G. E. Martin: *Transformation Geometry, An Introduction to Symmetry*, Springer Undergraduate Texts in Math, 1982.

[8] MathWorld Website: `http://mathworld.wolfram.com/`. A google search on any term from elementary geometry (like “Gergonne Point”) usually brings up this websiite.

[9] J. Robbin, *Matrix Algebra, Using MINImal MATlab*, A.K. Peters, 1995.