Math 320 Spring 2009
Part III – Linear Systems of Diff EQ

JWR

May 14, 2009

1 Monday March 30

The Existence and Uniqueness Theorem for Ordinary Differential Equations which we studied in the first part of the course has a vector version which is sill valid. Here is the version (for a single equation) from the first part of the course, followed by the vector version.

Theorem 1 (Existence and Uniqueness Theorem). *Suppose that $f(t, y)$ is a continuous function of two variables defined in a region R in (t, y) plane and that the partial $\partial f/\partial y$ exists and is continuous everyhere in R. Let (t_0, y_0) be a point in R. Then there is a solution $y = y(t)$ to the initial value problem*

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0$$

defined on some interval I about t_0. The solution is unique in the sense that any two such solutions of the initial value problem are equal where both are defined.

Theorem 2 (Existence and Uniqueness Theorem for Systems). *Assume that $\mathbf{f}(t, \mathbf{x})$ is a (possibly time dependent) vector field on $\mathbb{R}^n$, i.e. a function which assigns to each time t and each vector $\mathbf{x} = (x_1, \dots, x_n)$ in $\mathbb{R}^n$ a vector $\mathbf{f}(t, \mathbf{x})$ in $\mathbb{R}^n$. Assume that $\mathbf{f}(t, \mathbf{x})$ is continuous in $(t, \mathbf{x})$ and that the partial derivatives in the variables x_i are continuous. Then for each initial time t_0 and each point $\mathbf{x}_0$ in $\mathbb{R}^n$ the initial value problem*

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

defined on some interval I about t_0. The solution is unique in the sense that any two such solutions of the initial value problem are equal where both are defined.

Proof. See Theorem 1 on page 682 in Appendix A.3 and Theorem 1 on page 683 in Appendix A.4 of the text. $\square$

3. For the rest of this course we will study the special case of linear system where the vector field $\mathbf{f}(t, \mathbf{x})$ has the following form In that case the vector field $\mathbf{f}(t, \mathbf{x})$ has the form

$$\mathbf{f}(t, \mathbf{x}) = -\mathbf{A}(t)\mathbf{x} + \mathbf{b}(t)$$

where $\mathbf{A}(t)$ is a continous $n \times n$ matrix valued function of t and $\mathbf{b}(t)$ is a continuous vector valued function of t with values in $\mathbb{R}^n$. A system of this form is called a **linear system of differential equations**. We shall move the term $-\mathbf{A}(t)\mathbf{x}$ to the other side so the system takes the form

$$\frac{d\mathbf{x}}{dt} + \mathbf{A}(t)\mathbf{x} = \mathbf{b}(t) \tag{1}$$

In case the the right hand side $\mathbf{b}$ is identically zero we say that the system is **homogeneous** otherwise it is called **inhomogeneous** or **non homogeneous**. If

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Then $\mathbf{f} = (f_1, \dots, f_n)$ where $f_i = -a_{i1}(t)x_1 - a_{i2}(t)x_2 - \dots - a_{in}(t)x_n + b_i(t)$ and the system (1) can be written as n equations

$$\frac{dx_i}{dt} + a_{i1}(t)x_1 + a_{i2}(t)x_2 + \dots + a_{in}(t)x_n = b_i(t), \qquad i = 1, 2, \dots n \tag{2}$$

in n unknowns $x_1, \dots, x_n$ where the a_{ij} and $b_i(t)$ are given functions of t. For the most part we shall study the case where the coefficients a_{ij} are constant. Using matrix notation makes the theory of equation (1) look very much like the theory of linear first order differential equations from the first part of this course.

4. In the case of linear systems of form (2) the partial derivatives are automatically continuous. This is because $\partial f_i/\partial x_j = -a_{ij}(t)$ and the matrix $\mathbf{A}(t)$ is assumed to be a continuous function of t. Hence the Existence and Uniqueness Theorem 2 apples. But something even better happens. In the general case of a nonlinear system solutions can become infinite in finite time. For example (with $n = 1$) the solution to the nonlinear equation $dx/dt = x^2$ is $x = x_0/(1 - x_0 t)$ which becomes infinite when $t = 1/x_0$. But the following theorem says that in the linear case this doesn't happen.

Theorem 5 (Existence and Uniqueness for Linear Systems)**.** *Let t_0 be a real number and $\mathbf{x}_0$ be a point in $\mathbb{R}^n$ then the differential equation (1) has a unique solution $\mathbf{x}$ defined for all t satisfying the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$.*

Proof. See Theorem 1 on page 399 of the text and Theorem 1 page 681 of Appendix A.2. □

Definition 6. A **nth order linear differential equation** is of form

$$\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = f(t) \tag{3}$$

where the functions $p_1(t), \ldots, p_n(t)$, and $f(t)$ are given and the function y is the unknown. If the function $f(t)$ vanishes identically, the equation is called **homogeneous** otherwise it is called **inhomogeneous**. For the most part we shall study the case where the coefficients $p_1, \ldots, p_n$ are constant.

7. The text treats nth order equations in chapter 5 and systems in chapter 7, but really the former is a special case of the latter. This is because after introducing the variables

$$x_i = \frac{d^{i-1}y}{dt^{i-1}}$$

the equation (3) becomes the system

$$\frac{dx_i}{dt} - x_{i+1} = 0, \qquad i = 1, \ldots, n-1$$

and

$$\frac{dx_n}{dt} + p_1(t)x_{n-1} + \cdots + p_{n-2}(t)x_2 + p_n(t)x_1 = f(t).$$

For example the 2nd order equation

$$\frac{d^2y}{dt^2} + p_1(t)\frac{dy}{dt} + p_2(t)y = f(t)$$

becomes the linear system

$$\begin{cases} \dfrac{dx_1}{dt} - x_2 & = 0 \\ \dfrac{dx_2}{dt} + p_1(t)x_2 + p_2(t)x_1 & = f(t) \end{cases}$$

in the new variables $x_1 = y$, $x_2 = dy/dt$. In matrix notation this linear system is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ p_2(t) & p_1(t) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}.$$

For this reason the terminology and theory in chapter 5 is essentially the same as that in chapter 7.

Theorem 8 (Existence and Uniqueness for Higher Order ODE's)**.** *If the functions $p_1(t), \dots, p_{n-1}(t), p_n(t), f(t)$ are continuous then for any given numbers $t_0, y_0, \dots, y_{n-1}$ the nth order system (3) has a unique solution defined for all t satisfying the initial condition*

$$y(t_0) = y_0, \quad y'(t_0) = y_1, \quad \dots, \quad y^{(n-1)}(t_0) = y_{n-1}$$

Proof. This is a corollary of Theorem 5. See Theorem 2 page 285, Theorem 2 page 297. □

Theorem 9 (Superposition)**.** *Suppose $\mathbf{A}(t)$ is a continuous $n \times n$ matrix valued function of t. Then the solutions of the homogeneous linear system*

$$\frac{d\mathbf{x}}{dt} + \mathbf{A}(t)\mathbf{x} = \mathbf{0} \tag{4}$$

of differential equations form a vector space In particular, the solutions of a higher order homogeneous linear differential equation form a vector space.

Proof. The show that the set of solutions is a vector space we must check three things:

1. The constant function $\mathbf{0}$ is a solution.
2. The sum $\mathbf{x}_1 + \mathbf{x}_2$ of two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ is a solution.
3. The product $c\mathbf{x}$ of a constant c and a solution $\mathbf{x}$ is a solution.

(See Theorem 1 page 283, Theorem 1 page 296, and Theorem 1 page 406 of the text.) □

Theorem 10 (Principle of the Particular Solution)**.** *Let* $\mathbf{x}_p$ *is a solution is a particular solution of the non homogeneous linear system*

$$\frac{d\mathbf{x}}{dt} + \mathbf{A}(t)\mathbf{x} = \mathbf{b}(t).$$

Then if $\mathbf{x}$ *is a solution of the corresponding homogeneous linear system*

$$\frac{d\mathbf{x}}{dt} + \mathbf{A}(t)\mathbf{x} = \mathbf{0}$$

then $\mathbf{x} + \mathbf{x}_p$ *solves the nonhomogeneous system and conversely every solution of the nonhomogeneous system has this form.*

Proof. The proof is the same as the proof of the superposition principle. The text (see page 490) says it a bit differently: *The general solution of the nonhomogeneous system is a particilar solution of the nonhomogeneous system plus the general solution of the corresponding homogeneous system.* □

Corollary 11. *Let* y_p *be a solution is a particular solution of the non homogeneous higher order differential equation*

$$\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = f(t).$$

Then if y *is a solution of the corresponding homogeneous higher order differential equation*

$$\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = 0$$

then $y + y_p$ *solves the nonhomogeneous differential equation and conversely every solution of the nonhomogeneous differential equation has this form.*

Proof. Theorem 4 page 411. □

Theorem 12. *The vector space of solutions of (4) has dimension* n*. In particular, the vector space of solutions of (3) has dimension* n*.*

Proof. Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^n$. By Theorem 2 there is a unique solution $\mathbf{x}_i$ of (4) satisfying $\mathbf{x}_i(0) = \mathbf{e}_i$. We show that these solutions form a basis for the vector space of all solutions.

The solutions $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ *span the space of solutions.* If $\mathbf{x}$ is any solution, the vector $\mathbf{x}(0)$ is a vector in $\mathbb{R}^n$ and is therefore a linear combination $\mathbf{x}(0) = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \cdots + c_n\mathbf{e}_n$ of $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$. Then $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t)$ is a solution (by the Superposition Principle) and agrees with $\mathbf{x}$ at $t = 0$ (by construction) so it must equal $\mathbf{x}(t)$ for all t (by the Existence and Uniqueness Theorem).

The solutions $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ *are independent.* If $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) = 0$ then evaluating at $t = 0$ gives $c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \cdots + c_n\mathbf{e}_n = 0$ so $c_1 = c_2 = \cdots = c_n = 0$ as $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ are independent. □

2 Wednesday April 1, Friday April 3

13. To proceed we need to understand the complex exponential function. Euler noticed that when you substitute $z = i\theta$ into the power series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{5}$$

you get

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{i^n\theta^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{i^{2k}\theta^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{i^{2k+1}\theta^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k\theta^{2k}}{(2k)!} + i\sum_{k=0}^{\infty} \frac{(-1)^k\theta^{2k+1}}{(2k+1)!} \\ &= \cos\theta + i\sin\theta. \end{aligned}$$

This provides a handy way of remembering the trigonometric addition formulas:

$$e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

and

$$\begin{aligned} e^{i\alpha}e^{i\beta} &= \big(\cos\alpha + i\sin\alpha\big)\big(\cos+\beta + i\sin\beta\big) \\ &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \end{aligned}$$

so equating the real and imaginary parts we get

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta, \quad \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Because $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$ we have

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}.$$

Note the similarity to

$$\cosh t = \frac{e^t+e^t}{2}, \qquad \sinh t = \frac{e^t-e^{-t}}{2}.$$

It is not difficult to prove that the series (5) converges for complex numbers z and that $e^{z+w} = e^z e^w$. In particular, if $z = x+iy$ where x and y are real then

$$e^z = e^x e^{iy} = e^x(\cos y + i\sin y) = e^x\cos y + ie^x\sin y$$

so the real and imaginary parts of e^z are given

$$\Re e^z = e^x\cos y = \frac{e^z+e^{\bar z}}{2}, \qquad \Im e^z = e^x\sin y = \frac{e^z-e^{\bar z}}{2i}.$$

14. We will use the complex exponential to find solutions to the nth order linear homgeneous constant coefficient differential equation

$$\frac{d^ny}{dt^n} + p_1\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}\frac{dy}{dt} + p_ny = 0 \tag{6}$$

As a warmup let's find all the solutions of the linear homogeneous 2nd order equation

$$a\frac{d^y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

where a, b, c are constants and $a \neq 0$. (Dividing by a puts the equation in the form (3) with $n = 1$ and $p_1 = b/a$ and $p_2 = c/a$ constant.) As an ansatz we seek a solution of form $y = e^{rt}$. Substituting gives

$$(ar^2 + br + c)e^{rt} = 0$$

so $y = e^{rt}$ is a solution iff

$$ar^2 + br + c = 0, \tag{7}$$

i.e. if $r = r_1$ or $r = r_2$ where

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The equation (7) is called the **charactistic equation** (or sometimes the **auxilliary equation**). There are three cases.

1. $b^2 - 4ac > 0$. In this case the solutions r_1 and r_2 are distinct and real and for any real numbers c_1 and c_2 the function $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ satisfies the differential equation.

2. $b^2 - 4ac < 0$. In this case the roots are complex conjugates:

$$r_1 = r = \frac{-b}{2a} + i\frac{\sqrt{4ac - b^2}}{2a}, \qquad r_2 = \bar{r} = \frac{-b}{2a} - i\frac{\sqrt{4ac - b^2}}{2a}.$$

   The functions e^{rt} and $e^{\bar{r}t}$ are still solutions of the equation because calculus and algebra works the same way for real numbers as for complex numbers. The equation is linear so linear combinations of solutions are solutions so the real and imaginary parts

$$y_1 = \frac{e^{rt} + e^{\bar{r}t}}{2}, \qquad y_2 = \frac{e^{rt} - e^{\bar{r}t}}{2i}$$

   of solutions are solutions and for any real numbers c_1 and c_2 the function $y = c_1 y_1 + c_2 y_2$ is a real solution.

3. $b^2 - 4ac = 0$. In this case the characteristic equation (7) has a double root so $aD^2 + bD + c = a(D - r)^2$ where $r = b/(2a)$. (For the moment interpret D as an indeterminate; we'll give another interpretation later.) It is easy to check that both e^{rt} and te^{rt} are solutions so $y = c_1 e^{rt} + c_2 te^{rt}$ is a solution for any constants c_1 and c_2.

Example 15. For any real numbers c_1 and c_2 the function

$$y = c_1 e^{2t} + c_2 e^{3t}$$

is a solution the differential equation

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 0.$$

Example 16. For any real numbers c_1 and c_2 the function

$$y = c_1 e^t \cos t + c_2 e^t \sin t$$

is a solution the differential equation

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = 0.$$

Example 17. For any real numbers c_1 and c_2 the function

$$y = c_1 e^t + c_2 t e^t$$

is a solution the differential equation

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 0.$$

3 Friday April 3 – Wednesday April 8

18. It is handy to introduce **operator notation**. If

$$p(r) = r^n + p_1 r^{n-1} + \cdots + p_{n-1} r + p_n$$

and y is a function of t, then

$$p(D)y := \frac{d^n y}{dt^n} + p_1 \frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}\frac{dy}{dt} + p_n y$$

denotes the left hand side of equation (6). This gives the differential equation the handy form $p(D)y = 0$. The characteristic equation of this differential equation is the algebraic equation $p(r) = 0$, i.e.

$$y = e^{rt} \implies p(D)y = p(D)e^{rt} = p(r)e^{rt} = p(r)y$$

so $y = e^{rt}$ is a solution when $p(r) = 0$. When $p(x) = x^k$ we have that $p(D)y = D^k y$, i.e.

$$D^k y = \frac{d^k y}{dt^k} = \left(\frac{d}{dt}\right)^k y$$

is the result of differentiating k times. Here is what makes the whole theory work:

Theorem 19. *Let pq denote the product of the polynomial p and the polynomial q, i.e.*

$$(pq)(r) = p(r)q(r).$$

Then for any function y of t we have

$$(pq)(D)y = p(D)q(D)y$$

where $D = d/dt$.

Proof. This is because $D(cy) = cDy$ if c is a constant. $\square$

Corollary 20. $p(D)q(D)y = q(D)p(D)y$.

Proof. $pq = qp$. $\square$

21. If $q(D) = 0$ then certainly $pq(D)y = 0$ by the theorem and if $p(D)y = 0$ then $pq(D)y = 0$ by the theorem and the corollary. This means that we can solve a homogeneous linear constant coefficient equation by factoring the characteristic polynomial.

Theorem 22 (Fundamental Theorem of Algebra)**.** *Every polynomial has a complex root.*

Corollary 23. *Every real polynomial $p(r)$ factors as a product of (possibly repeated) linear factors $r - c$ where c is real and quadratic factors $ar^2 + br + c$ where $b^2 - 4ac < 0$.*

24. A basis for the solution space of $(D - r)^k y = 0$ is

$$e^{rt}, te^{rt}, \ldots, t^{k-1}e^{rt}.$$

A basis for the solution space of $(aD^2 + bD + c)^k y = 0$ is

$$e^{pt}\cos(qt), e^{pt}\sin(qt), \ldots, t^{k-1}e^{pt}\cos(qt), t^{k-1}e^{pt}\sin(qt)$$

where $r = p \pm qi$ are the roots of $ar^2 + br + c$, i.e. $p = \frac{-b}{2a}$ and $q = \frac{\sqrt{b^2-4ac}}{2a}$.

Example 25. A basis for the solutions of the equation

$$(D^2 + 2D + 2)^2(D - 1)^3 y = 0$$

is

$$e^t \cos t, e^t \sin t, te^t \cos t, te^t \sin t,\; e^t, te^t, t^2e^t.$$

26. This reasoning can also be used to solve inhomogeneous constant coefficient linear differential equation

$$p(D)y = f(t)$$

where the homogeneous term $f(t)$ itself solves solves a homogeneous constant coefficient linear differential equation

$$q(D)f = 0.$$

This is because any solution y of $p(D)y = f(t)$ will then solve the homogeneous equation $(qp)(D)y = 0$ and we can compute which solutions of $(qp)(D)y = 0$ also satisfy $p(D)y = f(t)$ by the **method of undetermined coefficients**. Here's an

Example 27. Solve the initial value problem

$$\frac{dy}{dt} + 2y = e^{3t}, \qquad y(0) = 7.$$

Since e^{3t} solves the problem $(D-3)e^{3t} = 0$ we can look for the solutions to $(D+2)y = e^{3t}$ among the solutions of $(D-3)(D+2)y = 0$. These all have the form

$$y = c_1 e^{-2t} + c_2 e^{3t}$$

but not every solution homogeneous second order equation solves the original first order equation. To see which do, we plug in

$$(D+2)(c_1 e^{-2t} + c_2 e^{3t}) = (D+2)c_2 e^{3t} = (3c_2 + 2c_2)e^{3t} = e^{3t}$$

if and only if $c_2 = 1/5$. Thus $y_p = e^{3t}/5$ is a *particular solution* of the inhomogeneous equation. By the *Principle of the Particular Solution* above the *general solution* to the inhomogeneous equation is the particular solution plus the general solution to the homogeneous problem, i.e.

$$y = c_1 e^{-2t} + \frac{e^{3t}}{5}$$

To satisfy the initial condition $y(0) = 7$ we must have $7 = c_1 + 1/5$ or $c_1 = 6.8$. the solution we want is $y = 6.8e^{-2t} + 0.2e^{3t}$.

4 Friday April 10, Monday April 13

28. The **displacement from equilibrium** y of a object suspended from the ceiling by a spring is governed by a second order differential equation

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F(t) \tag{8}$$

where m is the **mass** of the object, c is a constant called the **damping constant**, k is a constant of proportionality called the **spring constant**, and $F(t)$ is a (generally time dependent) **external force**. The constants m and k are positive and c is nonnegative. This is of course Newton's law

$$F_{\text{tot}} = ma, \qquad a := \frac{d^2y}{dt^2}$$

where the total force F_{tot} is the sum $F_{\text{tot}} = F_S + F_R + F_E$ of three terms, viz. the spring force

$$F_S = -ky, \tag{9}$$

the resistive force

$$F_R = -c\frac{dy}{dt}, \tag{10}$$

and the external force

$$F_E = F(t).$$

The spring force F_S is the combined force of gravity and the force the spring exerts to restore the mass to equilibrium. Equation (9) says that the sign of the spring force F_S is opposite to the sign of the displacement y and hence F_S pushes the object towards equilibrium. The resistive F_R is the resistive force proportional to the velocity of the object. Equation (10) says that the sign of the resistive force F_R is opposite to the sign of the velocity dy/dt and hence F_R slows the object.. In the text, the resistive force F_R is often ascribed to a **dashpot** (e.g. the device which prevents a screen door from slamming) but in problems it might be ascribed to friction or to the viscosity of the surrounding media. (For example, if the body is submersed in oil, the resistive force F_R might be significant.) The external force will generally arise from some oscillation, e.g.

$$F(t) = F_0\cos(\omega t) \tag{11}$$

and might be caused by the oscillation of the ceiling or of a flywheel. (See figure 5.6.1 in the text.)

29. Equation 9 is called **Hooke's law**. It can be understood as follows. The force F_S depends solely of the position y of the spring. For y near y_0 the *Fundamental Idea of Calculus* says that the force $F_S(y)$ is well approximated by the linear function whose graph is the tangent line to the graph of F_S at y_0, i.e.

$$F_S(y) = F_S(y_0) + F_S'(y_0)(y - y_0) + o(y - y_0) \tag{12}$$

where the error term $o(y - y_0)$ is small in the sense that

$$\lim_{y \to y_0} \frac{o(y - y_0)}{y - y_0} = 0.$$

The fact that equilibrium occurs at $y = y_0$ means that $F_S(y_0) = 0$, i.e. (assuming F_E is zero) i.e. if the object is at rest at position $y = y_0$ then Newton's law $F = ma$ holds. The assertion that y is *displacement from equilibrium* means that $y_0 = 0$, i.e. we are measuring the position of the object as its signed distance from its equilibrium position as opposed to say its height above the floor or distance from the ceiling. From this point of view Hooke's law is the approximation that arises when we ignore the error term $o(y - y_0)$ in (12). This is the same reasoning that leads to the equation

$$\frac{d^2\theta}{dt^t} + \frac{g}{L}\theta = 0$$

as an approximation to the equation of motion

$$\frac{d^2\theta}{dt^t} + \frac{g}{L}\sin\theta = 0$$

for the simple pendulum. (See page 322 of the text.)

On the other hand, some of the assigned problems begin with a sentence like *A weight of 5 pounds stretches a spring 2 feet.* In this problem there are apparently two equilibria, one where the weight is not present and another where it is. In this situation the student is supposed to assume that the force F_S is a linear (not just approximately linear) function of the position, so that the spring constant k is 5/2 (i.e. the slope of the line) and that the equilibrium position occurs where the weight is suspended at rest.

30. The **energy** is defined as the sum

$$E := \frac{mv^2}{2} + \frac{ky^2}{2}, \qquad v := \frac{dy}{dt}$$

of the **kinetic energy** $mv^v2/2$ and the **potential energy** $ky^2/2$. If the external force F_E is zero, the energy satisfies

$$\frac{dE}{dt} = mv\frac{dv}{dt} + ky\frac{dy}{dt} = v\left(m\frac{d^2y}{de^2} + ky\right) = -c\left(\frac{dy}{dt}\right)^2$$

When c (and F_E) are zero, $dE/dt = 0$ so E is constant (energy is conserved) while if c is positive, $dE/dt < 0$ so E is decreasing (energy is dissipated). When we solve the equation (with $F_E = 0$) we will see that the motion is periodic if $c = 0$ and tends to equilibrium as t becomes infinite if $c > 0$.

31. It is important to keep track of the units in doing problems of this sort if for no other reason than that it helps avoid errors in computation. We never add two terms if they have different units and whenever a quantity appears in a nonlinear function like the sine or the exponential function, that quantity must be unitless. In the metric system the various terms have units as follows:

- y has the units of length: meters (m).
- $\dfrac{dy}{dt}$ has the units of velocity: meters/sec.
- $\dfrac{d^2y}{dt^2}$ has the units of acceleration: meters/sec^2.
- m has the units of mass: kilograms (kg).
- F has the units of force: newtons = kg·m/sec^2.

Using the principle that quantities can be added or equated only if they have the same units and that the units of a product (or ratio) of two quantities is the product (or ratio) of the units of the quantities we see that c has the units of kg·m/sec and that k has the units of kg/sec^2. The quantity

$$\omega_0 := \sqrt{\frac{k}{m}}$$

thus has the units of frequency: sec^{-1} which is good news: When $c = 0$ and $F_E = 0$ equation (8) becomes the **harmonic oscillator** equation

$$\frac{d^2y}{dt^2} + \omega_0^2 y = 0 \tag{13}$$

(we divided by m) and the solutions are

$$y = A\cos(\omega_0 t) + B\sin(\omega_0 t). \tag{14}$$

(The input $\omega_0 t$ to the trigonometric functions is unitless.)

Remark 32. When using English units (lb, ft, etc.) you need to be a bit careful with equations involving mass. Pounds (lb) is a unit of force, not mass. Using $mg = F$ and g=32 ft/sec^2 we see that an object at the surface of the earth which weighs 32 lbs (i.e. the force on it is 32 lbs) will have a mass of 1 slug[1] So one slug weighs 32 lbs at the surface of the earth (or lb = (1/32)·slug·ft/sec^2). When using metric units, kilogram is a unit of mass not force or weight. A 1 kilogram mass will weigh 9.8 newtons on the surface of the earth. (g= 9.8 m/sec^2 and newton = kg·m/sec^2). Saying that a mass "weighs" 1 kilogram is technically incorrect usage, but it is often used. What one really means is that it has 1 kilogram of mass and therefore weighs 9.8 newtons.

33. Consider the case where the external force F_E is not present. In this case equation (8) reduces to the homogeneous problem

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0 \tag{15}$$

The roots of the characteristic equation are

$$r_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m}, \qquad r_1 = \frac{-c - \sqrt{c^2 - 4mk}}{2m}.$$

We distinguish four cases.

1. Undamped: $c = 0$. The general solution is

$$y = A\cos(\omega_0 t) + B\sin(\omega_0 t), \qquad \omega_0 := \sqrt{\frac{k}{m}}$$

2. Under damped: $c^2 - 4mk < 0$. The general solution is

$$y = e^{-pt}\left(A\cos(\omega_1 t) + B\sin(\omega_0 t)\right), \qquad p := \frac{c}{2m}, \qquad \omega_1 := \sqrt{\omega_0^2 - p^2}$$

[1] The unit of mass in the English units is called the slug – really!

3. Critically damped: $c^2 - 4mk = 0$. The general solution is

$$y = e^{-pt}(c_1 + c_2 t)$$

4. Over damped: $c^2 - 4mk$. The general solution is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

In the undamped case (where $c = 0$) the motion is oscillatory and the limit of $y(t)$ as t becomes infinite does not exists (except of course when $A = B = 0$). In the three remaining cases (where $c > 0$) we have $\lim_{t\to\infty} y = 0$. In case 3 this is because

$$\lim_{t\to\infty} te^{-pt} = 0$$

(in a struggle between an exponential and a polynomial the exponential wins) while case 4 we have $r_2 < r_1 < 0$ because $\sqrt{c^2 - 4mk} < c$. In cases 1 and 2 we can define α and C by

$$C := \sqrt{A^2 + B^2}, \qquad \cos\alpha = \frac{A}{C}, \qquad \sin\alpha = \frac{B}{C}$$

and the solution takes the form

$$y = Ce^{-pt}\cos(\omega_1 t - \alpha).$$

In the undamped case $p = 0$ and the y is a (shifted) sine wave with **amplitude** C, while in the under damped case the graph bounces back and forth between the two graphs $y = \pm e^{-pt}$.

34. Now consider the case of **forced oscillation** where the external force is given by (11). The right hand side $F(t) = F_0\cos(\omega t)$ of (8) solves the ODE $(D^2 + \omega^2)F = 0$ so we can solve using the method of undetermined coefficients. We write equation (8) in the form

$$(mD^2 + cD + k)y = F_0\cos\omega t \tag{16}$$

and observe that every solution of this inhomogeneous equation satisfies the homogeneous equation

$$(D^2 + \omega^2)(mD^2 + cD + k)y = 0.$$

The solutions of this last equation all can be written as $y + y_p$ where $(mD^2 + cD + k)y = 0$ and $(D^2 + \omega^2)y_p$. We know all the solutions of the former equation by the previous paragraph and the most general solution of the latter is

$$y_p = b_1 \cos \omega t + b_2 \sin \omega t.$$

We consider three cases.

(i) $c = 0$ and $\omega \neq \omega_0$. The function y_p satisfies (16) iff

$$(k - m\omega^2) b_1 \cos \omega t + (k - m\omega^2) b_2 \sin \omega t = F_0 \cos \omega t$$

which (since the functions $\cos \omega t$ and $\sin \omega t$ are linearly independent) can only be true if $b_2 = 0$ and $b_1 = F_0/(k - m\omega^2)$. Our particular solution is thus

$$y_p = \frac{F_0 \cos \omega t}{k - m\omega^2} = \frac{F_0 \cos \omega t}{m(\omega_0^2 - \omega^2)}$$

(ii) $c = 0$ and $\omega = \omega_0$. In this case we can look for a solution ihe form $y_p = b_1 t \cos \omega_0 t + b_2 t \sin \omega_0 t$ but it is easier to let ω tend to ω_0 in the solution we found in part (i). Then by l'Hôpital's rule (differentiate top and bottom with respect to ω) we get

$$y_p = \lim_{\omega \to \omega_0} \frac{F_0 \cos \omega t}{m(\omega_0^2 - \omega^2)} = \frac{F_0 t \sin \omega_0 t}{2m\omega_0}$$

The solution y_p bounces back and forth between the two lines $y = \pm F_0 t/(2m)$. The general solution in this case is (by the principle of the particular solution) the general solution of the homogeneous system plus the solution y_p and the former remains bounded. Thus *every solution* oscillates wildly as t becomes infinite. This is the phenomenon of **resonance** and is the cause of many engineering disasters. (See the text page 352.)

(iii) $c > 0$. The high school algebra in this case is the most complicated but at least we know that there are no repeated roots since the roots of $r^2 + \omega^2 = 0$ are pure imaginary and the roots of $mr^2 + cr + k = 0$ are not. The function y_p satisfies (16) iff

$$\Big((k - m\omega^2) b_1 + c\omega b_2 \Big) \cos \omega t + \Big(-c\omega b_1 + (k - m\omega^2) b_2 \Big) \sin \omega t = F_0 \cos \omega t$$

so

$$(k - m\omega^2) b_1 + c\omega b_2 = F_0 \qquad \text{and} \qquad (-c\omega b_1 + (k - m\omega^2) b_2 = 0.$$

In matrix form this becomes

$$\begin{bmatrix} k - m\omega^2 & c\omega \\ -\omega & k - m\omega^2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

and the solution is

$$\begin{aligned} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} &= \begin{bmatrix} k - m\omega^2 & c\omega \\ -c\omega & k - m\omega^2 \end{bmatrix}^{-1} \begin{bmatrix} F_0 \\ 0 \end{bmatrix} \\ &= \frac{1}{(k - m\omega^2)^2 + (c\omega)^2} \begin{bmatrix} k - m\omega^2 & -c\omega \\ c\omega & k - m\omega^2 \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix} \\ &= \frac{F_0}{(k - m\omega^2)^2 + (c\omega)^2} \begin{bmatrix} k - m\omega^2 \\ c\omega \end{bmatrix} \end{aligned}$$

so our particular solution is

$$y_p = \frac{F_0}{(k - m\omega^2)^2 + (c\omega)^2} \big((k - m\omega^2)\cos\omega t + c\omega \sin\omega t\big).$$

As above this can writen as

$$y_p = \frac{F_0 \cos(\omega t - \alpha)}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

where

$$\cos\alpha = \frac{k - m\omega^2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}, \qquad \sin\alpha = \frac{c\omega}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}.$$

The general solution in this case is (by the principle of the particular solution) the general solution of the homogeneous system plus the solution y_p. The former decays to zero as t becomes infinite, and the latter has the same frequency as the external force $F_0 \cos\omega t$ but has a smaller amplitude and a "phase shift" α. (See the text page 355.)

5 Wednesday April 15 - Friday April 17

35. It's easy to solve the initial value problem

$$\frac{dx_1}{dt} = 3x_1, \qquad \frac{dx_2}{dt} = 5x_2, \qquad x_1(0) = 4, \qquad x_2(0) = 7.$$

The answer is

$$x_1 = 4e^{3t}, \qquad x_2 = 7e^{5t}.$$

The reason this is easy is that this really isn't a system of two equations in two unknowns, it is two equations each in one unknown. When we write this system in matrix notation we get

$$\frac{d\mathbf{x}}{dt} = \mathbf{D}\mathbf{x}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}.$$

The problem is easy because the matrix $\mathbf{D}$ is diagonal.

36. Here is a system which isn't so easy.

$$\frac{dy_1}{dt} = y_1 + 4y_2, \qquad \frac{dy_2}{dt} = -2y_1 + 7y_2, \qquad y_1(0) = 15, \quad y_2(0) = 11.$$

To solve it we make the change of variables

$$y_1 = 2x_1 + x_2, \qquad y_2 = x_1 + x_2; \qquad x_1 = y_1 - y_2, \qquad x_2 = -y_1 + 2y_2.$$

Then

$$\frac{dx_1}{dt} = \frac{dy_1}{dt} - \frac{dy_2}{dt} = (y_1 + 4y_2) - (-2y_1 + 7y_2) = 3y_1 - 3y_2 = 3x_1,$$

$$\frac{dx_2}{dt} = -\frac{dy_1}{dt} + 2\frac{dy_2}{dt} = -(y_1 + 4y_2) + 2(-2y_1 + 7y_2) = 5(-y_1 + 2y_2) = 5x_2,$$

$$x_1(0) = y_1(0) - y_2(0) = 4, \qquad x_2(0) = -y_1(0) + 2y_2(0) = 7.$$

The not so easy problem 36 has been transformed to the easy problem 35 and the solution is

$$y_1 = 2x_1 + x_2 = 8e^{3t} + 7e^{5t}, \qquad y_2 = x_1 + x_2 = 4e^{3t} + 7e^{5t}.$$

It's magic!

37. To see how to find the magic change of variables rewrite problem 36 in matrix form

$$\frac{d\mathbf{y}}{dt} = \mathbf{A}\mathbf{y}, \qquad \mathbf{A} = \begin{bmatrix} 1 & 4 \\ -2 & 7 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

and the change of variables as

$$\mathbf{y} = \mathbf{P}\mathbf{x}, \qquad \mathbf{x} = \mathbf{P}^{-1}\mathbf{y}.$$

In the new variables we have

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}^{-1}\frac{d\mathbf{y}}{dt} = \mathbf{P}^{-1}\mathbf{A}\mathbf{y} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{x}.$$

so we want to find a matrix $\mathbf{P}$ so that the matrix

$$\mathbf{D} := \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{17}$$

is diagonal. Once we have done this the not so easy initial value problem

$$\frac{d\mathbf{y}}{dt} = \mathbf{A}\mathbf{y}, \qquad \mathbf{y}(0) = \mathbf{y}_0 := \begin{bmatrix} 15 \\ 11 \end{bmatrix}$$

is transformed into the easy intial value problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{D}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0 := \mathbf{P}^{-1}\mathbf{y}_0$$

as follows. Let $\mathbf{x}$ be the solution to the easy problem and define $\mathbf{y} := \mathbf{P}\mathbf{x}$. Since the matrix $\mathbf{P}$ is constant we can differentiate the relation $\mathbf{y} = \mathbf{P}\mathbf{x}$ to get

$$\frac{d\mathbf{y}}{dt} = \mathbf{P}\frac{d\mathbf{x}}{dt} = \mathbf{P}^{-1}\mathbf{D}\mathbf{x} = \mathbf{P}^{-1}\mathbf{D}\mathbf{P}\mathbf{y} = \mathbf{A}\mathbf{y}.$$

Also since the equation $\mathbf{y} := \mathbf{P}\mathbf{x}$ holds for all t it holds in particular for $t = 0$, i.e.

$$\mathbf{y}(0) = \mathbf{P}\mathbf{x}(0) = \mathbf{P}\mathbf{x}_0 = \mathbf{P}\mathbf{P}^{-1}\mathbf{y}_0 = bfy_0.$$

This shows that $\mathbf{y} := \mathbf{P}\mathbf{x}$ solves the not so easy problem when $\mathbf{x}$ solves the easy problem.

38. So how do we find a matrices $\mathbf{P}$ and $\mathbf{D}$ satisfying (17)? Multiplying the equation $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ on the right by $\mathbf{P}$ gives $\mathbf{A}\mathbf{P} = \mathbf{D}\mathbf{P}$. Let $\mathbf{v}$ and $\mathbf{w}$ be the columns of $\mathbf{P}$ so that $\mathbf{P} = \begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix}$. Then as matrix multiplication distributes over concatentation we have

$$\mathbf{A}\mathbf{P} = \begin{bmatrix} \mathbf{A}\mathbf{v} & \mathbf{A}\mathbf{w} \end{bmatrix}.$$

But

$$\mathbf{PD} = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 & \lambda_2 w_1 \\ \lambda_1 v_2 & \lambda_2 w_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 \mathbf{v} & \lambda_2 \mathbf{w} \end{bmatrix}.$$

Thus the equation $\mathbf{AP} = \mathbf{DP}$ can be written as

$$\begin{bmatrix} \mathbf{Av} & \mathbf{Aw} \end{bmatrix} = \begin{bmatrix} \lambda_1 \mathbf{v} & \lambda_2 \mathbf{w} \end{bmatrix},$$

i.e. the single matrix equation $\mathbf{AP} = \mathbf{DP}$ becomes to two vector equations $\mathbf{Av} = \lambda_1 \mathbf{v}$ and $\mathbf{Aw} = \lambda_2 \mathbf{w}$.

Definition 39. Let $\mathbf{A}$ be an $n \times n$ matrix. We say that λ is an **eigenvalue** for $\mathbf{A}$ iff there is a nonzero vector $\mathbf{v}$ such that

$$\mathbf{Av} = \lambda \mathbf{v}. \tag{18}$$

Any nonzero solution $\mathbf{v}$ of equation (18) is called an **eigenvector** of $\mathbf{A}$ corresponding to the eigenvector λ. The set of all solutions to (18) (including $\mathbf{v} = \mathbf{0}$) is called the **eigenspace** corresponding to the eigenvalue λ. Equation (18) can be rewritten is a homgeneous system

$$(\lambda \mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0}$$

which has a nonzero solution $\mathbf{v}$ if and only if

$$\det(\lambda \mathbf{I} - \mathbf{A}) = 0.$$

The polynomial $\det(\lambda \mathbf{I} - \mathbf{A})$ is called the **characteristic polynomial** of $\mathbf{A}$ and so

> *The eigenvalues of a matrix are the roots of its characteristic polynomial.*

40. We now find the eigenvalues of the matrix $\mathbf{A}$ from section 37. The characteristic equation is

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \det \begin{bmatrix} \lambda - 1 & -4 \\ 2 & \lambda - 7 \end{bmatrix} = (\lambda - 1)(\lambda - 7) + 8 = \lambda^2 - 8\lambda + 15$$

so the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 5$. The eigenvectors $\mathbf{v}$ corresponding to the eigenvalue $\lambda_1 = 3$ are the solutions of the homogeneous system

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3-1 & -4 \\ 2 & 3-7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

i.e. the multiples of the column vector $\mathbf{v} = (2,1)$. The eigenvectors $\mathbf{w}$ corresponding to the eigenvalue $\lambda_2 = 5$ are the solutions of the homogeneous system

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5-1 & -4 \\ 2 & 5-7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

i.e. the multiples of the column vector $\mathbf{v} = (1,1)$. A solution to (17) is given by

$$\mathbf{P} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$$

and the change of variables $\mathbf{y} = \mathbf{P}\mathbf{x}$ used in problem 36 is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ x_1 + x_2 \end{bmatrix}.$$

Remark 41. The solution of the diagonalization problem (17) is never unique because we can always find another solutions be replacing each eigenvector (i.e. column of $\mathbf{P}$) by a nonzero multiple of itself. For example if $c_1 \neq 0$ and $c_2 \neq 0$, $\mathbf{Av} = \lambda_1\mathbf{v}$, $\mathbf{Aw} = \lambda_2\mathbf{w}$, then also $\mathbf{A}(c_1\mathbf{v}) = \lambda_1(c_1\mathbf{v})$, $\mathbf{A}(c_2\mathbf{w}) = \lambda_2(c_2\mathbf{w})$ so the matrix $\begin{bmatrix} c_1\mathbf{v} & c_2\mathbf{w} \end{bmatrix}$ should work as well as the matrix $\mathbf{P} = \begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix}$ used above. Indeed

$$\begin{bmatrix} c_1\mathbf{v} & c_2\mathbf{w} \end{bmatrix} = \mathbf{PC} = \text{ where } \mathbf{C} = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$$

and $\mathbf{CD} = \mathbf{DC}$ (since both are diagonal) so

$$(\mathbf{PC})\mathbf{D}(\mathbf{PC})^{-1} = \mathbf{PCDC}^{-1}\mathbf{P}^{-1} = \mathbf{PDCC}^{-1}\mathbf{P}^{-1} = \mathbf{PDP}^{-1}$$

which shows that if $\mathbf{A} = \mathbf{PDP}^{-1}$ then also $\mathbf{A} = (\mathbf{PC})\mathbf{D}(\mathbf{PC})^{-1}$.

6 Monday April 20 - Wednesday April 22

Definition 42. The matrix $\mathbf{A}$ is **similar** to the matrix $\mathbf{B}$ iff there is an invertible matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{P}\mathbf{B}\mathbf{P}^{-1}$. A square matrix $\mathbf{D}$ is said to be **diagonal** if its off diagonal entries are zero, i.e. iff it has the form

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

A square matrix is said to be **diagonzlizable** iff it is similar to a diagonal matrix.

Remark 43. (i) Every matrix is similar to itself and hence a diagonal matrix is diagonalizable. (ii) If $\mathbf{A}$ is similar to $\mathbf{B}$, then $\mathbf{B}$ is similar to $\mathbf{A}$ (because $\mathbf{A} = \mathbf{P}\mathbf{B}\mathbf{P}^{-1} \implies \mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$). (iii) If $\mathbf{A}$ is similar to $\mathbf{B}$ and $\mathbf{B}$ is similar to $\mathbf{C}$, then $\mathbf{A}$ is similar to $\mathbf{C}$ (because $\mathbf{A} = \mathbf{P}\mathbf{B}\mathbf{P}^{-1}$ and $\mathbf{B} = \mathbf{Q}\mathbf{C}\mathbf{Q}^{-1} \implies \mathbf{A} = (\mathbf{P}\mathbf{Q})\mathbf{C}(\mathbf{P}\mathbf{Q})^{-1}$).

Theorem 44. *Similar matrices have the same characteristic polynomial.*

Proof. If $\mathbf{A} = \mathbf{P}\mathbf{B}\mathbf{P}^{-1}$ then

$$\lambda\mathbf{I} - \mathbf{A} = \lambda\mathbf{I} - \mathbf{P}\mathbf{B}\mathbf{P}^{-1} = \lambda\mathbf{P}\mathbf{I}\mathbf{P}^{-1} - \mathbf{P}\mathbf{B}\mathbf{P}^{-1} = \mathbf{P}(\lambda\mathbf{I} - \mathbf{B})\mathbf{P}^{-1}$$

so

$$\det(\lambda\mathbf{I}-\mathbf{A}) = \det\big(\mathbf{P}(\lambda\mathbf{I}-\mathbf{B})\mathbf{P}^{-1}\big) = \det(\mathbf{P})\det(\lambda\mathbf{I}-\mathbf{B})\det(\mathbf{P})^{-1} = \det(\lambda\mathbf{I}-\mathbf{B})$$

as the determinant of the product is the producr of the determinants and the determinant of the inverse is the inverse of the determinant. □

Remark 45. It follows that similar matrices have the same eigenvalues as the eigenvalues are the roots of the characteristic polynomial. Of course they don't necessarily have the same eigenvectors, but if $\mathbf{A} = \mathbf{P}\mathbf{B}\mathbf{P}^{-1}$ and $\mathbf{w}$ is an eigenvector for $\mathbf{B}$ then $\mathbf{P}\mathbf{w}$ is an eigenvector for $\mathbf{A}$:

$$\mathbf{A}(\mathbf{P}\mathbf{w}) = (\mathbf{P}\mathbf{B}\mathbf{P}^{-1})(\mathbf{P}\mathbf{w}) = \mathbf{P}(\mathbf{B}\mathbf{W}) = \mathbf{P}(\lambda\mathbf{w}) = \lambda(\mathbf{P}\mathbf{W}).$$

Theorem 46. *An $n \times n$ matrix $\mathbf{A}$ is diagonalizable iff there is a linearly independent sequence $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ of eigenvectors of $\mathbf{A}$*

Proof. The matrix $\mathbf{P} = \left[\ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\ \right]$ is invertible if and only if its columns $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent and the matrix equation $\mathbf{AP} = \mathbf{PD}$ holds with $\mathbf{D}$ is diagonal if and only if the columns of $\mathbf{P}$ are eigenvectors of $\mathbf{A}$. (See Theorem 2 on page 376 of the text.) □

Theorem 47. *Assume that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are nonzero eigenvectors of $\mathbf{A}$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, i.e. $\mathbf{A}\mathbf{v}_i = \lambda_i$, $\mathbf{v}_i \neq \mathbf{0}$, and $\lambda_i \neq \lambda_j$ for $i \neq j$. Then the vector $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.*

Proof. By induction on k. The one element sequence $\mathbf{v}_1$ is independent because we are are assuming $\mathbf{v}_1$ is non zero. Assume as the hypothesis of induction that $\mathbf{v}_2, \ldots, \mathbf{v}_k$ are independent. We must show that $\mathbf{v}1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are independent. For this assume that

$$c_1\mathbf{v}1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

(We must show that $c_1 = c_2 = \cdots = c_k = 0$.) Multiply by $\lambda_1\mathbf{I} - \mathbf{A}$:

$$c_1(\lambda_1\mathbf{I} - \mathbf{A})\mathbf{v}_1 + c_2)\lambda_1\mathbf{I} - \mathbf{A})\mathbf{v}_2 + \cdots + c_k(\lambda_1\mathbf{I} - \mathbf{A})\mathbf{v}_k = \mathbf{0}.$$

Since $\mathbf{A}\mathbf{v}_i = \lambda_i\mathbf{v}_i$ this becomes

$$c_1(\lambda_1 - \lambda_1)\mathbf{v}_1 + c_2(\lambda_1 - \lambda_2)\mathbf{v}_2 + \cdots + c_k(\lambda_1 - \lambda_k)\mathbf{v}_k = \mathbf{0}.$$

Since $\lambda_1 - \lambda_1 = 0$ this becomes

$$c_2(\lambda_1 - \lambda_2)\mathbf{v}_2 + \cdots + c_k(\lambda_1 - \lambda_k)\mathbf{v}_k = \mathbf{0}.$$

As $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are independent (by the induction hypothesis) we get

$$c_2(\lambda_{-}\lambda_2) = \cdots = c_k(\lambda_1 - \lambda_k) = 0$$

as the eigenvalues are distinct we have $\lambda_1 - \lambda_i \neq 0$ for $i > 1$ so $c_2 = \cdots = c_k = 0$. But now $c_1\mathbf{v}_1 = \mathbf{0}$ so $c_1 = 0$ as well (since $\mathbf{v}_1 \neq \mathbf{0}$) so the c_i are all zero as required. (See Theorem 2 on page 376 of the text.) □

Corollary 48. *If an $n \times n$ matrix $\mathbf{A}$ has n distinct real eigenvalues, then it is diagonalizable.*

Proof. A square matrix is invertible if and only if its columns are independent. (See Theorem 3 on page 377 of the text.) □

Remark 49. The analog of the corollary remains true if we allow complex eigenvalues and assume that all matrices are complex.

Example 50. The characteristic polynomial of the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$$

is

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \det\left(\begin{bmatrix} \lambda - 3 & -4 \\ 4 & \lambda - 3 \end{bmatrix}\right) = (\lambda - 3)^2 + 16 = \lambda^2 - 6x + 25$$

and the roots are $3 \pm 4i$. The eigenvalues aren't real so the eigenvectors can't be real either. However, the matrix $\mathbf{A}$ can be diagonalized if we use complex numbers: For $\lambda = 3 + 4i$ the solutions of

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = \begin{bmatrix} 4i & 4 \\ -4 & 4i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 4 \begin{bmatrix} iv_1 - v_2 \\ -v_1 + iv_2 \end{bmatrix} = \mathbf{0}$$

arei $(v_1, v_2) = c(1, -i)$ while for $\lambda = 3 - 4i$ the solutions of

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = \begin{bmatrix} -4i & 4 \\ -4 & -4i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 4 \begin{bmatrix} -iv_1 - v_2 \\ -v_1 - iv_2 \end{bmatrix} = \mathbf{0}$$

are $(v_1, v_2) = c(1, i)$. Hence we have $\mathbf{A} = \mathbf{PDP}^{-1}$ where

$$\mathbf{D} = \begin{bmatrix} 3 + 4i & 0 \\ 0 & 3 - 4i \end{bmatrix}, \qquad \mathbf{P} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}, \qquad \mathbf{P}^{-1} = \frac{1}{2i} \begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix}.$$

Example 51. Not every matrix is diagonalizable even if we use complex numbers. For example, the matrix

$$\mathbf{N} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is not diagonalizable. This is because $\mathbf{N}^2 = \mathbf{0}$ but $\mathbf{N} \neq \mathbf{0}$. If $\mathbf{N} = \mathbf{PDP}^{-1}$ then

$$\mathbf{0} = \mathbf{N}^2 = (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1}) = \mathbf{PD}^2\mathbf{P}^{-1}$$

so $\mathbf{D}^2 = \mathbf{0}$. But if $\mathbf{D}$ is diagonal, then

$$\mathbf{D}^2 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^2 = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}$$

is also diagonal so $\lambda_1^2 = \lambda_2^2 = 0$ so $\lambda_1 = \lambda_2 = 0$ so $\mathbf{D} = \mathbf{0}$ so $\mathbf{N} = \mathbf{0}$ contradicting $\mathbf{N} \neq \mathbf{0}$.

7 Friday April 24

52. The **Wronskian** of an n functions $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ taking values in $\mathbb{R}^n$ is the determinant

$$W(t) = \det(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$$

of the $n \times n$ matrix whose ith column is $\mathbf{x}_i$. Of course for each value of t it is the case that $W(t) \neq 0$ if and only if the vectors $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ are linearly independent and certainly it can happen that $W(t)$ is zero for some values of t and non-zero for other values of t. But if the functions $\mathbf{x}_i$ are solution of a matrix differential equation, this is not so:

Theorem 53. *If $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ are solutions of the homogeneous linear system $\dfrac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x}$ then either $W(t) \neq 0$ for all t or $W(t) = 0$ for all t.*

Proof. This is an immediate consequence of the Existence and Uniqueness Theorem: If $W(t_0) = 0$ then there are constants $c_1, c_2, \ldots, c_n$ such that $c_1\mathbf{x}_1(t_0) + c_2\mathbf{x}_2(t_0) + \cdot + c_n, \mathbf{x}_n(t_0)$. Now $\mathbf{x}(t) := c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdot + c_n, \mathbf{x}_n(t)$ satisfies the equation and $\mathbf{x}(t_0) = \mathbf{0}$ so (by uniqueness) $\mathbf{x}(t) = \mathbf{0}$ for all t. $\square$

Remark 54. As explained in paragraph 7 this specializes to higher order differential equations. The **Wronskian** of n functions $x_1(t), x_2(t), \ldots, x_n(t)$ is Wronskian of of the coresponding sequence

$$\mathbf{x}_i = \left(x_i, x_i', x_i'', \ldots, x_i^{(n-1)}\right)$$

of vectors. For example in $n = 2$ the Wronskian of $x_1(t), x_2(t)$ is

$$W(x_1, x_2) = \det \begin{bmatrix} x_1 & x_1 \\ x_1' & x_2' \end{bmatrix} = x_1 x_2' - x_2 x_1'$$

where $x' := dx/dt$.

Definition 55. The **trace** $\mathrm{Tr}(\mathbf{A})$ of a square matrix $\mathbf{A}$ is the sum of its diagonal entries.

Theorem 56. *Let $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ be solutions of the homogeneous linear system $\dfrac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x}$. Then the Wronskian $W(t)$ satisfies the differential equation*

$$\frac{dW}{dt} = \mathrm{Tr}\big(\mathbf{A}(t)\big)W(t).$$

Proof. We do the 2×2 case

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad \mathbf{x}_1 = \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix}.$$

Let $x' = dx/dt$. Writing out the equations $\mathbf{x}_1' = A\mathbf{x}_1$ and $\mathbf{x}_2' = \mathbf{A}\mathbf{x}_2$ gives

$$\begin{aligned} x_{11}' &= a_{11}x_{11} + a_{12}x_{21}, \qquad & x_{12}' &= a_{11}x_{12} + a_{12}x_{22}, \\ x_{21}' &= a_{21}x_{11} + a_{22}x_{21}, \qquad & x_{22}' &= a_{21}x_{12} + a_{22}x_{22}. \end{aligned}$$

Since $W = x_{11}x_{22} - x_{12}x_{21}$ we get

$$\begin{aligned} W' &= x_{11}'x_{22} + x_{11}x_{22}' - x_{12}'x_{21} - x_{12}x_{21}' \\ &= (a_{11}x_{11} + a_{12}x_{21})x_{22} + x_{11}(a_{21}x_{12} + a_{22}x_{22}) \\ &\quad -(a_{11}x_{12} + a_{12}x_{22})x_{21} - x_{12}(a_{21}x_{11} + a_{22}x_{21}) \\ &= (a_{11} + a_{22})(x_{11}x_{22} - x_{12}x_{21}) \\ &= \operatorname{Tr}\big(\mathbf{A}\big)W. \end{aligned}$$

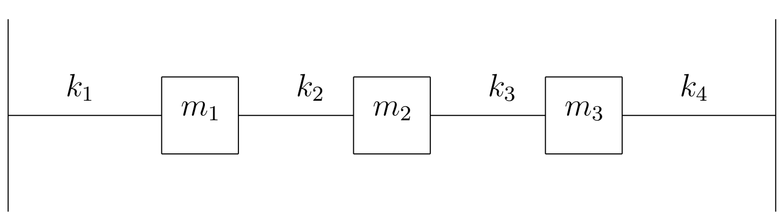
as claimed. $\square$

8 Monday April 29

57. Consider a system of tanks containing brine (salt and water) connected by pipes through which brine flows from one tank to another.

9 Wednesday April 29

58. Consider a collection of masses on a track each connected to the next by a spring with the first and last connected to opposite walls.



There are n masses lying on the x axis. The left wall is at $x = a$, the right wall at $x = b$, and the ith mass is at X_i so $a < X_1 < X_2 < \cdots X_n < b$. The spring constant of the ith spring is k_i and the ith mass is m_i. The first spring connects the first mass to the left wall, the last ($(n+1)$st) spring connects the last mass to the right wall, and the $(i+1)$st spring ($i = 1, 2, \ldots, n-1$) connects the ith mass to the $(i+1)$st mass. We assume that there is an **equilibrium** configuration $a < X_{0,1} < X_{0,2} < \cdots X_{0,n} < b$ where the masses are at rest and define the **displacement from equilibrium** $x_1, x_2, \ldots, x_n$ by

$$x_i := X_i - X_{0,i}.$$

note that the distance $X_{i+1} - X_i$ between two adjacent masses is related to the difference $x_{i+1} - x_i$ of the displacements by the formula

$$X_{i+1} - X_i = (X_{0,i+1} - X_{0,i}) + (x_{i+1} - x_i). \tag{19}$$

With $n = 3$ (as in the diagram above) the equations of motion for this system are

$$\begin{aligned}
m_1 \frac{d^2 x_1}{dt^2} &= -(k_1 + k_2)x_1 + k_2 x_2 \\
m_2 \frac{d^2 x_2}{dt^2} &= k_2 x_1 - (k_2 + k_3)x_2 + k_3 x_3 \\
m_3 \frac{d^2 x_3}{dt^2} &= k_3 x_2 - (k_3 + k_4)x_3
\end{aligned}$$

or in matrix notation

$$\mathbf{M} \frac{d^2 \mathbf{x}}{dt^2} = \mathbf{K}\mathbf{x} \tag{20}$$

with

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} -(k_1 + k_2) & k_2 & 0 \\ k_2 & -(k_2 + k_3) & k_3 \\ 0 & k_3 & -(k_3 + k_4) \end{bmatrix}.$$

In the general case (arbitrary n) the matrix $\mathbf{M}$ is diagonal with diagonal entries $m_1, m_2, \ldots, m_n$ and the matrix $\mathbf{K}$ is symmetric "tridiagonal" with entries $-(k_1 + k_2) \ldots, -(k_n + k_{n+1})$ on the diagonal and entries $k_2, \ldots, k_n$ on the sub and super diagonal.

59. To derive the system of equations from first principles[2] let T_i ($i = 1, 2, \ldots, n+1$) denote the tension in ith spring. This means that the force exerted by the ith spring on the mass attached to its left end if T_i and the force exerted by the ith spring on the mass attached to its right end is $-T_i$. The tension depends on the length $X_i - X_{i-1}$ of the ith spring so by linear approximation and (19)

$$T_i = T_{i,0} + k_i(x_{i+1} - x_i) + o(x_{i+1} - x_i)$$

where $T_{i,0}$ denotes the tension in the ith spring when the system is in equilibrium and k_i is the derivative of the tension at equilibrium. (The tension is assumed positive meaning that each spring is trying to contract, so the mass on its left is pulled to the right, and the mass on the right is pulled to the left.) We ignore the small error term $o(x_{i+1} - x_i)$. The net force on the ith mass is

$$T_{i+1} - T_i = T_{0,i+1} + k_{i+1}(x_{i+1} - x_i) - T_{0,i} - k_i(x_i - x_{i-1}).$$

At equilibrium the net force on each mass is zero: $T_{i+1,0} - T_{i,0} = 0$ so $T_{i+1} - T_i$ simplifies to

$$T_{i+1} - T_i = k_{i-1}x_{i-1} - (k_{i+1} + k_i)x_i + k_{i+1}x_{i+1}.$$

Remark 60. If the masses are hung in a line from the ceiling with the lowest mass attached only to the mass directly above, the system of equations is essentially the same: one takes $k_{n+1} = 0$.

61. Assume for the moment that all the masses are equal to one. Then the system takes the form

$$\frac{d^2\mathbf{x}}{dt^2} = \mathbf{K}\mathbf{x}. \tag{21}$$

The eigenvalues of $\mathbf{K}$ are negative so we may write them as the negatives of squares of real numbers. If

$$\mathbf{K}\mathbf{v} = -\omega^2\mathbf{v}$$

then for any constants A and B the function

$$\mathbf{x} = (A\cos\omega t + B\sin\omega t)\mathbf{v}$$

is a solution of (21). The following theorem says that $\mathbf{K}$ is diagonalizable so this gives $2n$ independent solutions of (21).

[2]This is not in the text: I worked it out to fulfill an inner need of my own.

Theorem 62. *Assume* $\mathbf{A}$ *is a symmetric real matrix, i.e.* $\mathbf{A}^{\mathsf{T}} = \mathbf{A}$*. Then* $\mathbf{A}$ *is diagonalizable. If in addition,* $\langle \mathbf{A}\mathbf{v}, \mathbf{v} \rangle > 0$ *for all* $\mathbf{v} \neq \mathbf{0}$ *then the eigenvalues are positive.*

Remark 63. This theorem is sometimes called the **Spectral Theorem**. It is often proved in Math 340 but I haven't found it in our text. It is also true that for a symmetric matrix, eigenvectors belonging to distinct eigenvalues are orthogonal. In fact there is an **orthonormal basis** of eigenvectors i.e. a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ so that $|\mathbf{v}_i| = 1$ and $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for $i \neq j$. (We can always make a non zero eigenvector into a unit vector by dividing it by its length.)

Remark 64. We can always make a change of variables to convert (20) to (21) as follows. Let $\mathbf{M}^{1/2}$ denote the diagonal matrix whose entries are $\sqrt{m_i}$. Then multiplying (20) by the inverse $\mathbf{M}^{-1/2}$ of $\mathbf{M}^{1/2}$ gives

$$\mathbf{M}^{1/2}\frac{d^2\mathbf{x}}{dt^2} = \mathbf{M}^{-1/2}\mathbf{K}\mathbf{x}.$$

Now make the change of variables $\mathbf{y} = \mathbf{M}^{1/2}\mathbf{x}$ to get

$$\frac{d^2\mathbf{y}}{dt^2} = \mathbf{M}^{1/2}\frac{d^2\mathbf{x}}{dt^2} = \mathbf{M}^{-1/2}\mathbf{K}\mathbf{x} = \big(\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}\big)\mathbf{y}.$$

It is easy to see that $\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}$ (the "new" $\mathbf{K}$) is again symmetric.

10 Friday May 1

65. You can plug square matrix into a polynomial (or more generally a power series) just as if it is a number. For example, if $f(x) = 3x^2 + 5$ then $f(\mathbf{A}) = 3\mathbf{A}^2 + 5\mathbf{I}$. Since you add or multiply diagonal matrices by adding or multiplying corresponding diagonal entries we have

$$f\left(\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}\right) = \begin{bmatrix} f(\lambda) & 0 \\ 0 & f(\mu) \end{bmatrix}. \tag{22}$$

Finally, since $\mathbf{P}(\mathbf{A}+\mathbf{B})\mathbf{P}^{-1} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1} + \mathbf{P}\mathbf{B}\mathbf{P}^{-1}$ and $(\mathbf{P}\mathbf{A}\mathbf{P}^{-1})^k = \mathbf{P}\mathbf{A}^k\mathbf{P}^{-1}$ we have

$$f(\mathbf{P}\mathbf{A}\mathbf{P}^{-1}) = \mathbf{P}f(\mathbf{A})\mathbf{P}^{-1}. \tag{23}$$

66. This even works for power series. For numbers the exponential function has the power series expansion

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

so for square matrices we make the definition

$$\exp(\mathbf{A}) := \sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{A}^k.$$

Replacing $\mathbf{A}$ by $t\mathbf{A}$ gives

$$\exp(t\mathbf{A}) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{A}^k.$$

(Some people write $e^{t\mathbf{A}}$.) Differentiating term by term gives

$$\begin{aligned} \frac{d}{dt}\exp(t\mathbf{A}) &= \sum_{k=0}^{\infty} k\frac{t^{k-1}}{k!}\mathbf{A}^k \\ &= \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!}\mathbf{A}^k \\ &= \sum_{j=0}^{\infty} \frac{t^j}{j!}\mathbf{A}^{j+1} \\ &= \mathbf{A}\left(\sum_{j=0}^{\infty} \frac{t^j}{j!}\mathbf{A}^j\right) \\ &= \mathbf{A}\exp(t\mathbf{A}) \end{aligned}$$

Since $\exp(t\mathbf{A}) = \mathbf{I}$ when $t = 0$ this means that the solution to the initial value problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0$$

is

$$\mathbf{x} = \exp(t\mathbf{A})\mathbf{x}_0.$$

67. The moral of the story is that matrix algebra is just like ordinary algebra and matrix calculus is just like ordinary calculus except that the commutative law doesn't always hold. However the commutative law does hold for powers of a single matrix:

$$\mathbf{A}^p\mathbf{A}^q = \mathbf{A}^{p+q} = \mathbf{A}^{q+p} = \mathbf{A}^q\mathbf{A}^p.$$

68. You can compute the exponential of a matrix using equations (22) and (23). If

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}, \qquad \mathbf{D} = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix},$$

then

$$\exp(t\mathbf{A}) = \mathbf{P}\exp(t\mathbf{D})\mathbf{P}^{-1}, \qquad \exp(t\mathbf{D}) = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix}.$$

Example 69. In paragraph 40 we saw how to diagonalize the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ -2 & 7 \end{bmatrix}.$$

We found that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ where

$$\mathbf{P} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{P}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}.$$

Now

$$\exp(t\mathbf{D}) = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{5t} \end{bmatrix}$$

so $\exp(t\mathbf{A}) = \mathbf{P}\exp(t\mathbf{D})\mathbf{P}^{-1}$ is computed by matrix mutiplication

$$\begin{aligned}
\exp(t\mathbf{A}) &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{5t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{3t} & -e^{3t} \\ -e^{5t} & 2e^{5t} \end{bmatrix} \\
&= \begin{bmatrix} 2e^{3t} - e^{5t} & -2e^{3t} + 2e^{5t} \\ e^{3t} - e^{5t} & -e^{3t} + 2e^{5t} \end{bmatrix}.
\end{aligned}$$

Example 70. If $\mathbf{N}^2 = \mathbf{0}$ then

$$\exp(t\mathbf{N}) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{N}^k = \sum_{k=0}^{1} \frac{t^k}{k!}\mathbf{N}^k = \mathbf{I} + t\mathbf{N}.$$

In particular,

$$\exp\left(\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Theorem 71. *If* $\mathbf{AB} = \mathbf{BA}$ *then* $\exp(\mathbf{A}+\mathbf{B}) = \exp(\mathbf{A})\exp(\mathbf{B})$.

Proof. The formula $e^{a+b} = a^a e^b$ holds for numbers. Here is a proof using power series. By the Binomial Theorem

$$(a+b)^k = \sum_{p+q=k} \frac{k!}{p!q!} a^p b^q$$

so

$$\sum_{k=0}^{\infty} \frac{(a+b)^k}{k!} = \sum_{k=0}^{\infty} \sum_{p+q=k} \frac{a^p b^q}{p!q!} = \left(\sum_{p=0}^{\infty} \frac{a^p}{p!}\right)\left(\sum_{q=0}^{\infty} \frac{b^q}{q!}\right).$$

If $\mathbf{AB} = \mathbf{BA}$ the same proof works to prove the theorem. $\square$

Example 72. (Problem 4 page 487 of the text) We compute $\exp(t\mathbf{A})$ where

$$\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}.$$

The characteristic polynomial is

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \det\begin{bmatrix} \lambda-3 & 1 \\ -1 & \lambda-1 \end{bmatrix} = (\lambda-3)(\lambda-1)+1 = (\lambda-2)^2$$

and has a double root. But the null space of $(2\mathbf{I} - \mathbf{A})$ is one dimensional so we can't diagonalize the matrix. However

$$(\mathbf{A} - 2\mathbf{I})^2 = \begin{bmatrix} 3-2 & -1 \\ 1 & 1-2 \end{bmatrix}^2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}^2 = \mathbf{0}$$

so $\exp\big(t(\mathbf{A}-2\mathbf{I})\big) = \mathbf{I} + t(\mathbf{A}-2\mathbf{I})$ as in Example 70. But the matrices $2t\mathbf{I}$ and $t(\mathbf{A}-2\mathbf{I})$ commute (the identity matrix commutes with every matrix) and $t\mathbf{A} = 2t\mathbf{I} + t(\mathbf{A}-2\mathbf{I})$ so

$$\exp(t\mathbf{A}) = \exp(t\mathbf{I})\exp\big(t(\mathbf{A}-2\mathbf{I})\big) = e^{2t}\big(\mathbf{I} + t(\mathbf{A}-2\mathbf{I})\big)$$

i.e.

$$\exp(t\mathbf{A}) = e^{2t}\begin{bmatrix} 1+t & -t \\ t & 1-t \end{bmatrix}.$$

11 Monday May 4

73. A solution of a two dimensional system

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x})$$

is a parametric curve in the plane $\mathbb{R}^2$. The collection of all solutions is called the **phase portrait** of the system. When we draw the phase portrait we only draw a few representative solutions. We put arrows on the solutions to indicate the direction of the parameterization just like we did when we drew the phase line in the frst part of this course. We shall only draw phase portraits for linear systems

$$\mathbf{F}(\mathbf{x}) = \mathbf{A}\mathbf{x}$$

where $\mathbf{A}$ is a (constant) 2×2 matrix.

12 Wednesday May 6

74. Consider the inhomogeneous system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(y) \tag{24}$$

where $\mathbf{A}(t)$ is a continuous $n \times n$ matrix valued function, $\mathbf{f}(t)$ is a continuous function with values in $\mathbb{R}^n$ and the unknown $\mathbf{x}$ also takes values in $\mathbb{R}^n$. We shall call the system

$$\frac{d\mathbf{v}}{dt} = \mathbf{A}(t)\mathbf{v} \tag{25}$$

the **homogeneous system corresponding to the inhomogenoous system** (24).

75. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be n linearly independent solutions to the homogeneous system (25) and form the matrix

$$\mathbf{\Phi}(t) := \begin{bmatrix} \mathbf{v}_1(t) & \mathbf{v}_2(t) & \cdots & \mathbf{v}_n(t) \end{bmatrix}.$$

The matrix $\mathbf{\Phi}$ is called a **fundamental matrix** for the system (25); this means that the columns are solutions of (25) and they form a basis for $\mathbb{R}^n$ for some (and hence[3] every) value of t.

[3]by Theorem 53

Theorem 76. *A fundamental matrix satisfies the matrix differential equation*

$$\frac{d\mathbf{\Phi}}{dt} = \mathbf{A}\mathbf{\Phi}$$

Proof.

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} &= \begin{bmatrix} \dfrac{d\mathbf{v}_1}{dt} & \dfrac{d\mathbf{v}_2}{dt} & \dots & \dfrac{d\mathbf{v}_n}{dt} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{A}\mathbf{v}_1 & \mathbf{A}\mathbf{v}_2 & \cdots & \mathbf{A}\mathbf{v}_n \end{bmatrix} \\ &= \mathbf{A}\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}.\end{aligned}$$

□

Theorem 77. *If* $\mathbf{\Phi}$ *is a fundamental matrix for the system (25) then the function*

$$\mathbf{v}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}(t_0)^{-1}\mathbf{v}_0$$

is the solution of the initial value problem

$$\frac{d\mathbf{v}}{dt} = \mathbf{A}(t)\mathbf{v}, \qquad \mathbf{v}(t_0) = \mathbf{v}_0.$$

Proof. The columns of $\mathbf{\Phi}(t)\mathbf{\Phi}(t_0)^{-1}$ are linear combinations of the columns of $\mathbf{\Phi}(t)$ (see the Remark 79 below) and hence are solutions of the homogeneous system. The initial condition holds because $\mathbf{\Phi}(t_0)\mathbf{\Phi}(t_0)^{-1} = \mathbf{I}$ □

Theorem 78. *If the matrix* $\mathbf{A}$ *is constant the matrix*

$$\Phi(t) = \exp(t\mathbf{A})$$

is a fundamental matrix for the system $\dfrac{d\mathbf{v}}{dt} = \mathbf{A}\mathbf{v}$.

Proof. $\det(\mathbf{\Phi}(0)) = \det(0\mathbf{A}) = \det(\mathbf{I}) = 1 \neq 0.$ □

Remark 79. The proof of Theorem 77 asserted that *The columns of* $\mathbf{PC}$ *are linear combnations of the columns of* $\mathbf{P}$. Because

$$\mathbf{P}\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \dots \mathbf{c}_k \end{bmatrix} = \begin{bmatrix} \mathbf{P}\mathbf{c}_1 & \mathbf{P}\mathbf{c}_2 & \dots \mathbf{P}\mathbf{c}_k \end{bmatrix}$$

it is enough to see that this is true when $\mathbf{C}$ is a single column, i.e. when $\mathbf{C}$ is $n \times 1$. In this case it is the definition of matrix multiplication.

80. Now we show how to solve the inhomogeneous system (24) once we have a fundamental matrix for the corresponding homogeneous system (25). By the Superposition Principle (more precisely the Principle of the Particular Solution) it is enough to find a particular solution $\mathbf{x}_p$ of (24) for then the general solution of (24) is the particular solution plus the general solution of (25). The method we use is called **variation of parameters**. We already saw a one dimensional example of this in the first part of the course. We use the Ansatz

$$\mathbf{x}_p(t) = \mathbf{\Phi}(t)\mathbf{u}(t)$$

Then

$$\frac{d\mathbf{x}_p}{dt} = \frac{d\mathbf{\Phi}}{dt}\mathbf{u} + \mathbf{\Phi}\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{\Phi}\mathbf{u} + \mathbf{\Phi}\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{x}_p + \mathbf{\Phi}\frac{d\mathbf{u}}{dt}$$

which solves (24) if

$$\mathbf{\Phi}\frac{d\mathbf{u}}{dt} = \mathbf{f}$$

so we can solve by integration

$$\mathbf{u}(t) = \mathbf{u}(0) + \int_0^t \mathbf{u}'(\tau)\,d\tau = \mathbf{u}(0) + \int_0^t \mathbf{\Phi}(\tau)^{-1}\mathbf{f}(\tau)\,d\tau.$$

(Since we only want one solution not all the solutions we can take $\mathbf{u}(0)$ to be anything.) The solution of the initial value problem

$$\mathbf{\Phi}\frac{d\mathbf{u}}{dt} = \mathbf{f}, \qquad \mathbf{x}(0) = \mathbf{x}_0$$

is

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{u}(t) + \mathbf{\Phi}(t)\Big(\mathbf{\Phi}(0)^{-1}\mathbf{x}_0 - \mathbf{u}(0)\Big). \tag{26}$$

Example 81. We solve the inhomogeneous system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{f}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ -2 & 7 \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

In Example 69 we found the fundamental matrix

$$\mathbf{\Phi}(t) = \exp(t\mathbf{A}) = \begin{bmatrix} 2e^{3t} - e^{5t} & -2e^{3t} + 2e^{5t} \\ e^{3t} - e^{5t} & -e^{3t} + 2e^{5t} \end{bmatrix}$$

for the corresponding homogeneous system. We take

$$\frac{d\mathbf{u}}{dt} = \mathbf{\Phi}(t)^{-1}\mathbf{f} = \exp(t\mathbf{A})^{-1}\mathbf{f} = \exp(-t\mathbf{A})\mathbf{f} = \begin{bmatrix} 2e^{-3t} - e^{-5t} \\ e^{-3t} - e^{-5t} \end{bmatrix}$$

Integrating gives

$$\mathbf{u} = \begin{bmatrix} -\dfrac{2e^{-3t}}{3} + \dfrac{e^{-5t}}{5} \\ -\dfrac{e^{-3t}}{3} + \dfrac{e^{-5t}}{5} \end{bmatrix} = \frac{1}{15}\begin{bmatrix} -10e^{-3t} + 3e^{-5t} \\ \\ -5e^{-3t} + 3e^{-5t} \end{bmatrix}$$

so a particular solution is

$$\begin{aligned}
\mathbf{x}_p = \exp(t\mathbf{A})\mathbf{u} &= \begin{bmatrix} 2e^{3t} - e^{5t} & -2e^{3t} + 2e^{5t} \\ e^{3t} - e^{5t} & -e^{3t} + 2e^{5t} \end{bmatrix} \frac{1}{15} \begin{bmatrix} -10e^{-3t} + 3e^{-5t} \\ -5e^{-3t} + 3e^{-5t} \end{bmatrix} \\
&= \frac{1}{15}\begin{bmatrix} (2e^{3t} - e^{5t})(-10e^{-3t} + 3e^{-5t}) + (-2e^{3t} + 2e^{5t})(-5e^{-3t} + 3e^{-5t}) \\ (e^{3t} - e^{5t})(-10e^{-3t} + 3e^{-5t}) + (-e^{3t} + 2e^{5t})(-5e^{-3t} + 3e^{-5t}) \end{bmatrix} \\
&= \frac{1}{15}\begin{bmatrix} (-23 + 10e^{2t} + 6e^{3t}) + (16 - 10e^{2t} - 6e^{-2t}) \\ (-13 + 10e^{2t} + 6e^{-2t}) + (11 - 10e^{2t} - 3e^{-2t}) \end{bmatrix} \\
&= \frac{1}{15}\begin{bmatrix} -7 \\ -2 \end{bmatrix}.
\end{aligned}$$

Whew! This means that if we haven't made a mistake, the functions

$$x_1 = \frac{-7}{15}, \qquad x_2 = \frac{-2}{15}$$

should satisfy

$$\frac{dx_1}{dt} = x_1 + 4x_2 + 1, \qquad \frac{dx_2}{dt} = -2x_1 + 7x_2.$$

Let's check:

$$\frac{dx_1}{dt} = 0 = \left(\frac{-7}{15}\right) + 4\left(\frac{-2}{15}\right) + 1 = x_1 + 4x_2 + 1$$

$$\frac{dx_2}{dt} = 0 = -2\left(\frac{-7}{15}\right) + 7\left(\frac{-2}{15}\right) = -2x_1 + 7x_2$$

Using equation (26) we see that the solution of the initial value problem

$$\frac{dx_1}{dt} = x_1 + 4x_2 + 1, \quad \frac{dx_2}{dt} = -2x_1 + 7x_2, \qquad x_1(0) = 17, \quad x_2(0) = 29$$

is $\mathbf{x} = \exp(t\mathbf{A})\mathbf{u}(t) + \exp(t\mathbf{A})\Big(\mathbf{x}_0 - \mathbf{u}(0)\Big)$ i.e.

$$\begin{aligned} x_1 &= \tfrac{-7}{15} + (2e^{3t} - e^{5t})(17 + \tfrac{7}{15}) + (-2e^{3t} + 2e^{4t})(29 + \tfrac{2}{15}), \\ x_2 &= \tfrac{-2}{15} + (e^{3t} - e^{5t})(17 + \tfrac{7}{15}) + (-e^{3t} + 2e^{5t})(29 + \tfrac{2}{15}) \end{aligned}$$

Remark 82. This particular problem can be more easily solved by differentiating the equation $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$. Since $\mathbf{f}$ is constant we get $\mathbf{x}'' = \mathbf{A}\mathbf{x}'$ which is a homogeneoous equation.

Final Exam 07:45 A.M. THU. MAY 14