Sample First Exam for Calculus 223

1. Find dw/dt at $t = 3$ if

$$w = \frac{x}{z} + \frac{y}{x}, \qquad x = \cos^2 t, \quad y = \sin^2 t, \quad z = \frac{1}{t}.$$

Answer:

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\ &= \left(\frac{1}{z} - \frac{y}{x^2}\right)(-2\cos t \sin t) + \left(\frac{1}{x}\right)(2\sin t\cos t) + \left(\frac{-x}{z^2}\right)\left(\frac{-1}{t^2}\right) \\ &= \left(3 - \frac{\sin^2 3}{\cos^4 3}\right)(-2\cos 3\sin 3) + \left(\frac{1}{\cos^2 3}\right)(2\sin 3\cos 3) + \left(-9\cos^2 3\right)\left(\frac{-1}{9}\right)\end{aligned}$$

at $t = 3$.

2. Find an equation for the tangent plane to the surface

$$x^3 z + y^2 x^2 + \sin(yz) + 54 = 0$$

at the point $P_0 = (3, 0, -2)$.

Answer:

$$\nabla f = (3x^2 z + 2y^2 x)\mathbf{i} + (2yx^2 + z\cos yz)\mathbf{j} + (x^3 + y\cos yz)\mathbf{k}$$

so

$$\nabla f|_{(3,0,-2)} = -54\mathbf{i} - 2\mathbf{j} + 27\mathbf{k}.$$

The tangent plane is

$$-54(x-3) - 2(y-0) + 27(z+2) = 0.$$

3. The function $f(x, y)$ is defined by

$$f(x, y) = uv, \qquad x = u, \quad y = v + u^2.$$

Find the gradient ∇f at the point $(x_0, y_0) = (3, 13)$. Express your answer in the form $a\mathbf{i} + b\mathbf{j}$.

Answer: We can use implicit differentiation, but it is a little easier to express f in terms of x and y directly using $u = x$ and $v = y - u^2 = y - x^2$ so

$$f(x, y) = x(y - x^2) = xy - x^3.$$

Hence

$$\nabla f = (y - 3x^2)\mathbf{i} + x\mathbf{j}.$$

4. For the function $h(x, y) = x^3 + y^3 - 9xy$:

(i) Find all points $P = (x, y)$ where the gradient ∇h is zero.

(ii) For each point in part (i) say whether it is a local minimum, a local maximum, or a saddle. *Don't guess: an answer without a reason receives no credit.*

Answer: $h_x = 3x^2 - 9y$, $h_y = 3y^2 - 9x$, so $h_x = h_y = 0$ when $x^2 = 3y$ and $y^2 = 3x$. So $x^4 = 9y^2 = 27x$ so $x = 0$ (and $y = 0$) or $x = 3$ (and $y = 3$). At $(x, y) = (0, 0)$ we have $h_{xx}h_{yy} - h_{xy}^2 = -81 < 0$ so we have a saddle. At $(x, y) = (3, 3)$ we have $h_{xx}h_{yy} - h_{xy}^2 = (6x)(6y) - (-9)^2 = 18^2 - 9^2 > 0$ and $h_{xx} = 18 > 0$ so we have a (local) minimum.

5. Find the maximum and minimum values of $x^2 + y^2$ subject to the constraint $x^2 - 2x + y^2 - 4y = 0$.

Answer: The equations $f_x = \lambda g_x$ and $f_y = \lambda g_y$ are

$$2x = \lambda(2x - 2), \qquad 2y = \lambda(2y - 4).$$

Dividing gives

$$\frac{x}{y} = \frac{2x-2}{2y-4} \qquad \text{or} \qquad 2xy - 4x = 2xy - 2y$$

so $y = 2x$. Substitute into the constraint:

$$x^2 - 2x + 4x^2 - 8x = 5x(x - 2)$$

so $x = y = 0$ or $x = 2$ and $y = 4$. $f(0, 0) = 0$ is the minimum and $f(2, 4) = 20$ is the maximum.

6. Find the polynomial $p(x, y)$ of degree two which best approximates the function $f(x, y) = x^2y^3$ near the point $(x_0, y_0) = (1, -1)$.

Answer:

$$\begin{array}{ll}
f(x,y) = x^2y^3 & f(1,-1) = -1 \\
f_x(x,y) = 2xy^3 & f_x(1,-1) = -2 \\
f_y(x,y) = 3x^2y^2 & f_y(1,-1) = 3 \\
f_{xx}(x,y) = 2y^3 & f_{xx}(1,-1) = -2 \\
f_{xy}(x,y) = 6xy^2 & f_y(1,-1) = 6 \\
f_{yy}(x,y) = 6x^2y & f_{yy}(1,-1) = -6
\end{array}$$

so the Taylor polynomial is

$$p(x, y) = -1 - 2(x - 1) + 3(y + 1) - (x - 1)^2 + 6(x - 1)(y + 1) - 3(y + 1)^2.$$

7. Find the extreme values of the function $f(x, y) = x^2 + 3y^2 + 2y$ on the unit disk $x^2 + y^2 \leq 1$. Hint: On the boundary $x^2 = 1 - y^2$.

Answer: The critical points are given by

$$0 = \frac{\partial f}{\partial x} = 2x = 0, \qquad 0 = \frac{\partial f}{\partial y} = 6y + 2,$$

so $x = 0$ and $y = -1/3$. The point $(x, y) = (0, -1/3)$ lies in the region $x^2 + y^2 \leq 1$ so (by the first derivative test) this is a candidate for an extremum. At this point the discriminant is

$$f_{xx} f_{yy} - f_{xy}^2 = 1 > 0$$

and f_{xx} and f_{yy} are both positive so the point is a local minimum by the second derivative test. The value at the critical point is $f(0, -1/3) = 0 + 1/3 - 2/3 = -1/3 < 0$. The maximum occurs on the boundary. We can find the maximum using Lagrange multipliers (Maximize $x^2 + 3y^2 + 2y$ subject to $x^2 + y + 2 + 1$), *or* by maximizing (using Calc 221)

$$f(\cos\theta, \sin\theta) = \cos^2\theta + 3\sin^2\theta + 2\sin\theta,$$

or by using the hint.

Here is how to finish the problem using the hint. On the boundary $x^2 = y^2 - 1$, $-1 \leq y \leq 1$, and $f = (1 - y^2) + 3y^2 + 2y = 2y^2 + 2y + 1 = F(y)$. We must maximize $F(y)$ on the interval $-1 \leq y \leq 1$. This is a calculus 221 problem. The critical point occurs at $F'(y) = 4y + 2 = 0$ so $y = -1/2$ and $F(-1/2) = 1/2 - 1 + 1 = 1/2$. At the endpoints $F(-1) = 1$ and $F(1) = 5$.

In summary: the minimum value $f(0, -1/3) = -1/3$ occurs at at the interior point $(x, y) = (0, -1/3)$, and the maximum value $f(0, 1) = F(1) = 5$ occurs at the boundary point $(x, y) = (0, 1)$.

8. Suppose that

$$w = x^2 - y^2 + 4z + t, \quad \text{and} \quad x + 2z + t = 25.$$

Show that the equations

$$\frac{\partial w}{\partial x} = 2x - 1 \qquad \frac{\partial w}{\partial x} = 2x - 2$$

each give $\partial w/\partial x$ depending on which variables are chosen to be dependent and which are chosen to be independent. Identify (using thermodynamic notation) the independent variables in each case.

Answer: From $x + 2z + t = 25$ we conclude that

$$1 + 2\left(\frac{\partial z}{\partial x}\right)_{yt} = 1 + \left(\frac{\partial t}{\partial x}\right)_{yz} = 0.$$

Hence

$$\left(\frac{\partial w}{\partial x}\right)_{yt} = 2x + 4\left(\frac{\partial z}{\partial x}\right)_{yt} = 2x + 4\left(-\frac{1}{2}\right) = 2x - 2$$

$$\left(\frac{\partial w}{\partial x}\right)_{yz} = 2x + \left(\frac{\partial t}{\partial x}\right)_{yz} = 2x - 1.$$

The two equations do not define w and y as functions of x, z, and t since $x + 2z + t = 25$.

9. (a) Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$.

Answer: Along the line $y = mx$ we have

$$\frac{x^2 - y^2}{x^2 + y^2} = \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \frac{1 - m^2}{1 + m^2}.$$

There is a different limit along each line through the origin; the limit as $(x, y) \to (0, 0)$ does not exist.

(b) Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x^2 - y^2)^2}{x^2 + y^2}$.

Answer: Since

$$0 \leq (x^2 - y^2)^2 \leq (x^2 + y^2)^2$$

we have

$$0 \leq \frac{(x^2 - y^2)^2}{(x^2 + y^2)} \leq (x^2 + y^2).$$

But

$$\lim_{(x,y)\to(0,0)} (x^2 + y^2) = 0$$

so

$$\lim_{(x,y)\to(0,0)} \frac{(x^2 - y^2)^2}{(x^2 + y^2)} = 0$$

by the sandwich theorem.