Math 320 Spring 2009
Part I – Differential Equations

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The text is *Differential Equations & Linear Algebra* (Second Edition) by Edwards & Penney.

1 Wednesday Jan 21

1. In first year calculus you learned to solve a linear differential equation like

$$\frac{dy}{dt} = 2y + 3, \qquad y(0) = 5 \tag{1}$$

This semester you will learn to solve a system of linear differential equations like:

$$\frac{dx}{dt} = 3x + y + 7, \quad \frac{dy}{dt} = x + 5y - 2, \qquad (x(0), y(0)) = (4, 8). \tag{2}$$

Note that if you can solve systems of equations like (2) you can also solve higher order equations like

$$\frac{d^2y}{dt^2} = 3\frac{dy}{dt} + y + 7, \qquad y(0) = 4, \quad \left.\frac{dy}{dt}\right|_{t=0} = 8. \tag{3}$$

You can change (3) into a system:

$$\frac{dy}{dt} = v, \quad \frac{dv}{dt} = 3v + y + 7, \qquad y(0) = 4, \quad v(0) = 8 \tag{4}$$

2. An **ODE** (**ordinary differential equation**) of **order** n looks like

$$F\big(t, y, y', y'', \dots, y^{(n)}\big) = 0 \tag{5}$$

The unknown is a function $y = y(t)$ of the independent variable t and

$$y' := \frac{dy}{dt}, \qquad y'' := \frac{d^2y}{dt^2}, \quad \dots, \quad y^{(n)} := \frac{d^ny}{dt^n}.$$

When the equation looks like

$$y^{(n)} = G\big(t, y, y', y'', \dots, y^{(n-1)}\big) \tag{6}$$

we say it is in **normal form**. It may be impossible to rewrite equation (5) as equation (6). A **system of differential equations** is the same thing with the single unknown y replaced by the vector $\mathbf{y} := (y_1, y_2, \dots, y_m)$.

Remark 3. As our first examples will show, the independent variable often has the interpretation of time which is why the letter t is used. Un this case the ODE represents the time evolution of a **dynamical system**. For example the 2nd order system

$$m\ddot{\mathbf{r}} = -\frac{GMm}{r^3}$$

describes the motion of a planet of mass m moving about a sun of mass M. The sun is at the origin, $\mathbf{r}$ is the posiition vector of the planet, and $r = |\mathbf{r}|$ is the length of $\mathbf{r}$, i.e. the distance from the planet to the sun. Sometimes the ODE has a geometric interpretation in which case the letter x is often used for the independent variable.

Example 4. *Swimmer crossing a river* (Text page 15.) Let the banks of a river be the vertical lines $x = \pm a$ in the (x, y) plane and suppose that the river flows up so that the velocity v_R of the river at the point (x, y) is

$$v_R = v_0 \left(1 - \frac{x^2}{a^2}\right).$$

The formula says that $v_R = 0$ on the banks where $x = \pm a$ and $v_R = v_0$ in the center of the river where $x = 0$ (the y-axis). The swimmer swims with constant velocity v_S towards the closest point on the opposite shore. The system

$$\frac{dx}{dt} = v_S, \qquad \frac{dy}{dt} = v_R = v_0\left(1 - \frac{x^2}{a^2}\right)$$

is a dynamical system describing the position of the swimmer. Dividing the two equations and using $\frac{dy}{dx} = \frac{dy/dt}{dy/dx}$ gives the geometric equation

$$\frac{dy}{dx} = \frac{v_0}{v_S}\left(1 - \frac{x^2}{a^2}\right)$$

which describes the trajectory of the swimmer.

Example 5. *Newton's law of cooling.* This says that rate of change of the temperature T of a body (e.g. a cup of coffee) is proportional to difference $A - T$ bewteen the ambient temperature (i.e. room temperature) and the temperature of the body. The ODE is

$$\frac{dT}{dt} = k(A - T).$$

In a tiny time from t to $t + h$ of duration $\Delta t = (t + h) - t = h$ the change in the temperature is $\Delta T = T(t + h) = T(t)$ so the rate of change id $\Delta T/\Delta t$. By Newton's law of cooling we have (approximately)

$$\frac{\Delta T}{\Delta t} \approx k(A - T).$$

It doesn't matter much if we use $T = T(t)$ or $T = T(t + h)$ on the right hand side because T is continuous and h is small. By the definition of the derivative

$$\frac{dT}{dt} = \lim_{\Delta t \to 0} \frac{\Delta T}{\Delta t} = \lim_{h \to 0} \frac{T(t + h) = T(t)}{h}$$

so we get the exact form of Newton's law of cooling as the limit as $h \to 0$ in the approximate form.

2 Friday Jan 23

Theorem 6 (Existence and Uniqueness Theorem)**.** *Suppose that $f(t, y)$ is a continuous function of two variables defined in a region R in (t, y) plane and* *that the partial $\partial f/\partial y$ exists and is continuous everyhere in R. Let (t_0, y_0) be a point in R. Then there is a solution $y = y(t)$ to the initial value problem*

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0$$

defined on some interval I about t_0. The solution is unique in the sense that any two such solutions of the initial value problem are equal where both are defined.

Remark 7. The theorem is stated on page 23 of the text and proved in an appendix. The same theorem holds for systems and hence higher order equations. We usually solve an ODE by doing an integration. Then an arbitrary constant C arises and we choose it to satisfy the initial condition $y(t_0) = y_0$. The Existence and Uniqueness Theorem tells us that this is the only answer.

8. The first order ODE

$$\frac{dx}{dt} = f(t, x)$$

has an important special case, where the function $f(t, x)$ factors as a product

$$f(t, x) = g(x)h(t)$$

of a function$g(x)$ of x and a function $h(t)$ of t. Then we can write the ODE $dx/dt = g(x)h(t)$ as $dx/g(x) = h(t)\,dt$, integrate to get

$$\int \frac{dx}{g(x)} = \int h(t)\,dt,$$

and solve for x in terms of t. When $g(x)$ is identically one, the equation is $dx/dt = h(t)$ so the answer is $x = \int h(t)\,dt$. When $h(t)$ is is identically one, the system is autonomous, i.e. $dx/dt = g(x)$. In this case can find out a lot about the solutions from the *phase diagram.* [1]

Example 9. *Braking a car.* A car going at speed v_0 skids to a stop at a constant deceleration k in time T leaving skid marks of length L. We find each of the four quantities in terms of the other three. Let the brakes be applied at time $t = 0$, so the car stops at time $t = T$, and let $v = v(t)$ denote

[1] We'll study this later in Section 2.2 of the text. See Figure 2.2.7 on page 93.

the velocity at time t, and $x = x(t)$ denote the distance travelled over the time interval $[0, t]$. Then the statement of the problem translates into the equations

$$\frac{dv}{dt} = -k, \quad v = \frac{dx}{dt}, \quad v(0) = v_0, \quad v(T) = 0, \quad x(0) = 0, \quad x(T) = L.$$

Integrating the first differential equation gives

$$\int \frac{dv}{dt}\, dt = \int -k\, dt = -kt + C,$$

so $C = v(0) = v_0$ and $v(t) = v_0 - kt$ so $0 = v(T) = v_0 - kT$ so $v_0 = kT$, $k = v_0/T$, and $T = v_0/k$. Integrating the second differential equation gives

$$L = x(T) - x(0) = \int_0^T \frac{dx}{dt}\, dt = \int_0^T v(t)\, dt \int_0^T (v_0 - kt)\, dt = v_0 T - \tfrac{1}{2} k T^2.$$

From $T = v_0/k$ we get $L = v_0^2/k - \frac{1}{2} v_0^2/k = \frac{1}{2} v_0^2/k$. (See problems 30-32 page 17 of the text.)

Remark 10. Mathematically this is the same problem as the problem of a falling body on earth: If y is the height of the body, $v = dy/dt$ is its speed, $a = dv/dt = d^2y/dt^2$ is the acceleration, then Newton's 3rd law is $F = ma = -mg$ where $g = 32\text{ft/sec}^2 = 9.8\text{m/sec}^2$ is the acceleration due to gravity so

$$v = \frac{dy}{dt} = -gt + v_0, \qquad y = -\frac{gt^2}{2} + v_0 t + y_0.$$

Example 11. *Population equation (exponential growth and decay).* The differential equation

$$\frac{dP}{dt} = kP$$

says that the rate of growth (or decay if $k < 0$) of a quantity P is proportional to its size. We solve by separation of variables: $dP/P = dt$ so

$$\ln P = \int \frac{dP}{P} = \int k\, dt = kt + C = kt + \ln P_0$$

so $P = P_0 e^{kt}$.

12. *A single linear homogeneous equation.* The more general equation

$$\frac{dy}{dt} = R(t)y$$

is solved the same way: $dy/y = R(t)\,dt$ so

$$\ln y = \int \frac{dy}{y} = \int R(t)\,dt$$

and exponentiating this equation gives

$$y = e^{\int R(t)\,dt}.$$

Note that the additive constant in $\int R(t)\,dt$ becomes a multiplicative constant after exponentiating. For example, integrating the equation

$$\frac{dy}{dt} = ty$$

gives

$$\ln y = \int \frac{dy}{y} = \int t\,dt = \tfrac{1}{2}t^2 + C$$

so exponentiating gives

$$y = \exp\left(\tfrac{1}{2}t^2 + C\right) = \exp\left(\tfrac{1}{2}t^2\right)\exp(C) = y_0 \exp\left(\tfrac{1}{2}t^2\right)$$

where $y_0 = e^C$. (For typographical reasons the exponential function is often denoted as $\exp(x) := e^x$.)

Example 13. Consider the function $f(y) = |y|^p$. On the region where $y \neq 0$ the the derivative $f'(y)$ is continuous so the Existence and Uniqueness Theorem applies to solutions which stay in this region. We solve by separation of variables where $y > 0$

$$\frac{y^{1-p}}{1-p)} = \int \frac{dy}{y^p} = t - c.$$

so (as long as $t - c > 0$)

$$y = [(1-p)(t-c)]^{1/(1-p)}.$$

When $y < 0$ we have $|y| = -y$ and

$$y = -[(1-p)(|t-c|]^{1/(1-p)}.$$

Funny things happen when $y = 0$. If $p > 1$ the derivative $f'(y)$ is continuous so by the Existence an Uniqueness Theorem the only solution with $y(t_0) = 0$ is $y \equiv 0$. (This is reflected in the fact that the above formula for y becomes infinite when $t = c$.) If $0 < p < 1$ however a solution can remain at zero for a finite amount of time and follow one of the above solutions to the left and right. For example for $p = \frac{1}{2}$ and *any choice* of $c_1 < 0$ and $c_2 > 0$ the function

$$y(t) = \begin{cases} \frac{1}{4}(c_1 - t)^2 & \text{for } t < c_1 \\ 0 & \text{for } c_1 \le t \le c_2 \\ \frac{1}{4}(t - c_2)^2 & \text{for } c_2 < t \end{cases}$$

solves the ODE and the initial condition $y(0) = 0$, so the solution is not unique. This is essentially the example of Remark 2 on page 23 of the text.

3 Monday Jan 26

14. *Slope fields and phase diagrams.* To draw the **slope field** of an ODE $dy/dx = f(x, y)$ raw a little line segment of slope $f(x, y)$ and many points (x, y) in the (x, y)-plane. The curves tangent to these little line segments are the graphs of the solution curves. This is a lot of work unless you have a computer and it is often not very helpful. In the case of an autonomous ODE $dy/dt = f(y)$ the **phase diagram** (see e.g. Figures 2.2.8n and 2.2.9 on page 94 of the text.) is more helpful. This is a line representing the y-axis with the zeros of f indicated and the intervals in between the zeros marked with an arrow indicating the sign of $f(y)$ for y in that interval.

Example 15. *Swimmer crossing river.* Recall from last Wednesday the dynamical system

$$\frac{dx}{dt} = v_S, \qquad \frac{dy}{dt} = v_R = v_0\left(1 - \frac{x^2}{a^2}\right)$$

describing the position of the swimmer. Dividing the two equations and using

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dy/dx}$ gives the geometric equation

$$\frac{dy}{dx} = k\left(1 - \frac{x^2}{a^2}\right), \qquad k := \frac{v_0}{v_S}$$

which describes the trajectory of the swimmer. Take $k = 1$, $a = 1$. The solution curves are

$$y = x - \frac{x^3}{3} + C.$$

They are vertical translates of one another. The solution with $C = 0$ starts at the point $(x, y) = \left(-1, -\frac{2}{3}\right)$ and ends at $(x, y) = \left(1, \frac{2}{3}\right)$.

Example 16. *The slope field for $dy/dx = x - y$* The slope is horizontal on the line $y = x$, negative to the left and positive to the right. The picture in the text (page 20) suggests that the solutions are asymptotic as $x \to \infty$. We'll check this in the next lecture.

17. *The phase diagram for $dy/dt = (y - a)(y - b)$.* Assume that $a < b$ so $dy/dt > 0$ for $y < a$ and for $b < y$ while $dy/dt < 0$ for $a < y < b$. The phase diagram is

From the diagram we can see that

$$\begin{aligned}
&\lim_{t\to\infty} y(t) = a && \text{if } y(0) < a \\
&y(t) = a && \text{if } y(0) = a \\
&\lim_{t\to-\infty} y(t) = a && \text{if } a < y(0) < b \\
&y(t) = b && \text{if } y(0) = b \\
&\lim_{t\to-\infty} y(t) = a && \text{if } y(0) < b \\
&\lim_{t\to T_1-} y(t) = \infty && \text{if } b < y(0) \\
&\lim_{t\to T_2+} y(t) = -\infty && \text{if } y(0) < a
\end{aligned}$$

The diagram does not tell us whether T_1 and T_2 are finite. For this we will solve the equation by separation of variables and partial fractions.

$$\frac{1}{(y-a)(y-b)} = \frac{1}{b-a}\left(\frac{1}{y-b} - \frac{1}{y-a}\right)$$

$$\int \frac{dy}{(y-a)(y-b)} = \int dt$$

$$\ln\frac{|y-b|}{|y-a|} = (b-a)t + c.$$

$$\frac{|y-b|}{|y-a|} = Ce^{(b-a)t}, \qquad C := e^c.$$

What to do about the absolute values? Well certainly

$$\frac{y-b}{y-a} = \pm Ce^{(b-a)t},$$

$y = y_0$ when $t = 0$, and the exponential is positive so we must have

$$\pm C = \frac{y_0 - b}{y_0 - a}, \qquad y_0 := y(0).$$

Now we can solve for y. We introduce the abbreviation $u := \pm Ce^{(b-a)t}$ to save writing:

$$\frac{y-b}{y-a} = u \implies y = (y-a)u + b \implies y\big(1-u\big) = b - au \implies y = \frac{b-au}{1-u}.$$

Now plug back in the values of u and $\pm C$ and multiply top and bottom of the resulting fraction by $y_0 - a$ to simplify:

$$y = \frac{(y_0-a)b - a(y_0-b)e^{(b-a)t}}{(y_0-a) - (y_0-b)e^{(b-a)t}}.$$

As a check we plug in $t = 0$. We get

$$y = \frac{(y_0-a)b - a(y_0-b)}{(y_0-a)-(y_0-b)} = \frac{(b-a)y_0}{b-a} = y_0.$$

as expected. Now

$$\lim_{t\to\infty} y(t) = \frac{-a(y_0-b)}{-(y_0-b)} = a, \qquad \lim_{t\to-\infty} y(t) = \frac{(y_0-a)b}{(y_0-a)} = b$$

but if $y_0 < a$ there is a negative value of t (namely $t = T_2$ above) where the denominator vanishes and similarly if $y_0 > b$ there is a positive value of t (namely $t = T_1$ above) where the denominator vanishes.

4 Wednesday Jan 28

18. *Three ways to solve $dy/dt + 2y = 3$.* A linear first order ODE is one of form

$$\frac{dy}{dt} + P(t)y = Q(t). \tag{1}$$

If P and Q are constants we can solve by separation of variables. For example to solve $dy/dt + 2y = 3$ we write

$$\frac{\ln(2y-3)}{2} = \int \frac{dy}{2y-3} = \int dt = t + c$$

so $2y - 3 = e^{2t}C$ (where $C = e^c$) and hence $y = (3 + e^{2t}C)/2$. This doesn't work if either P or Q is not a constant. In the **method of integrating factors** we multiply the ODE (1) by a function ρ to get

$$\rho(t)\frac{dy}{dt} + \rho(t)P(t)y = \rho(t)Q(t)$$

and then choose ρ so that

$$\frac{d\rho}{dt} = \rho(t)P(t). \tag{2}$$

The ODE (1) then takes the form

$$\frac{d}{dt}\big(\rho y\big) = \rho Q \tag{3}$$

which can be solved by integration. In the **method of variation of parameters** we look for a solution of the form

$$y = \Phi(t)u(t)$$

so the ODE (1) takes the form

$$\frac{d\Phi}{dt}u + \Phi\frac{du}{dt} + P\Phi u = Q.$$

Then once we solve

$$\frac{d\Phi}{dt} + P\Phi = 0 \tag{4}$$

the ODE (1) simplifies to

$$\frac{du}{dt} = \Phi^{-1}Q. \tag{5}$$

In either method we first reduce to a homogeneous linear ODE (either (2) or (4)) and then do an integration problem (either (3) or (5)).

Remark 19. Because equation (3) is the homogeneous linear ODE corresponding to the inhomogeneous linear ODE (1), the general solution of (3) is of form $\Phi(t)C$ where C is an arbitrary constant. Having solved this problem by separating variables we solve (1) by trying to find a solution where the constant C is replaced by a variable u. For this reason the method of variation of parameters is also called the **method of variation of constants**. The text uses the method of integrating factors for a single ODE in section 1.5 page 50 and the method of variation of constants for systems on section 8.2 page 493.

Example 20. To solve $dy/dx = x - y$ rewrite it as $dy/dx + y = x$. Multiply by $\rho(x) = e^x$ to get

$$\frac{dy}{dx}e^x + ye^x = xe^x.$$

Then

$$\frac{d}{dx}\big(ye^x\big) = \frac{dy}{dx}e^x + ye^x = xe^x$$

so. integrating by parts,

$$ye^x = \int xe^x\,dx = xe^x - e^x + C$$

so

$$y = x - 1 + Ce^{-x}.$$

Note that the general solution is asymptotic to the particular ($C = 0$) solution $y = x - 1$.

21. *The Superposition Principle.* **Important!** If

$$\frac{dy_1}{dt} + P(t)y_1 = Q_1(t) \qquad \text{and} \qquad \frac{dy_2}{dt} + P(t)y_2 = Q_2(t)$$

and if $y = y_1 + y_2$ and $Q = Q_1 + Q_2$, then

$$\frac{dy}{dt} + P(t)y = Q(t).$$

In particular (take $Q_2 = 0$ and $Q = Q_1$) this shows that the general solution of an inhomogeneous linear equation $\dfrac{dy}{dt} + P(t)y = Q(t)$ is the general solution of the corresponding homogeneous equation $\dfrac{du}{dt} + P(t)u = 0$ plus a particular solution of the inhomogeneous linear equation.

Example 22. When we discussed the slope field of $dy/dx = x - y$ (text figure 1.3.6 page 20) we observed that it looks like all the solutions are asymptotic. Indeed if $dy_1/dx = x - y_1$ and $dy_2/dx = x - y_2$ then

$$\frac{d}{dx}(y_1 - y_2) = -(y_1 - y_2)$$

so $y_1 - y_2 = Ce^{-x}$ so $\lim_{x\to\infty}(y_1 - y_2) = 0$. This proves that all the solutions are asymptotic without solving the equation. The argument works more generally if x is replaced by $Q(x)$, i.e. for the equation $dy/dx = Q(x) - y$.

5 Friday January 30

23. *Mixture problems.* Let x denote the **amount of solute** in volume of size V and c denote its **concentration**. Then

$$c = x/V.$$

In a mixture problem, any of these may vary in time. Thus if a fluid with concentration c_{in} (units = mass/volume) flows into a tank at a rate of r_{in} (units = volume/time) the amount of solute added in time dt is $c_{in}\, r_{in}\, dt$. Similarly if a fluid with concentration c_{out} (units = mass/volume) flows out of the tank at a rate of r_{out} (units = volume/time) the amount of solute removed in time dt is $c_{in}\, r_{in}\, dt$. (The book uses the subscript i as an abbreviation for *in* and the subscript o as an abbreviation for *out*.) Hence the differential equation

$$\frac{dx}{dt} = c_{in}\, r_{in} - c_{out}\, r_{out}.$$

In such problems one generally assumes that c_{in}, r_{in}, and r_{out} are constant but x, c_{in}, and possibly also the volume V of the tank vary.

Example 24. A tank contains V liters of pure water. A solution that contains c_{in} kg of sugar per liter enters a tank at the rate r_{in} Liters/min. The solution is mixed and drains from the tank at the same rate.

(a) How much sugar is in the tank initially?

(b) Find the amount of sugar x in the tank after t minutes.

(c) Find the concentration of sugar in the solution in the tank after 78 minutes.

In this problem $r_{in} = r_{out}$ so the volume V of the tank is constant. In a time interval dt, $c_{in}\ r_{in}\, dt$ kg of sugar enters the tank and $x(t)/V\, dt$ kg of sugar leaves the tank so we have an inhomogeneous linear ODE

$$\frac{dx}{dt} = c_{in} r_{in} - \frac{x}{V} r_{out}$$

with initial value $x(0) = 0$. To save writing we abbreviate $c := c_{in}$, $r := r_{in} = r_{out}$ so the ODE is

$$\frac{dx}{dt} = \left(c - \frac{x}{V}\right) r.$$

Solve by separation of variables

$$-V \ln(Vc - x) = \int \frac{V\, dx}{Vc - x} = \int r\, dt = rt + K.$$

Since the tank initially holds pure water we have $x = 0$ when $t = 0$, hence $K = -V \ln(Vc)$ so $-K/V = \ln(Vc)$. Solving for x gives

$$\ln(Vc - x) = -\frac{rt}{V} + \ln(Vc) \implies x = Vc\left(1 - \exp\left(-\frac{rt}{V}\right)\right)$$

Remark 25. When x is small, the term x/V is even smaller so the equation is roughly $dx/dt = c_{in} r_{out}$ and the answer small values of t is roughly $x = (c_{in} r_{in})t$. For small values of t the amount of sugar x is also small and the approximation $x = (c_{in} r_{in})t$ is very accurate – so accurate that it may fool WeBWorK – but it is obviously wrong for large values of t. The reason is that $\lim_{t\to\infty}(c_{in} r_{in})t = \infty$ whereas $\lim_{t\to\infty} x = c_{in} V$ so that the limiting concentration of the sugar in the tank is the same as the concentration of solution flowing in.

Remark 26. One student was assigned this problem in WeBWorK with values of $V = 2780$, $c = 0.06$ and $r = 3$ and complained to me that WeBWorK rejected the answer. I typed

```
2780*0.06[1-exp(-3t/2780)]
```

and WeBWorK accepted the answer. The student had typed the value

```
(-2780/3)(exp((-3(t+1589))/2780)-.18)
```

and WeBWorK rejected that answer. The two answers would agree if

```
exp(-3*1589/2780)=0.18
```

but this isn't exactly true. I typed `exp(-3*1589/2780)` into the answer box for the part 1 of the question to see what WeBWorK thinks is the value and WeBWork said the value is `0.180009041024602`. (The answer to part 1 is 0, but when I hit the Preview Button WeBWorK did the computation.) I replaced `0.18` by this value in the student's answer and WeBWorK accepted it.

6 Monday February 2

Here are some tricks for solving special equations. The real trick is to find a trick for remembering the trick.

27. *Linear substitutions.* To solve

$$\frac{dy}{dx} = (ax + by + c)^p$$

try $v = ax + by + c$ so

$$\frac{dv}{dx} = a + b\frac{dy}{dx} = a + bv^p$$

28. *Homogeneous equations.* A linear equation is called *homogeneous* if a scalar multiple of a solution is again a solution. A function $h(x, y)$ is called **homogeneous of degree** n if

$$h(\lambda x, \lambda y) = \lambda^n h(x, y).$$

In particular, f is homogeneous of degree 0 iff $f(\lambda x, \lambda y) = f(x, y)$. Then

$$f(x, y) = F\left(\frac{y}{x}\right), \qquad F(u) := f(1, u)$$

To solve

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Try the substitution $v = y/x$.

29. *Bernoulli equations.* This is an equation of form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

Try $y = v^p$ and solve for a value of p which makes the equation simpler.

30. *Exact equations.* The equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is **exact** if there is a function $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = M, \qquad \frac{\partial F}{\partial y} = N \tag{$**$}$$

then

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{$*$}$$

because

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}.$$

In Math 234 you learn that converse is true (if M and N are defined for all (x, y). Exactness implies that the solutions to the ODE

$$M(x, y) + N(x, y)\,\frac{dy}{dx} = 0$$

are the curves

$$F(x, y) = c$$

for various values of c. To find $F(x, y)$ satisfying $(**)$ choose (x_0, y_0) and integrate from (x_0, y_0) along any path joining (x_0, y_0) to (x, y). Condition $(*)$ guarantees that the integral is independent of the choice of the path.

Example 31. Write the ODE

$$3x^2y^5 + 5x^3y^4\frac{dy}{dx} = 0$$

as

$$3x^2y^5\,dx + 5x^3y^4\,dy = 0.$$

The exactness condition $(*)$ holds as

$$\frac{\partial}{\partial y}3x^2y^5 = 15x^3y^4 = \frac{\partial}{\partial x}5x^3y^4.$$

Let $(x_0, y_0) = (0, 0)$ and compute $F(x, y)$ by integrating first along the y-axis (where $dx = 0$) from $(0, 0)$ to $(0, y)$ and the along the horizontal line from $(0, y)$ to (x, y) (where $dy = 0$). We get

$$\begin{aligned} F(x,y) &= \int_{t=0}^{t=y} N(0,t)\,dt + \int_{t=0}^{t=x} M(t,y)\,dt \\ &= \int_{t=0}^{t=y} 5(0^3)t^4\,dt + \int_{t=0}^{t=x} 3t^2y^5\,dt \\ &= 0 + x^3y^5 = x^3y^5. \end{aligned}$$

so the solutions of the ODE are the curves $x^3y^5 = C$. Because the exactness condition holds it doesn't matter which path we use to compute $F(x, y)$ so long as it goes from $(0, 0)$ to (x, y). For example, integrating first along the x-axis (where $dy = 0$) from $(0, 0)$ to $(x, 0)$ and the along the vertical line from $(x, 0)$ to (x, y) (where $dx = 0$) gives

$$\begin{aligned} F(x,y) &= \int_{t=0}^{t=y} N(x,t)\,dt + \int_{t=0}^{t=x} M(t,0)\,dt \\ &= \int_{t=0}^{t=y} 5(x^3)t^4\,dt + \int_{t=0}^{t=x} 3t^2(0^5)\,dt \\ &= x^3y^5 + 0 = x^3y^5. \end{aligned}$$

Along the diagonal line from $(0, 0)$ to (x, y) we have $dx = x\,dt$ and $dy = y\,dt$ with t running from 0 to 1 so

$$\begin{aligned} F(x,y) &= \int_{t=0}^{t=1} N(tx,ty)y\,dt + \int_{t=0}^{t=1} M(tx,ty)x\,dt \\ &= \int_{t=0}^{t=1} 5t^7x^3y^4y\,dt + \int_{t=0}^{t=1} 3t^7x^2(y^5)x\,dt \\ &= \tfrac{5}{8}x^3y^5 + \tfrac{3}{8}x^3y^5 = x^3y^5. \end{aligned}$$

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32. *Reducible second order equations.* A second order ODE where either the unknown x or or its derivative dx/dt is missing can be reduced the equation to a first order equation. If x is missing the equation is already first order in dx/dt. The case where both t and dx/dt are missing is like a conservative force field in physics, i.e. a force field which is the negative gradient of a potential energy function U so Newton's third law takes the form

$$m\frac{d^2\mathbf{x}}{dt^2} = -\boldsymbol{\nabla} U$$

In this case the energy

$$E := \frac{m|\mathbf{v}|^2}{2} + U, \qquad \mathbf{v} := \frac{d\mathbf{x}}{dt}$$

is conserved (constant along solutions). When the number of dimensions is one (but not in higher dimensions) every force field is a gradient and we can use this fact to reduce the order. To solve

$$\frac{d^2x}{dt^2} = f(x)$$

take $U = -\int f(x)\,dx$ and $v = dx/dt$ so the equation becomes

$$\tfrac{1}{2}\left(\frac{dx}{dt}\right)^2 + U(x) = E$$

which can be solve by separation of variables.

Example 33. Consider the equation

$$m\frac{d^2x}{dt^2} = -kx.$$

Define the **velocity** v, and the **total energy** E by

$$v := \frac{dx}{dt}, \qquad E := \frac{mv^2}{2} + \frac{kx^2}{2}.$$

(The total energy is the sum of the **kinetic energy** $mv^2/2$ and the **potential energy** $U(x) := kx^2/2$.) Now

$$\frac{dE}{dt} = mv\frac{dv}{dt} + kx\frac{dx}{dt} = \left(m\frac{dv}{dt} + kx\right)v = 0,$$

so the total energy E is constant along solutions. Then

$$\frac{dx}{dt} = v = \pm\sqrt{\frac{2E - kx^2}{m}}.$$

We solve the initial value problem $v(0) = 0$ and $x(0) = x_0$. Then $2E = kx_0^2$ so

$$\frac{dx}{dt} = \mu\sqrt{x_0^2 - x^2}, \qquad \mu := \pm\sqrt{\frac{k}{m}},$$

so

$$\frac{dx}{\sqrt{x_0^2 - x^2}} = \mu\, dt$$

so

$$-\cos^{-1}\left(\frac{x}{x_0}\right) = \int \frac{dx}{\sqrt{x_0^2 - x^2}} = \int \mu\, dt = \mu t + C.$$

When $t = 0$, we have $x = x_0$ so $x.x_0 = 1$ so (since $\cos(0) = 1$) C=0 and hence so $x = x_0 \cos(\mu t)$.

Remark 34. On page 70 of the text, the problem is treated a little differently. The unknown x is viewed as the independent variable and the substitution

$$v = \frac{dx}{dt}, \qquad \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot v$$

is used to transform the equation

$$m\frac{d^2x}{dt^2} + kx = 0$$

into the equation

$$m\frac{dv}{dx}v + kx = 0.$$

Solving this by separation of variables gives

$$m\int v\, dv + k\int x\, dx = 0$$

which is the conservation law $\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E$ from before. The book uses the letters x, y, p where I have used t, x, v. (I deviated from the book's notation to emphasize the connection with physics.)

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35. *Peak Oil.* In 1957 a geologist named M. K. Hubbert plotted the annual percentage rate of increase in US oil production against total cumulative US oil production and discovered that the data points fell (more or less) on a straight line. Specifically

$$\frac{dQ/dt}{aQ} + \frac{Q}{b} = 1$$

where $Q = Q(t)$ us the total amount of oil (in billions of barrels) produced by year t, $a = 0.055$, $b = 220$ with the initial condition $Q(1958) = 60$.[2] The ODE for Q can be written as

$$\frac{dQ}{dt} = aQ - kQ^2, \qquad k = \frac{a}{b}.$$

This equation is called the **Logistic Equation**. (We solved a similar equation $dy/dt = (y-a)(y-b)$ above.) By solving this equation Hubbert predicted that annual US oil production would peak (i.e. dQ/dt would become negative) in the year 1975. The peak actually occurred in 1970 but this went unnoticed because by this time the US had begun to import much of its oil. A similar calculation for world oil production produced a prediction of a peak in the year 2005.

36. *First order autonomous quadratic equations.* Consider the equation

$$\frac{dx}{dt} = Ax^2 + Bx + C.$$

The right hand side will have either two zeros, one (double) zero, or no (real) zeros depending on whether $B^2 - 4AC$ is positive, zero, or negative. If there are two zeros, say

$$p := \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \qquad q := \frac{-B - \sqrt{B^2 - 4AC}}{2A},$$

then the equations may be written as

$$\frac{dx}{dt} = A(x-p)(x-q)$$

[2] I got these figures from page 155 (see also page 201) of the very entertaining book: Kenneth S. Deffeyes, *Hubbert's Peak*, Princeton University Press, 2001. I estimated the initial condition from the graph, so it may not be exactly right.

and the limiting behavior can be determined from the phase diagram as we did last week. If there are no zeros, all solutions reach $\pm\infty$ is finite time. After completing the square and rescaling x and t the equation has one of the folloeing three forms:

Example 37. *Example with no zeros.* We solve the ODE

$$\frac{dx}{dt} = 1 + x^2, \qquad x(0) = x_0.$$

We separate variables and integrate:

$$\tan^{-1}(x) = \int \frac{dx}{1+x^2} = \int dt = t + c, \qquad c := \tan^{-1}(x_0),$$

so for $-\pi/2 < t < \pi/2$ we have

$$x = \tan(t + c) = \frac{\tan t + \tan c}{1 - \tan t \tan c} = \frac{\tan t + x_0}{1 - x_0 \tan t}.$$

The solution becomes infinite when $t = \tan^{-1}(1/x_0)$.

Example 38. *Example with two zeros.* We solve the ODE

$$\frac{dx}{dt} = 1 - x^2, \qquad x(0) = x_0.$$

We separate variables and integrate:

$$\tanh^{-1}(x) = \int \frac{dx}{1-x^2} = \int dt = t + c, \qquad c := \tanh^{-1}(x_0),$$

so for $-\pi/2 < t < \pi/2$ we have

$$x = \tanh(t + c) = \frac{\tanh t + \tanh c}{1 + \tanh t \tanh c} = \frac{\tanh t + x_0}{1 + x_0 \tanh t}.$$

For $-1 < x_0 < 1$ we have $\lim_{t\to\infty} x = 1$ and $\lim_{t\to-\infty} x = -1$.

Example 39. *Example with a double zero.* The solution of the ODE

$$\frac{dx}{dt} = x^2, \qquad x(0) = x_0$$

is $x = x_0/(x_0 - t)$. If $x_0 \neq 0$ it becomes infinite when $t = x_0$.

40. After a change of variables, every quadratic ODE

$$\frac{dy}{ds} = Ay^2 + By + C$$

takes one of these three forms. Divide by A and complete the square

$$\frac{1}{A}\frac{dy}{ds} = \left(y + \frac{B}{2A}\right)^2 - \frac{B^2 - 4AC}{4A^2}$$

Let $u = y + (B/2A)$ and $k^2 = |(B^2 - 4AC)/(4A^2)|$:

$$\frac{1}{A}\frac{du}{ds} = u^2 \pm k^2.$$

Finally (if $k \neq 0$) divide by k^2 and let $x = u/k$ and $t = -Ak^2 s$ to arrive at $dx/dt = 1 \pm x^2$. (If $k = 0$, take $x = u$ and $t = As$ to arrive at $dx/dt = x^2$.)

41. *Trig functions and hyperbolic functions.*

$$\sin(t) = \frac{e^{it} - e^{-it}}{2i} \qquad \sinh(t) = \frac{e^{t} - e^{-t}}{2}$$

$$\cos(t) = \frac{e^{it} + e^{-it}}{2} \qquad \cosh(t) = \frac{e^{t} + e^{-t}}{2}$$

$$i\tan(t) = \frac{e^{it} - e^{-it}}{e^{it} + e^{-it}} \qquad \tanh(t) = \frac{e^{t} - e^{-t}}{e^{t} + e^{-t}}$$

$$\cos^2(t) + \sin^2(t) = 1 \qquad \cosh^2(t) - \sinh^2(t) = 1$$

$$d\,\sin(t) = \cos(t)\,dt \qquad d\,\sinh(t) = \cosh(t)\,dt$$

$$d\,\cos(t) = -\sin(t)\,dt \qquad d\,\cosh(t) = \sinh(t)\,dt$$

$$\tan(t+s) = \frac{\tan(t) + \tan(s)}{1 - \tan(t)\tan(s)} \qquad \tanh(t+s) = \frac{\tanh(t) + \tanh(s)}{1 + \tanh(t)\tanh(s)}$$

42. *Bifurcation and dependence on parameters.* The differential equation

$$\frac{dx}{dt} = x(4 - x) - h$$

models a logistic population equation with harvesting rate h. The equilibrium points are

$$H = 2 - \sqrt{4 - h}, \qquad N = 2 + \sqrt{4 - h}$$

if $h < 4$, There is a **bifurcation** at $h = 4$. This means that the qualitative behavior of the system changes as h increases past 4. When $h = 4$ there is a double root ($H = N$) and for $h > 4$ there is no real root and all solutions reach $-\infty$ in finite time.

43. *Air resistance proportional to v.* The equation of motion is $F = ma$ where the force is $F = F_G + F_R$ with

$$a = \frac{dv}{dt}, \qquad v = \frac{dy}{dt}, \qquad F_G = -g, \qquad F_R = -kv.$$

This can be solved by separation of variables and there is a **terminal velocity**

$$v_\infty := \lim_{t\to\infty} v = -mg/k$$

whicsh is independent of the initial velocity.

44. *Air resistance proportional to v^2.* The equation of motion is $F = ma$ where the force is $F = F_G + F_R$ with

$$a = \frac{dv}{dt}, \qquad v = \frac{dy}{dt}, \qquad F_G = -g, \qquad F_R = -kv|v|.$$

Thus

$$m\frac{dv}{dt} = \begin{cases} -g - kv^2 & \text{when } v > 0 \\ -g + kv^2 & \text{when } v < 0. \end{cases}$$

After rescaling (i.e. a change of variable) we can suppose $m = g = k$ and we use the above. To find the height y we need to choose the constants of integration correctly.

45. *Escape velocity.* A spaceship of mass m is attracted to a planet of mass M by a gravitational force of magnitude GMm/r^2 so that (after cancelling m) the equation of motion (if gravity is the only force acting on the spaceship) is

$$\frac{dv}{dt} = \frac{d^2r}{dt^2} = -\frac{GM}{r^2}$$

where r is the distance of the spaceship to the center of the planet and $v = dr/dt$ is the velovity of the spaceship. As above, the energy

$$E := \frac{mv^2}{2} - \frac{GMm}{r}$$

is a constant of the motion so if $r(0) = r_0$ and $v(0) = v_0$ we have (after dividing by $m/2$)

$$v^2 - \frac{2GM}{r} = v_0^2 - \frac{2GM}{r_0}$$

from which follows

$$v^2 > v_0^2 - \frac{2GM}{r_0}.$$

The quantity $\sqrt{2GM/r_0}$ is called the **escape velocity**. If v_0 is greater than the esacape velocity then

$$r(t) = \int_0^t \frac{dr}{dt}\,dt = \int_0^t v\,dt > \int_0^t \sqrt{v_0^2 - \frac{2GM}{r_0}}\,dt = t\sqrt{v_0^2 - \frac{2GM}{r_0}}$$

so r becomes infinite in finite time.

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46. *Monthly Investing.* Mary starts a savings account. She plans to invest $100 + t$ dollars t months after opening the account. The account pays 6% annual interest. How much is in the account after t months? Denote by $S(t)$ the amount in the account after t months. Then $S(0) = 100$ and $S(t+1) = S(t)$+interest+deposit, i.e.

$$S(t+1) = S(t) + \frac{0.06}{12}S(t) + (100+t)$$

This equation can be written in the form

$$S(t+h) = S(t) + f(t, S(t))h$$

where $h = 1$ and $f(t, S) = \dfrac{0.06}{12}S(t) + (100+t)$. It can also be written

$$\Delta S = f(t, S(t))\Delta t$$

where $\Delta S = S(t+h) - S(t)$ and $\Delta t = h = 1$.

47. *Daily Investing.* Donald starts a savings account. He plans to invest daily at a rate of $100+t$ dollars per month after opening the account. The account pays 6% annual interest. How much is in the account after t months? Denote by $S(t)$ the amount in the account after t months. This is $n = 30t$ days. One day is h months where $h = 1/30$. Then $S(0) = 100$ and $S(t+h) = S(t)$+one day's interest+one day's deposit, i.e.

$$S(t+h) = S(t) + \frac{0.06}{12}S(t)h + (100+t)h, \qquad h = \frac{1}{30}, \qquad t = nh$$

This equation can be written in the form

$$S(t+h) = S(t) + f(t, S(t))h$$

where $h = 1/30$ and $f(t, S) = \dfrac{0.06}{12}S(t) + (100 + t)$. It can also be written

$$\Delta S = f(t, S(t))\Delta t$$

where $\Delta S = S(t+h) - S(t)$ and $\Delta t = h = \frac{1}{30}$.

48. *Hourly Investing.* Harold starts a savings account. He plans to invest hourly at a rate of $100 + t$ dollars per month after opening the account. The account pays 6% annual interest. How much is in the account after t months? Denote by $S(t)$ the amount in the account after t months. This is $n = 720t$ hours. One hour is h months where $h = 1/720$. Then $S(0) = 100$ and $S(t + h) = S(t)$+one hours's interest+one hours's deposit, i.e.

$$S(t+h) = S(t) + \frac{0.06}{12}S(t)h + (100+t)h, \qquad h = \frac{1}{720}, \qquad t = nh$$

This equation can be written in the form

$$S(t+h) = S(t) + f(t, S(t))h$$

where $h = 1/720$ and $f(t, S) = \dfrac{0.06}{12}S(t) + (100 + t)$. It can also be written

$$\Delta S = f(t, S(t))\Delta t$$

where $\Delta S = S(t+h) - S(t)$ and $\Delta t = h = \frac{1}{720}$.

49. *Continuous Investing.* Cynthia starts a savings account. She plans to invest continuously at a rate of $100 + t$ dollars per month after opening the account. The account pays 6% annual interest. How much is in the account after t months? Denote by $S(t)$ the amount in the account after t months. Then $S(0) = 100$ and the change dS in the account in an infinitessimal time interval of size dt at time t is

$$dS = \frac{0.06}{12}S(t)\,dt + (100+t)\,dt$$

This equation can be written in the form

$$dS = f(t, S(t))\,dt$$

where $f(t, S) = \dfrac{0.06}{12}S(t) + (100 + t)$.

Remark 50. Mary is getting an annual interest rate of 6% compounded monthly Donald is getting an annual interest rate of 6% compounded daily and is investing a little more each month than is Mary. Harold is getting an annual interest rate of 6% compounded hourly and is investing a little more each month than is Donald. Cynthia is getting an annual interest rate of 6% compounded continuously and is investing a little more each month than is Harold. The point is that all the answers are about the same. Here's why:

Theorem 51 (The Error in Euler's Method)**.** *Assume that $f(t, y)$ is continuously diffentiable. Let $y = y(t)$ be the solution to the initial value problem*

$$\frac{dy}{dt} = f(t, y), \qquad y(0) = y_0$$

and y_n be the solution to the difference equation

$$y_{n+1} = y_n + f(nh, y_n)h.$$

Then there is a constant $C = C(f, T)$ (dependent on T and f but independent of h) such that

$$|y(t) - y_n| \leq Ch,$$

for $t = nh$ and $0 \leq t \leq T$.

Remark 52. This theorem is stated on page 122 of the text. When I get a chance, I will put a formula for C in these notes and provide a proof. (Only motivated students should try to learn the proof.)