Math 461 Homework and syllabus

JWR

Monday August 29, 2005

When and where
> 12:05 MWF, B113 VAN VLECK

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Course Home Page
> http://www.math.wisc.edu/~robbin/461dir/461.html

My Office Hours
> 3:30 Monday, 9:00 Wednesday, 10:00 Friday.

Text
> Coxeter and Greitzer: *Geometry Revisited*, MAA.

Final Exam
> 07:45 A.M. THU. DEC 22

Policy

In both the exams and the homework problems, I am aiming for a high quality of writing and reasoning. *Do not give me messy or incoherent work.*

The secret to clear writing is to fix on what you have already told the reader before you write each sentence. Do not use a symbol before you define

it. If I write something like *What is A?* on your paper, it means you have violated this rule. Similarly avoid pronouns. If find your self saying "it" a lot, it may mean that you are not making key distinctions. Anything you write should be reasonably self contained. Write as if your reader does not know the statement of the problem you are solving. If you use a theorem from the book or refer to the statement of the problem or a diagram from the assignment sheet, so indicate. For example, you can write something like *Refer to figure 1.9B on page 24 of the text.* Write as if your reader is a student at your level who is not taking the course, not as if I am the audience. I hope this discipline will improve the writing.

The above advice on clear writing as applies to writing proofs. A proof is a sequence of steps each of which is either an axiom, a hypothesis, an unravelling of a definition, or follows from earlier steps by application of a previously proved general principle. When you write a proof, be sure that each step follows from earlier steps.

You are encouraged to collaborate since teaching another student something is a good way to learn it yourself. However, do not present someone else's work as your own: acknowledge any discussions you've had on any written work you do outside of class. (Write something like: "On this problem I benefitted from a discussion with Jane Doe." This is what mathematicians do when they write papers. If you learn something from a book or other written source, cite it. (This is also what mathematicians do.) Write any work you hand in yourself. There is no benefit in copying something verbatim.

Prior to each midterm and the final you will be given a list of questions from which the exam will be constructed. Again I am aiming for good writing rather than quick responses. My current intention is to give 4 exams (the last is the final), one for each of the major topics: Affine Geometry, Euclidean Geometry, Theorems in Euclidean Geometry, Projective Geometry.

The notes for this course are online at the course web site. Hard copy will be distributed but not all at once.

Assignment I

1. Explain the error in each of the three fallacies in Chapter 2 (pages 5-7) of the notes.

2. Find $a_1, \ldots, a_6$ such that

$$
\begin{aligned}
x &= a_1 x + a_2 y + a_3, \\
y &= a_4 x + a_5 y + a_6,
\end{aligned}
$$

for all x and y. Warning: Don't work too hard.

2. Assume that a transformation $(x', y') = A(x, y)$ is defined by the equations

$$\begin{aligned} x' &= a_1 x + a_2 y + a_3, \\ y' &= a_4 x + a_5 y + a_6, \end{aligned}$$

and a transformation $(x'', y'') = B(x', y')$ is defined by the equations

$$\begin{aligned} x'' &= b_1 x' + b_2 y' + b_3, \\ y'' &= b_4 x' + b_5 y' + b_6. \end{aligned}$$

Find $c_1, \ldots, c_6$ so that the transformation $(x'', y'') = B(A(x, y))$ is defined by the equations

$$\begin{aligned} x'' &= c_1 x + c_2 y + c_3, \\ y'' &= c_4 x + c_5 y + c_6. \end{aligned}$$

3. Assume that a transformation $(x', y') = A(x, y)$ is defined by the equations

$$\begin{aligned} x' &= a_1 x + a_2 y + a_3, \\ y' &= a_4 x + a_5 y + a_6, \end{aligned}$$

and that $a_1 a_5 - a_2 a_4 = 1$. Find $b_1, \ldots, b_6$ such that the inverse transformation $(x, y) = A^{-1}(x', y')$ is defined by the equations

$$\begin{aligned} x &= b_1 x' + b_2 y' + b_3, \\ y &= b_4 x' + b_5 y' + b_6. \end{aligned}$$

Assignment II

1. Let the corresponding sides of two triangles $\triangle ABC$ and $\triangle A'B'C'$ intersect in

$$X = BC \cap B'C', \qquad Y = CA \cap C'A', \quad Z = AB \cap A'B'.$$

Show that if the lines AA', BB', CC' are parallel, then the points X, Y, Z are collinear. Draw a picture. Hint: Choose coordinates (x, y) so that the lines AA', BB', CC' are the vertical lines $x = a$, $x = b$, $x = c$. Then $A = (a, p)$, $A' = (a, p')$, $B = (b, q)$, $B' = (b, q')$, $C = (c, r)$, $C' = (c, r')$. Calculate the coordinates of X, Y, Z, and use a theorem about determinants.

2. Assume that the three points A, B, C are collinear and that the three points $A', B,' C'$ are collinear. Let the lines joining them in pairs intersect as follows:

$$X = BC' \cap B'C, \qquad Y = CA' \cap C'A, \quad Z = AB' \cap A'B.$$

Show that if the lines ABC and $A'B'C'$ are parallel, then the points X,Y, Z are collinear. Draw a picture. Hint: Choose coordinates (x, y) so that the line ABC has equation $y = 0$ and the line $A'B'C'$ has equation $y = 1$. Then $A = (a, 0)$, $B = (b, 0)$, $C = (c, 0)$, $A' = (a', 1)$, $B' = (b', 1)$, $C' = (c', 1)$. Calculate $X = (x_1, x_2)$, $Y = (y_1, y_2)$, and $Z = (z_1, z_2)$ in terms and show that the slope of the line XY is the same as the slope of the line XZ.

3. Points P and Q are selected on two sides of $\triangle ABC$, as shown, and segments AQ and BP are drawn. Then QX PY are drawn parallel to BP and AQ, respectively. Show that $XY \| AB$.
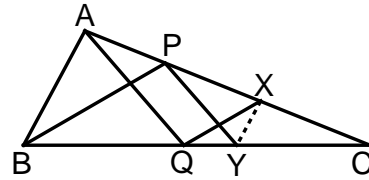
Assignment III

1. *(Addition in Affine Geometry)* Let $O = (0, 0)$, $A = (a, 0)$, $B = (b, 0)$, $C = (c, 0)$ be four points on the x-axis and (O, A', B', C') be a parallelogram such that the lines AA', BB', CC' are parallel, i.e. the lines OA' and $B'C'$ are parallel and the lines OB' and $A'C'$ are parallel. Show that $c = a + b$ Draw a picture.

2. *(Subtraction in Affine Geometry)* Let $O = (0, 0)$, $A = (a, 0)$, $B = (b, 0)$ be three points on the x-axis and (O, A', O', B') be a parallelogram such that the lines OO', AA'', BB' are parallel. Show that $b = -a$. Draw a picture.

3. *(Multiplication in Affine Geometry)* Let $O = (0, 0)$, $I = (1, 0)$, $A = (a, 0)$, $B = (b, 0)$, $C = (c, 0)$ be five points on the x-axis, and O', I', B' be three points such that (a) the lines OO', II', BB' are parallel, (b) the lines IB and $I'B'$ are parallel, (c) the points O', I', A are collinear, and (d) the points O', B', C are collinear. Show that $c = ab$. Draw a picture.

4. *(Division in Affine Geometry)* Let $O = (0, 0)$, $I = (1, 0)$, $A = (a, 0)$, $B = (b, 0)$ be four points on the x-axis, and O', I', B' be three points such that (a) the lines OO',BB', II' are parallel, (b) the lines IB and $I'B'$ are parallel, (c) the points O', I', A are collinear, and (d) the points O', B', I are collinear. Show that $b = 1/a$. Draw a picture.

Assignment IV

1. Show that the medians of a triangle divide it into six triangles of equal area. Hint: It is enough to prove this for an equilateral triangle.

2. Show that the centroid of a triangle divides each median into two segments one of which is twice as long as the other.

3. In the figure, vertices B and C of $\triangle ABC$ are joined to points P and Q on the opposite sides, and lines BP and CQ meet at point X. Suppose that $BX = (2/3)BP$ and $CX = (2/3)CQ$. Prove that BP and CQ are medians of $\triangle ABC$.
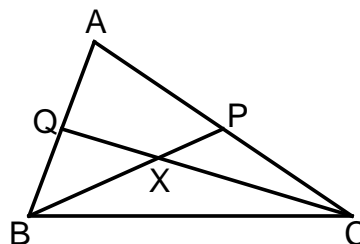
4. Show that there is no point P inside $\triangle ABC$ such that *every* line through P cuts the triangle into two pieces of equal area. Hint: Show that if there were such a point, it would have to lie on each median of the triangle.

5. In the figure, the side BC of $\triangle ABC$ is trisected by points R and S. Similarly, T and U trisect side AC and V and W trisect side AB. Each vertex of $\triangle ABC$ is joined to the two trisection points on the opposite side, and the intersections of these trisecting lines determine $\triangle XYZ$, as shown. Prove that the sides of $\triangle XYZ$ are parallel to the sides of $\triangle ABC$.
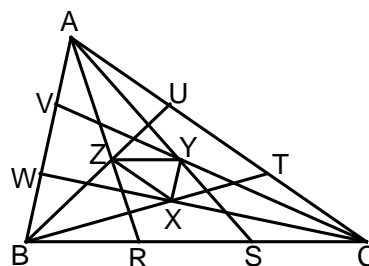
Assignment V

1. Points W, X, Y and Z are the midpoints of the sides of quadrangle $ABCD$ as shown, and P is the intersection of WY with XZ. Two of the four small quadrangles are shaded. Show that P is the midpoint of both WY and XZ and that the shaded area is exactly half of the area of quadrangle $ABCD$. Hint: For the second part, decompose the whole area into four triangles so that exactly half the area of each triangle is shaded.
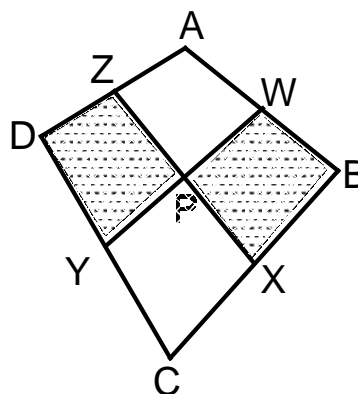
2. Points P and Q are chosen on two sides of $\triangle ABC$, as shown, and lines BP and QC meet at X. Show that X lies on the median from vertex A if and only if $QP \| BC$.
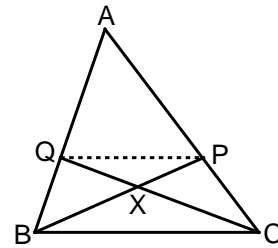
3. Given $\triangle ABC$, let A' be the point $1/3$ of the way from B to C, as shown. Similarly, B' is the point $1/3$ of the way from C to A and C' lies $1/3$ of the way from A to B. In this way, we have constructed a new triangle, $\triangle A'B'C'$ starting with an arbitrary triangle. Now apply the same procedure to $\triangle A'B'C'$, thereby creating $\triangle A''B''C''$. Show that the sides of $\triangle A''B''C''$ are parallel to the (appropriate) sides of $\triangle ABC$. What fraction of the area of $\triangle ABC$ is the area of $\triangle A''B''C''$?
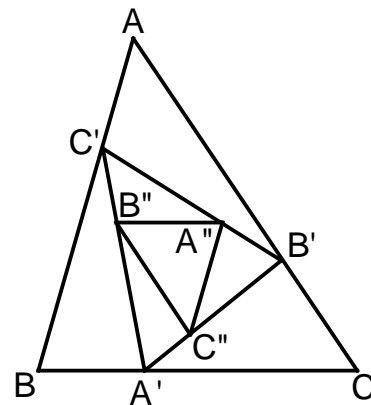
4. If we draw two medians of a triangle, we see that the interior of the triangle is divided into four pieces: three triangles and a quadrilateral. Prove that two of these small triangles have equal areas, and show that the other small triangle has the same area as the quadrilateral.