Math 320 – Exam I. Friday Feb 13, 09:55-10:45
Answers

I. (60 points.) (a) Find $x = x(t)$ if $\dfrac{dx}{dt} + \dfrac{x}{1+t^2} = 0$ and $x(0) = x_0$.

Answer: By separation of variables

$$\ln|x| = \int \frac{dx}{x} = -\int \frac{dt}{1+t^2} = -\tan^{-1}(t) + C$$

so (as $|x| = \pm x$) we get

$$x = \pm\exp\bigl(-\tan^{-1}(t) + C\bigr) = c\exp\bigl(-\tan^{-1}(t)\bigr), \qquad c = \pm e^C.$$

As $\tan^{-1}(0) = 0$ the initial condition gives $x_0 = c$. so

$$x = x_0 \exp\bigl(-\tan^{-1}(t)\bigr).$$

(b) Find $y = y(t)$ if $\dfrac{dy}{dt} + \dfrac{y}{1+t^2} = \exp(-\tan^{-1} t)$ and $y(0) = y_0$.

Answer: This is an inhomogeneous linear equation and the general solution

$$x(t) = c\Phi(t), \qquad \Phi(t) := \exp\bigl(-\tan^{-1} t\bigr).$$

of the corresponding homogeneous linear equation was found in part (a). We use as Ansatz

$$y(t) = c(t)\Phi(t).$$

Then

$$\begin{aligned}
\frac{dy}{dt} + \frac{y}{1+t^2} &= \Bigl(c'(t)\Phi(t) + c(t)\Phi'(t)\Bigr) + \frac{c(t)\Phi(t)}{1+t^2} \\
&= c'(t)\Phi(t) + c(t)\left(\Phi'(t) + \frac{\Phi(t)}{1+t^2}\right) \\
&= c'(t)\Phi(t) \\
&= \exp\bigl(-\tan^{-1} t\bigr) \qquad \text{if } c'(t) = 1
\end{aligned}$$

i.e. if $c(t) = t + \mathit{constant}$. Evaluating at $t = 0$ shows that the constant must be y_0 so $c(t) = t + y_0$ so the solution is

$$y(t) = (t + y_0)\exp\bigl(-\tan^{-1} t\bigr).$$

II. (40 points.) (a) State the Existence and Uniqueness Theorem for Ordinary Differential Equations.

Answer: If $f(t,x)$ has continuous partial derivatives then the initial value problem

$$\frac{dy}{dx} = f(x,y), \qquad y(x_0) = y_0$$

has a unique solution $y = y(x)$.

(b) State the Exactness Criterion for the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0.$$

Answer: If $M(x,y)$ and $N(x,y)$ have continuous partial derivatives, then there is a function $F = F(x,y)$ solving the equations

$$\frac{\partial F}{\partial x} = M, \qquad \frac{\partial F}{\partial y} = N$$

if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

(c) Does the differential equation

$$(1 + x^2 + x^4y^6)\,dx + (1 + x^2 + y^2)\,dy = 0$$

have a solution $y = y(x)$ satisfying the initial condition $y(0) = 5$? (Explain your answer.)

Answer: Yes. The differential equation can be written as

$$\frac{dy}{dx} = f(x,y), \qquad f(x,y) = -\frac{1 + x^2 + x^4y^6}{1 + x^2 + y^2}$$

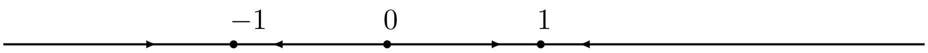
so the Existence and Uniqueness Theorem applies. (The Exactness Criterion is irrelevant here.)

III. (30 points.) Consider the differential equation $\dfrac{dx}{dt} = (1-x)x(1+x)$.

(a) Draw a phase diagram.

Answer:

	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x$
$(1-x)x(1+x)$	> 0	< 0	> 0	< 0



(b) Determine the limit $\lim_{t\to\infty} x(t)$ if $x(t)$ is the solution with $x(0) = -0.5$.

Answer: From the phase diagram $\lim_{t\to\infty} x(t) = -1$.

(b) Determine the limit $\lim_{t\to\infty} x(t)$ if $x(t)$ is the solution with $x(0) = -1$.

Answer: The constant function $x(t) = -1$ satisfies both the differential equation and initial condition $x(0) = -1$. Therefore it is the only solution by the the Existence and Uniqueness Theorem so $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} -1 = -1$. Similarly if $x(0) = 0$ then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} 0 = 0$.

IV. (60 points.) A 1200 gallon tank initially holds 900 gallons of salt water with a concentration of 0.5 pounds of salt per gallon. Salt water with a concentration of 11 pounds of salt per gallon flows into the tank at a rate of 8 gallons per minute and the well stirred mixture flows out of the tank at a rate of 3 gallons per minute. Write a differential equation for the amount $x = x(t)$ of salt in the tank after t minutes. (You need not solve the differential equation but do give the initial condition.) SHOW YOUR REASONING.

Answer: After t minutes $8t$ gallons of saltwater has flowed into the tank and $3t$ gallons has flowed out so the volume of the saltwater in the tank is $V = 900 + 5t$. The concentration of salt in this saltwater is x/V pounds per gallon. In a tiny time interval of size dt the amount of saltwater flowing out of the tank is $3\,dt$ gallons and the amount of salt in this saltwater is $(x/V) \times (3\,dt)$ pounds. In this same tiny time interval the amount of saltwater flowing in is $8\,dt$ gallons and the amount of salt in that saltwater is $11 \times 8\,dt = 88\,dt$ pounds. Hence the net change in the amount of salt is

$$dx = 88\,dt - \frac{3x\,dt}{V} = \left(88 - \frac{3x}{900+5t}\right) dt.$$

Initially $V = 900$ gallons so $x(0) = 0.5 \times 900$ pounds. Thus the ODE is

$$\frac{dx}{dt} = 88 - \frac{3x}{900+5t}, \qquad x(0) = 450.$$

The fact that the tank holds 1200 gallons means that it is full after 60 minutes. This problem (with different numbers) is Example 5 on page 53 of the text.

V. (60 points.) A projectile is launched straight upward from its intial position y_0 with initial velocity $v_0 > 0$. Air resistance exerts a force proportional to the square of the projectile's velocity so that Newton' second law gives that

$$\frac{dv}{dt} = \frac{F_G + F_R}{m} = -1 - v|v|.$$

(To simplify the problem we chose units where the gravitation constant is $g = 1$.) The projectile goes up for $0 \le t < T$ and then goes down.

(a) Find a formula for $v(t)$ for $0 < t < T$.

Answer:

$$\tan^{-1} v = \int \frac{dv}{1+v^2} = -\int dt = -t + \tan^{-1} v_0$$

so

$$v = \tan(-t + \tan^{-1} v_0) = \frac{v_0 - \tan t}{1 + v_0 \tan t}.$$

(b) Find a formula for $v(t)$ for $t > T$.

Answer:

$$\tanh^{-1} v = \int \frac{dv}{1-v^2} = -\int dt = -t + T$$

as $v = 0$ when $t = T$ so

$$v = \tanh(T - t).$$

(c) Find a formula T.

Answer: From part (a) $v_0 - \tan T = 0$ so $T = \tan^{-1}(v_0)$.

Table of Integrals and Identities[1]

$$\frac{a+u}{a-u} = e^{2aw} \iff \frac{u}{a} = \frac{e^{aw}-e^{-aw}}{e^{aw}+e^{-aw}}$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C. \qquad \int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C.$$

$$\sin(t) = \frac{e^{it}-e^{-it}}{2i} \qquad \sinh(t) = \frac{e^{t}-e^{-t}}{2}$$

$$\cos(t) = \frac{e^{it}+e^{-it}}{2} \qquad \cosh(t) = \frac{e^{t}+e^{-t}}{2}$$

$$i\tan(t) = \frac{e^{it}-e^{-it}}{e^{it}+e^{-it}} \qquad \tanh(t) = \frac{e^{t}-e^{-t}}{e^{t}+e^{-t}}$$

$$\cos^2(t)+\sin^2(t) = 1 \qquad \cosh^2(t)-\sinh^2(t) = 1$$

$$d\,\sin(t) = \cos(t)\,dt \qquad d\,\sinh(t) = \cosh(t)\,dt$$

$$d\,\cos(t) = -\sin(t)\,dt \qquad d\,\cosh(t) = \sinh(t)\,dt$$

$$\tan(t+s) = \frac{\tan(t)+\tan(s)}{1-\tan(t)\tan(s)} \qquad \tanh(t+s) = \frac{\tanh(t)+\tanh(s)}{1+\tanh(t)\tanh(s)}$$

Remark (added on answer sheet). The formulas

$$\int \frac{dv}{a^2+vu^2} = \frac{1}{a}\tan^{-1}\frac{v}{a} + C. \qquad \int \frac{du}{a^2-u^2} = \frac{1}{a}\tanh^{-1}\frac{u}{a} + C.$$

can be related as follows. The formula

$$\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$$

can be proved with partial fractions. Assume that $-a < u < a$ so that $(u+a)/(u-a) > 0$ and introduce the abbreviation

$$w := \frac{1}{2a}\ln\left(\frac{u+a}{u-a}\right).$$

Multiply by $2a$ and exponentiate to get

$$e^{2aw} = \frac{a+u}{a-u}$$

[1]This is the same table that was emailed to the class yesterday morning.

so $ae^{2aw} - ue^{2aw} = a + u$ so $a(e^{2aw} - 1) = u(e^{2aw} + 1)$ so

$$\frac{u}{a} = \frac{e^{2aw} - 1}{e^{2aw} + 1} = \frac{e^{aw} - e^{-aw}}{e^{aw} + e^{-aw}} = \tanh(aw)$$

by high school algebra. Thus $\tanh^{-1}(u/a) = aw$ so

$$w = \frac{1}{a} \tanh^{-1} \frac{u}{a}.$$

Now the trig functions and hyperbolic functions are related by the formulas

$$i \sin(t) = \sinh(it), \qquad \cos(t) = \cosh(it), \qquad i \tan(t) = \tanh(it)$$

so the substitution $u = iv$, $du = i\,dv$ gives

$$\int \frac{du}{a^2 + u^2} = \int \frac{i\,dv}{a^2 - v^2} = \frac{i}{a} \tanh^{-1} \frac{iv}{a} + C = \frac{1}{a} \tan^{-1} \frac{v}{a} + C.$$