Math 320 – Exam II. Friday Mar 27, 09:55-10:45

Answers

I. (50 points.) Complete the definition.

(i) A subset $W \subseteq V$ of a vector space V is called a **subspace** iff

Answer: A subset $W \subseteq V$ of a vector space V is called a **subspace** iff it is **closed** under the vector space operations, i.e. iff

(i) $\mathbf{0} \in W$,

(ii) $\mathbf{u}, \mathbf{v} \in W \implies \mathbf{u} + \mathbf{w} \in W$, and

(iii) $c \in \mathbb{R}, \mathbf{u} \in W \implies c\mathbf{u} \in W$.

(ii). The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are said to be **linearly independent** iff

Answer: The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are said to be **linearly independent** iff the only solution of the equation

$$x_1\mathbf{v}_2 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k = 0$$

is the **trivial solution** $x_1 = x_2 = \cdots = x_k = 0$.

(iii). The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are said to **span** V iff they lie in V and

Answer: The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are said to **span** V iff they lie in V and every vector in V is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, i.e. iff for every vector $\mathbf{v}$ in V there exist numbers $x_1, x_2, \ldots, x_k$ such that

$$\mathbf{v} = x_1\mathbf{v}_2 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k.$$

II. (50 points.) I have been assigned the task of finding matrices $\mathbf{P}$, $\mathbf{P}^{-1}$ and $\mathbf{W}$ so that $\mathbf{W}$ is in reduced echelon form and $\mathbf{PA} = \mathbf{W}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 & -2 \\ -2 & -4 & 6 & 4 \\ 5 & 10 & -14 & -8 \end{bmatrix}.$$

After some calculation I found matrices $\mathbf{B}$ and $\mathbf{N}$ such that $\mathbf{NA} = \mathbf{B}$ and

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \qquad \mathbf{N} = \begin{bmatrix} 0 & 7 & 3 \\ 3 & 4 & 1 \\ 1 & 3 & 1 \end{bmatrix}, \qquad \mathbf{N}^{-1} = \begin{bmatrix} 1 & 2 & -5 \\ -2 & -3 & 9 \\ 5 & 7 & -21 \end{bmatrix}.$$

Only one elementary row operation to go! Find $\mathbf{P}$, $\mathbf{P}^{-1}$, and $\mathbf{W}$.

Answer: Let $\mathbf{E}$ be the elementary matrix $\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ so $\mathbf{E}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $\mathbf{W} =$ $\mathbf{ENA} = \mathbf{EB} = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in reduced echelon form. We can take

$$\mathbf{P} = \mathbf{EN} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 7 & 3 \\ 3 & 4 & 1 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 7 & 3 \\ 3 & 4 & 1 \\ -2 & -1 & 0 \end{bmatrix}$$

so

$$\mathbf{P}^{-1} = \mathbf{N}^{-1}\mathbf{E}^{-1} = \begin{bmatrix} 1 & 2 & -5 \\ -2 & -3 & 9 \\ 5 & 7 & -21 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -5 \\ -2 & 6 & 9 \\ 5 & -14 & -21 \end{bmatrix}$$

III. (50 points.) Here are the matrices from Problem II again.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 & -2 \\ -2 & -4 & 6 & 4 \\ 5 & 10 & -14 & -8 \end{bmatrix}.$$

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \qquad \mathbf{N} = \begin{bmatrix} 0 & 7 & 3 \\ 3 & 4 & 1 \\ 1 & 3 & 1 \end{bmatrix}, \qquad \mathbf{N}^{-1} = \begin{bmatrix} 1 & 2 & -5 \\ -2 & -3 & 9 \\ 5 & 7 & -21 \end{bmatrix}.$$

(They still satisfy the equation $\mathbf{NA} = \mathbf{B}$.) True or false? *The second and fourth columns of* $\mathbf{A}$ *form a basis for its column space.* **Justify your answer.**

Answer: This is true. First note that the second and fourth coulumns of $\mathbf{B}$ form a basis for the column space of $\mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{bmatrix}$. This is because $\mathbf{b}_1 = \frac{1}{2}\mathbf{b}_2 + 0\mathbf{b}_4$ and $\mathbf{b}_3 = \frac{1}{2}\mathbf{b}_4 - \mathbf{b}_2$. Hence the span of $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$ (i.e. the column space of $\mathbf{B}$) is the same as the span of $\mathbf{b}_2, \mathbf{b}_4$. The vectors $\mathbf{b}_2, \mathbf{b}_4$ are independent since the submatrix $\begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}$ of $\begin{bmatrix} \mathbf{b}_2 & \mathbf{b}_4 \end{bmatrix}$ has nonzero determinant so the only soultion of $x\mathbf{b}_2 + y\mathbf{b}_4 = \mathbf{0}$ is $x = y = 0$. Because $\mathbf{a}_i = \mathbf{N}^{-1}\mathbf{b}_i$ we have $\mathbf{a}_1 = \frac{1}{2}\mathbf{a}_2 + 0\mathbf{a}_4$ and $\mathbf{a}_3 = \frac{1}{2}\mathbf{a}_4 - \mathbf{a}_2$ so the span of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ (i.e. the column space of $\mathbf{A}$) is the same as the span of $\mathbf{a}_2, \mathbf{a}_4$. If $x\mathbf{a}_2 + y\mathbf{a}_4 = \mathbf{0}$ then $x\mathbf{N}\mathbf{a}_2 + y\mathbf{N}\mathbf{a}_4 = x\mathbf{b}_2 + y\mathbf{b}_4\mathbf{0}$ so $x = y = 0$ so $\mathbf{a}_2, \mathbf{a}_4$ are independent. Thus $\mathbf{a}_2, \mathbf{a}_4$ is a basis for the column space of $\mathbf{A}$ and the dimension of this column space is two. (Both the algorithm from the book on pages 250-260 and the proof of Theorem 73 in the notes tell us that the first and third columns also form a basis, and the arguments we just used are the same as the arguments used there.)

IV. (50 points.) (i) Find the inverse of the matrix $\begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$. Hint: You can use the formula.

Answer: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ so $\begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$.

(ii) The vector $(-4, 1, b)$ is in the span of the vectors $(7, 2, 3)$ and $(3, 1, 4)$. What is b? Hint: You can save a little bit of work by using part (i).

Answer: If $\begin{bmatrix} -4 \\ 1 \\ b \end{bmatrix} = x_1\begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$ then $\begin{bmatrix} -4 \\ 1 \end{bmatrix} = x_1\begin{bmatrix} 7 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ so $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}^{-1}\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ 15 \end{bmatrix}$ so $b = 3x_1 + 4x_2 = -21 + 60 = 39$.

V. (50 points.) Find a basis for the subspace of $\mathbb{R}^5$ consisting of all vectors $\mathbf{w}$ which are orthogonal to the vectors $(1, 0, 2, 3, 4)$ and $(0, 1, 4, 5, 6)$. What is the dimension of this subspace?

Answer: This is the same as the null space of the matrix $\begin{bmatrix} 1 & 0 & 2 & 3 & 4 \\ 0 & 1 & 4 & 5 & 6 \end{bmatrix}$. The three vectors $(-1, -4, 1, 0, 0)$, $(-3, -5, 0, 1, 0)$, and $(-4, -6, 0, 0, 1)$, form a basis and so the dimension is three.