Calculus 234

Problems

May 15, 2003

A book reference marked [TF] indicates this semester's official text; a book reference marked [VPR] indicates the official text for next semester. These are

[TF] Thomas and Finney: *Calculus and Analytic Geometry*, Fifth Edition;

[VPR] Varberg, Purcell, Rigdon: *Calculus*, Eighth Edition.

Contents

Part I
Partial Derivatives

1 Review of vectors

1. Consider the point $P_0(2,-2,1)$ and the vector $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$.

(a) Find parametric equations for the line through P_0 parallel to $\mathbf{v}$.

(b) Find an equation of form

$$ax + by + cz = d$$

for the plane through P_0 perpendicular to this line.

2. Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$.

(a) Write the formula for the area of the parallelogram with edges $\mathbf{u}$ and $\mathbf{v}$.

(b) Write the formula for the volume of the parallelopiped with edges $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

3. Let $P_0 = (x_0, y_0, z_0)$, $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2)$, $P_3 = (x_3, y_3, z_3)$.

(a) Write the formula for the area of the parallelogram with edges P_0P_1 and P_0P_2.

(b) What is the fourth vertex of this parallelogram?

(c) Write the formula for the volume of the parallelopiped with edges P_0P_1, P_0P_2, P_0P_3.

2 Velocity, acceleration, and curvature

1. (a) Find the unit tangent vector $\mathbf{T}$ (as a function of t) for the space curve

$$x = e^t \cos t, \qquad y = e^t \sin t, \qquad z = e^t.$$

(b) Find the length of this parametric curve from $t = 0$ to $t = \pi$.

Answer: (a) The velocity vector is

$$\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j} + e^t\mathbf{k}.$$

Its length is

$$|\mathbf{v}| = \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2 + e^{2t}} = \sqrt{3}e^t$$

The unit tangent vector is

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\cos t - \sin t}{\sqrt{3}}\mathbf{i} + \frac{\sin t + \cos t}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$$

(b) The length is

$$\int ds = \int_0^\pi \frac{ds}{dt}\, dt = \int_0^\pi |\mathbf{v}|\, dt = \int_0^\pi \sqrt{3}e^t\, dt = \sqrt{3}e^\pi - \sqrt{3}e^0.$$

□

2. Here are the important quantities associated with a parametric curve.

- The velocity vector $\mathbf{v} = \dfrac{d\mathbf{r}}{dt}$.
- The acceleration vector $\mathbf{a} = \dfrac{d\mathbf{v}}{dt}$.
- The speed $\dfrac{ds}{dt} = |\mathbf{v}|$.
- The unit tangent vector $\mathbf{T} = \dfrac{d\mathbf{r}}{|d\mathbf{r}|} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$.
- The curvature vector $\dfrac{d\mathbf{T}}{ds}$.
- The curvature $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right|$.
- The unit normal vector $\mathbf{N} = \dfrac{d\mathbf{T}}{|d\mathbf{T}|} = \dfrac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|}$.

Evaluate them for the parameterization

$$x = a\cos t, \qquad y = a\sin t$$

of the circle of radius a.

3 Surfaces in three-space

1. Describe and sketch each of the following surfaces. What is each one called? (See [TF] pages 526-530.)

(a) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$ (b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z}{c}.$ (c) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}.$

(d) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1.$ (e) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1.$ (f) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \dfrac{z}{c}.$

2. Describe the graph $z = f(x, y)$ of the function $f(x, y) = y$.

Answer: The graph of $z = f(x, y) = y$ in the yz-plane is a line through the origin. Since f does not depend on x, all cross sections of the graph of f parallel to this one are identical. Thus the graph of f in xyz-space is a plane containing the x-axis. □

3. Describe the graph of the surface $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

4. Describe the graph of the surface $\dfrac{x^2}{a^2} + \dfrac{z^2}{c^2} = 1.$

The following problems introduce the idea of a parameterization

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v)$$

of a surface. This is analogous to parameterization

$$x = x(t), \qquad y = y(t), \qquad z = z(t)$$

of a curve. Here we will use curve parameterization to compute some arc lengths; in Section 20 of Part II we will use surface parameterization to compute some areas.

5. Show that the equations

$$x = \cos t, \qquad y = \sin t$$

give a parameterization of the unit circle $x^2 + y^2 = 1$. Is the parameterization one-to-one? Onto?

6. Show that the equations

$$x = \frac{t^2 - 1}{t^2 + 1}, \qquad y = \frac{2t}{t^2 + 1}$$

give a parameterization of the unit circle $x^2 + y^2 = 1$. Is the parameterization one-to-one? Onto? Which point is left out?

7. Show that the equations

$$x = \cosh t, \qquad y = \sinh t$$

give a parameterization of the hyperbola $x^2 - y^2 = 1$. Is the parameterization one-to-one? Onto?

8. Show that the equations

$$x = \frac{t}{2} + \frac{1}{2t}, \qquad y = \frac{t}{2} - \frac{1}{2t}$$

give a parameterization of the hyperbola $x^2 - y^2 = 1$. Is the parameterization one-to-one? Onto?

9. Show that the equations

$$x = \cos u \sin v, \qquad y = \sin u \sin v, \qquad z = \cos v$$

give a parameterization of the unit sphere $x^2 + y^2 + z^2 = 1$. Is the parameterization one-to-one? Onto?

10. Show that the equations

$$x = v \cos u - \sin u, \qquad y = v \sin u + \cos u, \qquad z = v$$

give a parameterization of the hyperboloid $x^2 + y^2 = z^2 + 1$. Is the parameterization one-to-one? Onto?

11. Use the parameterization of Problem 5 to find the arclength of the quarter circle from $(1, 0)$ to $(0, 1)$. Then find this same arclength using the parameterization of Problem 6. (Compare with Problem 9 of Part II.)

12. Use the parameterization of Problem 8 to set up an integral for the arclength of the portion of the hyperbola $x^2 - y^2 = 1$ between $(1, 0)$ and $(5/4, 3/4)$. Then do this using the parameterization of Problem 7. Do not attempt to evaluate either integral. (Compare with Problem 10 of Part II.)

4 Cylindrical and Spherical coordinates

1. Write the formula relating cylindrical coordinates (r, θ, z) and rectangular coordinates (x, y, z).

Answer: $x = r \cos \theta$, $y = r \sin \theta$, $z = z$. □

2. Write the formula relating spherical coordinates $(\rho, \theta\phi)$ and rectangular coordinates (x, y, z).

Answer: $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, $z = \rho \cos \phi$. □

3. Write the equation of the surface $\tan\phi = c/a$ in rectangular coordinates. What is it called?

4. Write the equation of the surface $r = 1$ in rectangular coordinates. What is it called?

5. Write the equation of the surface $\phi = \pi/6$ in rectangular coordinates. What is it called?

6. Write the equation of the surface $\theta = \pi/6$ in rectangular coordinates. What is it called?

7. Write the equation of the surface $\rho = 1$ in rectangular coordinates. What is it called?

5 Multivariate functions

1. Sketch the graph $z = f(x, y)$ of each of the following functions $f(x, y)$.

(a) $f(x, y) = 1$ (b) $f(x, y) = x$ (c) $f(x, y) = 2 - x - 2y$

(d) $f(x, y) = x^2$ (e) $f(x, y) = x^2 + y^2$ (f) $f(x, y) = x^2 - y^2$

2. For each function sketch the three contours (level curves) $f(x, y) = -1$, $f(x, y) = 0$, $f(x, y) = 1$.

(a) $f(x, y) = 3x + 4y$ (b) $f(x, y) = xy$ (c) $f(x, y) = x^2 + y^2$

(d) $f(x, y) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ (e) $f(x, y) = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$ (f) $f(x, y) = -\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$

3. Repeat Problem 2 with $f(x, y)$ replaced by $f(x, y) + 1$.

4. Repeat Problem 2 with $f(x, y)$ replaced by $f(x, y) - 1$.

5. Suppose $f(x, y) = xy$. What is $f(x + 3, y + 4)$? $f(x - y, x + y)$?

In Problems 6-10 you are asked to graph some level curves of $f(T(x, y))$ where you have previously graphed the corresponding level curves of $f(x, y)$. This is essentially the problem of graphing a general second degree equation (as in Math 222 except that here you are told how to translate and/or rotate the axes. Here is the general procedure. First introduce new variables $(\tilde{x}, \tilde{y})$ and graph the level curves of $f(\tilde{x}, \tilde{y})$ on a diagram where the horizontal axes is labelled $\tilde{x}$ and the vertical axes is labelled $\tilde{y}$. Then make a second diagram where the horizontal axes is labelled x and the vertical axes is labelled y, write the equations

$$(\tilde{x}, \tilde{y}) = T(x, y),$$

and draw (in the xy plane) the $\tilde{x}$-axis (i.e. the line $\tilde{y} = 0$) and the $\tilde{y}$-axis (i.e. the line $\tilde{x} = 0$.) Finally copy the graph you made in the $\tilde{x}\tilde{y}$ plane in the xy-plane using the appropriate axes.

6. Repeat Problem 2 with $f(x, y)$ replaced by $f(T(x, y))$ where

$$T(x, y) = (x - h, y - k), \qquad (h, k) = (1, 2).$$

7. Repeat Problem 2 with $f(x, y)$ replaced by $f(T(x, y))$ where

$$T(x, y) = (x \cos\alpha - y \sin\alpha, x \sin\alpha + y \cos\alpha), \qquad \alpha = \frac{\pi}{4}.$$

8. Repeat Problem 2 with $f(x, y)$ replaced by $f(T(x, y))$ where

$$T(x, y) = (x \cos\alpha - y \sin\alpha, x \sin\alpha + y \cos\alpha), \qquad \alpha = \frac{\pi}{3}.$$

9. Repeat Problem 2 with $f(x, y)$ replaced by $f(T(x, y))$ where

$$T(x, y) = (x \cos\alpha - y \sin\alpha, x \sin\alpha + y \cos\alpha), \qquad \alpha = -\frac{\pi}{6}.$$

10. Repeat Problem 2 with $f(x, y)$ replaced by $f(T(x, y))$ where

$$T(x, y) = (x \cos\alpha + y \sin\alpha, -x \sin\alpha + y \cos\alpha), \qquad \alpha = \frac{\pi}{6}.$$

11. Suppose that $f(x, y) = Ax^2 + Cy^2 + Dx + Ey$ where $A, C \neq 0$. Find a point (h, k) such that

$$f(x, y) = A(x - h)^2 + C(y - k)^2 + f(h, k).$$

Problems 12-15 show how to reverse the process that was illustrated by Problems 6-10. Here we start with a complicated quadratic polynomial $f(x, y)$ and find T so that $f(T(x, y))$ is simple.

12. Suppose that $f(x, y) = Ax^2 + Bxy + Cy^2$ and that

$$T(x, y) = (cx - sy, sx + cy).$$

(a) Find $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ so that $f(T(x, y)) = \tilde{A}x^2 + \tilde{B}xy + \tilde{C}y^2$. (The quantities $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are expressions in in A, B, C, c, s.)

(b) Show that $\tilde{B} = 0$ if $(c^2 - s^2)B = 2cs(A - C)$.

13. Suppose that $f(x,y) = Ax^2 + Bxy + Cy^2$ and that

$$T(x,y) = (x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha).$$

(a) Find $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ so that $f(T(x,y)) = \tilde{A}x^2 + \tilde{B}xy + \tilde{C}y^2$. (See [TF] page 428 or [VPR] page 537.)

(b) Show that $\tilde{B}^2 - 4\tilde{A}\tilde{C} = B^2 - 4AC$ and $\tilde{A} + \tilde{C} = A + C$. (See [TF] page 430.)

14. Suppose that $f(x,y) = Ax^2 + Bxy + Cy^2$ and that

$$T(x,y) = (x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha).$$

Show that there are numbers a, b, and α such that

$$f(T(x,y)) = \begin{cases} \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \\ -\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} \end{cases} \quad \text{if} \quad \begin{cases} B^2 - 4AC > 0; \\ B^2 - 4AC < 0, \quad A + C > 0; \\ B^2 - 4AC < 0, \quad A + C < 0; \end{cases}$$

15. Suppose that $f(x,y) = Ax^2 + Bxy + Cy^2 + Dx + Ey$ and that

$$T(x,y) = (x\cos\alpha - y\sin\alpha - h, x\sin\alpha + y\cos\alpha - k).$$

Show that there are numbers a, b, h, k, and α and a choice of signs such that

$$f(T(x,y)) = \pm\frac{(x-h)^2}{a^2} \pm \frac{(y-k)^2}{b^2} + f(h,k)$$

where the two signs $\pm$ need not be the same.

Answer: First write $f(x,y)$ as a sum of two functions $q(x,y)$ and $\ell(x,y)$ where

$$q(x,y) = Ax^2 + Bxy + Cy^2, \qquad \ell(x,y) = Dx + Ey.$$

By Problems 12 and 13 there is an angle α such that

$$q(T_1(x,y)) = \tilde{A}x^2 + \tilde{C}y^2$$

where

$$T_1(x,y) = (x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha).$$

Since $B^2 - 4AC = -4\tilde{A}\tilde{C} \neq 0$ it follows that $\tilde{A}$ and $\tilde{C}$ are not zero. Now

$$\ell(T_1(x,y)) = \tilde{D}x + \tilde{E}y$$

where $\tilde{D} = D\cos\alpha + E\sin\alpha$ and $\tilde{E} = -D\sin\alpha + E\cos\alpha$. Hence

$$f(T_1(x,y)) = q(T_1(x,y)) + \ell(T_1(x,y))$$

fits the form of Problem 11 so there are numbers $\tilde{h}$ and $\tilde{k}$ such that

$$f(T_1(T_2(x,y))) = \tilde{A}x^2 + \tilde{C}y^2$$

where

$$T_2(x,y) = (x - \tilde{h}, y - \tilde{k}).$$

Now take $T(x,y) = T_1(T_2(x,y))$, $a = 1|\tilde{A}|^{-1/2}$, $b = |\tilde{B}|^{-1/2}$, and $(h,k) = T_1(\tilde{h},\tilde{k})$. □

16. There is a theorem which says that for any quadratic polynomial $f(x,y)$ with nonzero discriminant there is a transformation $T(x,y)$ so that $f(T(x,y))$ has a very simple form. State this theorem and explain how to use it to describe the level curves of the function f.

17. Suppose that $(\tilde{x},\tilde{y}) = T(x,y)$ where

$$T(x,y) = (x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha).$$

Find T_1 so that $(x,y) = T_1(\tilde{x},\tilde{y})$.

18. Let $Q(m) = Am^2 + Bm + C$.

(a) Show that if $B^2 - 4AC > 0$, then the function $Q(m)$ is zero for two distinct values of m and takes both positive and negative values.

(b) Show that if $B^2 - 4AC < 0$, then the function $Q(m)$ is either always positive or always negative.

(c) Give an example (i.e. values for A, B, C) illustrating the situation of part (a) and two examples illustrating the situation of part (b), one where $Q(m)$ is always positive and another where $Q(m)$ is always negative. Draw the graphs of each of the three examples.

19. Let $f(x,y) = Ax^2 + Bxy + Cy^2$.

(a) Show that if $B^2 - 4AC > 0$, then the function $f(x,y)$ positive (except at the origin) along some lines $y = mx$ and negative along others. (Hint: Use Problem 18.)

(b) Show that if $B^2 - 4AC < 0$, then the function $f(x,y)$ is either positive (except at the origin) along all lines $y = mx$ or negative along all lines $y = mx$.

(c) Give an example (i.e. values for A, B, C) illustrating the situation of part (a) and two examples illustrating the situation of part (b), one where $f(x,y)$ is always positive and another where $f(x,y)$ is always negative. For each of the three examples draw the three level curves $f(x,y) = -1, 0, 1$.

6 Limits and continuity

1. Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ or show that there is no limit.

Answer: If we take $x = r\cos\theta, y = r\sin\theta$ we have

$$\frac{xy}{x^2+y^2} = \frac{r^2\cos\theta\sin\theta}{r^2} = \cos\theta\sin\theta = \frac{\sin 2\theta}{2}.$$

and this function takes all values between $-1/2$ and $1/2$ as the point (x, y), or (r, θ), moves around the origin, no matter how small r may be. Therefore the limit does not exist. The general principle here is that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r\cos\theta, r\sin\theta)$$

when the limit on the left exists. Hence if the limit on the left exists, the limit on the right must be independent of θ. □

2. Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$.

Answer: Along the line $y = mx$ we have

$$\frac{x^2-y^2}{x^2+y^2} = \frac{x^2-m^2x^2}{x^2+m^2x^2} = \frac{1-m^2}{1+m^2}.$$

There is a different limit along each line through the origin; the limit as $(x, y) \to (0, 0)$ does not exist. □

3. Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)^2}{x^2+y^2}$.

Answer: Since

$$0 \le (x^2-y^2)^2 \le (x^2+y^2)^2$$

we have

$$0 \le \frac{(x^2-y^2)^2}{(x^2+y^2)} \le (x^2+y^2).$$

But

$$\lim_{(x,y)\to(0,0)} (x^2+y^2) = 0$$

so

$$\lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)^2}{(x^2+y^2)} = 0$$

by the Sandwich Theorem. □

4. Is it always true that

$$\lim_{x \to 0} \lim_{y \to 0} f(x, y) = \lim_{y \to 0} \lim_{x \to 0} f(x, y)?$$

(Assume that the limits exist.)

5. Give an informal explanation of the notation $\lim\limits_{P \to P_0} f(P) = L$.

Answer: The notation $\lim_{P \to P_0} f(P) = L$ means that $f(P)$ is close to L whenever P is close to (but not equal to) P_0. Some people write this as

$$f(P) \approx L \qquad \text{when} \qquad P \approx P_0.$$

Alternate answer: The notation $\lim_{P \to P_0} f(P) = L$ means that $f(P)$ approaches L as P approaches P_0:

$$f(P) \to L \qquad \text{as} \qquad P \to P_0.$$

□

6. Give the formal definition of the notation $\lim\limits_{P \to P_0} f(P) = L$.

Answer: Let $f(P)$ be a real valued function defined on some subset of n-dimensional space, P_0 be a point in n-dimensional space (not necessarily in the domain of f), and L be a real number. Then we say that

$$\lim_{P \to P_0} f(P) = L$$

if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(P) - L| < \varepsilon$ whenever $0 < |P - P_0| < \delta$. Here the notation $|P - P_0|$ denotes the distance from P to P_0, i.e. the length of the vector $\overrightarrow{P_0 P}$. For example, if $n = 2$, $P = (x, y)$, and $P_0 = (x_0, y_0)$, then

$$|P - P_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}.$$

□

7. Write the definition of *continuity*.

Answer: Let $f(P)$ be a real valued function defined on some subset of n-dimensional space and P_0 be a point in its domain. We say that the function f is **continuous** at the point P_0 iff

$$\lim_{P \to P_0} f(P) = f(P_0).$$

A function f is said to be **continuous** on a set D iff it is defined and continuous at every point of D. □

7 Partial derivatives

1. Let $f(x,y) = 3x^2 + 7xy + 5y^2$ and $g(x) = f(x,3)$. Evaluate

(a) $g'(2)$ (b) $f_x(2,3)$ (c) $\left.\dfrac{\partial f}{\partial x}\right|_{(x,y)=(2,3)}$

2. Find $\left.\dfrac{\partial f}{\partial z}\right|_{(x,y,z)=(1,e,0)}$ where $f(x,y,z) = (xy)^z$. (This is Example 5 on page 580 of [TF].)

Answer: From $f(x,y,z) = (xy)^z = e^{z\ln(xy)}$ we get $f_z(x,y,z) = e^{z\ln(xy)}\ln(xy) = (xy)^z\ln(xy)$. At $(x,y,z) = (1,e,0)$ we have $f_z(1,e,0) = e^0 \ln e = 1$. □

3. Find $\left.\dfrac{\partial f}{\partial z}\right|_{(x,y,z)=(1,e^e,\pi)}$ where $f(x,y,z) = \ln(xy)^z$.

Answer: First note that $f(x,y,z) = \ln(xy)^z = z\ln(xy)$ so that $\partial f/\partial z = \ln(xy)$. At $(x,y,z) = (1,e^e,\pi)$ we have $\partial f/\partial z = \ln e^e = e\ln e = e$. □

4. In each of the following, find $\partial w/\partial x$ and $\partial w/\partial y$.

(a) $w = e^x \sin y$ (b) $w = e^x \cos y$ (c) $w = \tan^{-1}(y/x)$

(d) $w = \sin(xy^2)$ (e) $w = \sin x \sin^2 y$ (f) $w = \sin x(\sin y)^2$

5. Three resistor of resistance R_1, R_2, R_3 connected in parallel produce a resistance R given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

Find $\partial R/\partial R_1$. (This is Example 4 on page 580 of [TF].)

6. Express the polar coordinates (r,θ) in terms of the rectangular coordinates (x,y) and then calculate $\partial r/\partial x$, $\partial r/\partial y$, $\partial\theta/\partial x$, $\partial\theta/\partial x$.

7. Express the spherical coordinates (ρ,ϕ,θ) in terms of the rectangular coordinates (x,y) and then calculate $\partial\rho/\partial x$, $\partial\phi/\partial x$, $\partial\theta/\partial x$, $\partial\rho/\partial y$, $\partial\phi/\partial y$, $\partial\theta/\partial y$, $\partial\rho/\partial z$, $\partial\phi/\partial z$, $\partial\theta/\partial z$. (See Problems 19-24 on page 581 of [TF].)

8 Differentiability

1. Define the *linear approximation* to a function $f(x, y)$ of two variables.

Answer: The **linear approximation** to the function $f(x, y)$ at the point (x_0, y_0) is the linear polynomial

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

It is the unique linear polynomial having the same value and first derivatives as f at the point (x_0, y_0). □

2. Write the definition of *differentiability* for a function $f(x, y)$ of two variables.

Answer: A function $f(x, y)$ is **differentiable** at a point (x_0, y_0) of its domain iff the linear approximation at (x_0, y_0) is defined (i.e. the partial derivatives exist) and satisfies

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - L(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.$$

A function f is said to be **differentiable** on a set D iff it is defined and differentiable at every point of D. □

3. Prove that if the partial derivatives f_x and f_y of a function $f(x, y)$ exist and are continuous, then the function f is differentiable.

Answer: Choose (x_0, y_0) in the domain of f and let

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

be the linear approximation of f at (x_0, y_0). Then

$$f(x, y) - L(x, y) = R_1 + R_2$$

where

$$R_1 = f(x, y_0) - f(x_0, y_0) - f_x(x_0, y_0)(x - x_0)$$

and

$$R_2 = f(x, y) - f(x, y_0) - f_y(x_0, y_0)(y - y_0).$$

By the Mean Value Theorem from Calculus 221 there is a number x_1 between x_0 and x such that

$$f(x, y_0) - f(x_0, y_0) = f_x(x_1, y_0)(x - x_0)$$

so

$$R_1 = \big(f_x(x_1, y_0) - f_x(x_0, y_0)\big)(x - x_0).$$

Similarly there is a number y_1 between y_0 and y such that

$$f(x, y) - f(x, y_0) = f_x(x, y_1)(y - y_0)$$

so

$$R_2 = \big(f_y(x, y_1) - f_y(x, y_0)\big)(y - y_0).$$

Now

$$\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \le \frac{1}{|x-x_0|} \quad \text{and} \quad \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \le \frac{1}{|y-y_0|}$$

so

$$\begin{aligned}
\frac{|f(x,y)-L(x,y)|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} &= \frac{|R_1+R_2|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \\
&\le \frac{|R_1|+|R_2|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \\
&\le \frac{|R_1|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} + \frac{|R_2|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \\
&\le \frac{|R_1|}{|x-x_0|} + \frac{|R_2|}{|y-y_0|} \\
&= |f_x(x_1,y_0)-f_x(x_0,y_0)| + |f_y(x,y_1)-f_y(x,y_0)|
\end{aligned}$$

Since f_x and f_y are continuous,

$$\lim_{(x,y)\to(x_0,y_0)} |f_x(x_1,y_0)-f_x(x_0,y_0)| = \lim_{(x,y)\to(x_0,y_0)} f_y(x,y_1)-f_y(x,y_0)| = 0.$$

Hence

$$\lim_{(x,y)\to(x_0,y_0)} \frac{|f(x,y)-L(x,y)|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$

by the Sandwich Theorem [TF] pages 51 and 746, (called the *Squeeze Theorem* in [VPR] pages 75-76 and 432). □

4. Prove that a differentiable function is continuous.

Answer: Choose (x_0, y_0) in the domain of f and let

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

be the linear approximation of f at (x_0, y_0). Then

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)-L(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$

because $f(x,y)$ is differentiable. But

$$\lim_{(x,y)\to(x_0,y_0)} \sqrt{(x-x_0)^2+(y-y_0)^2} = 0$$

so

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)-L(x,y) = 0$$

by the Product Law for limits. But $f(x_0,y_0)$, $f_x(x_0,y_0)$, $f_y(x_0,y_0)$ are constants so

$$\lim_{(x,y)\to(x_0,y_0)} L(x,y) = f(x_0,y_0).$$

Hence

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

by the Addition Law for limits. □

9 Chain rule

1. Let $f(x, y) = 3x^2 + 5xy + y^2$ and let $g(t) = f(1 + 2t, 1 + 3t^2)$ be the value of f along the parametric curve

$$x = 1 + 2t, \qquad y = 1 + 3t^2.$$

(a) Simplify $g(t)$ and then find dg/dt.

(b) Find $\partial f/\partial x$, $\partial f/\partial y$, dx/dt, dy/dt, and then plug in to find

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

as an expression in t.

(c) What does this illustrate?

2. Find dw/dt in two different ways, first by expressing w explicitly as a function of t and differentiating, and then using the chain rule. Express your final answer in terms of t.

(a) $w = x^2 + y^2 + z^2$, $x = e^t \cos t$, $y = e^t \sin t$, $z = e^t$.

(b) $w = x^2 + y^2 + z^2$, $x = \sin t \cos t$, $y = \sin t \sin t$, $z = \cos t$.

(c) $w = x^2 + y^2 + z^2$, $x = t$, $y = 0$, $z = 0$.

(d) $w = xy^2z^3$, $x = t$, $y = t$, $z = 3t$.

3. State the Chain Rule for a function of form $f(x(t), y(t))$.

Answer: If both the function $f(x, y)$ and the parametric curve $(x(t), y(t))$ are differentiable, then so is their composition $f(x(t), x(t))$ and

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

□

4. Find dw/dt at $t = 3$ if

$$w = \frac{x}{z} + \frac{y}{x}, \qquad x = \cos^2 t, \quad y = \sin^2 t, \quad z = \frac{1}{t}.$$

Answer:

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\
&= \left(\frac{1}{z} - \frac{y}{x^2}\right)(-2\cos t \sin t) + \left(\frac{1}{x}\right)(2\sin t \cos t) + \left(\frac{-x}{z^2}\right)\left(\frac{-1}{t^2}\right) \\
&= \left(3 - \frac{\sin^2 3}{\cos^4 3}\right)(-2\cos 3 \sin 3) + \left(\frac{1}{\cos^2 3}\right)(2\sin 3\cos 3) + \left(-9\cos^2 3\right)\left(\frac{-1}{9}\right)
\end{aligned}$$

at $t = 3$.

□

5. Show that if $w = f(x+ct) + g(x-ct)$, then $\dfrac{\partial^2 w}{\partial t^2} = c^2 \dfrac{\partial^2 w}{\partial x^2}$.

6. Find $\partial z/\partial u$ and $\partial z/\partial v$ when

(a) $z = x^2 + 3y^2$, $x = u\cos v$, $y = u \sin v$.

(b) $z = x^3 y^5$, $x = u^7 v^{11}$, $y = u^{13} v^{17}$.

(c) $z = e^x \cos y$, $x = u^2 - v^2$, $y = 2uv$.

(d) $z = xy$, $x = e^u \cos v$, $y = e^u \sin v$.

7. Assume that $h(t) = f(g(t))$. Find a formula for $h''(t)$ in terms of f', f'', g', g''.

8. Assume that $w = w(u, v)$, and that $u = u(x, y)$, $v = v(x, y)$. Find a formula for $w_{xx} + w_{yy}$ in terms of the derivatives of w with respect to u and v and the derivatives of u and v with respect to x and y.

9. Assume that $w = w(u, v)$, that $u = u(x, y)$, $v = v(x, y)$, that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

and that

$$\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2} = 0.$$

Show that

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0.$$

Hint: Use subscript notation for derivatives to make your work easier to follow.

10 Directional derivatives and gradients

1. Find a normal vector to the surface $z = x^2 - xy - y^2$ at the point $P_0 = (1, -1, 1)$.

Answer: The gradient $\nabla F(P)$ is perpendicular[1] to the surface $F(x, y, z) =$constant at a point P on this surface. To use this we write the given equation in the form

$$F(x, y, z) = x^2 - xy - y^2 - z = 0.$$

The gradient at $P = (x, y, z)$ is

$$\nabla F = (2x - y)\mathbf{i} + (-x - 2y)\mathbf{j} - \mathbf{k}$$

so the vector

$$\nabla F(P_0) = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$$

is normal to the surface at P_0. Any nonzero multiple of this vector is also a correct answer. □

[1] In this context the words *normal* and *perpendicular* are synonymous

2. Find the directional derivative of $f(x,y,z) = e^x \cos(yz)$ at $P_0 = (0,0,\pi)$ in the direction of $\mathbf{A} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

Answer: The partial derivatives of f are

$$f_x = e^x \cos(yz), \qquad f_y = -ze^x \sin(yz), \qquad f_z = -ye^x \sin(yz)$$

so the gradient ∇f of f at the point $P = (x,y,z)$ is

$$\nabla f(P) = e^x cos(yz)\mathbf{i} - ze^x \sin(yz)\mathbf{j} - ye^x \sin(yz)\mathbf{k}.$$

The length of $\mathbf{A}$ is

$$|\mathbf{A}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$$

so the normalization (i.e. unit vector in the same direction as $\mathbf{A}$) is

$$\mathbf{v} = \frac{2}{\sqrt{6}}\,\mathbf{i} - \frac{1}{\sqrt{6}}\,\mathbf{j} + \frac{1}{\sqrt{6}}\,\mathbf{k}$$

so answer is

$$\nabla f(P_0) \cdot \mathbf{v} = \mathbf{i} \cdot \mathbf{v} = \frac{2}{\sqrt{6}}.$$

□

3. Find the directional derivative at $P_0 = (0,0,0)$ of the function $f(x,y,z) = e^x \cos(yz)$ in the direction of the vector $\mathbf{A} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. (See [TF] page 604.)

4. Find the directional derivative at $P_0 = (1,1,1)$ of the function $f(x,y,z) = \ln\sqrt{x^2 + y^2 + z^2}$ in the direction of the vector $\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

5. Find the directional derivative at $P_0 = (0,0,0)$ of the function $f(x,y,z) = ax + by + cz$ in the direction of the vector $\mathbf{A} = m\mathbf{i} + n\mathbf{j} + p\mathbf{k}$.

6. In which direction does the derivative at $(1,1)$ of $f(x,y) = (x^2 - y^2)/(x^2 + y^2)$ vanish?

7. Find the electric intensity vector $\mathbf{E} = -\nabla V$ from the given potential function V at the given point: (See [TF] page 599.)

(a) $V = x^2 + y^2 - 2z^2, \quad (1,1,1).$ (b) $V = e^x \cos y, \quad (0,0,1).$

(c) $V = \ln\sqrt{x^2 + y^2}, \ (1,0,0).$ (d) $V = 1/\sqrt{x^2 + y^2 + z^2}, \quad (1,0,0).$

8. Verify that each of the four potential functions V in Problem 7 satisfies Laplace's equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

9. (a) Find the derivative of $F(x,y,z) = x^2 + xy + xyz$ at the point $(1,2,3)$ in the direction

$$\mathbf{u} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}.$$

(b) In what direction is this function increasing the fastest at the point $(1, 2, 3)$?

10. Let f be a differentiable function, P a point in its domain, and $\nabla f(p)$ be the value of the gradient of f at the point P. Assume $\nabla f(P) \neq \mathbf{0}$ and let

$$\mathbf{u} = |\nabla f(P)|^{-1}\nabla f(P)$$

be the **normalization** of $\nabla f(P)$, i.e. the unit vector in the direction $\nabla f(P)$.

(a) Prove that if $\mathbf{v}$ is any unit vector distinct from $\mathbf{u}$, then the directional derivatives satisfy the inequality

$$\nabla f(P) \cdot \mathbf{v} < \nabla f(P) \cdot \mathbf{u}.$$

Hint: What is the geometric definition of the dot product?

(b) In which direction is f increasing the fastest?

11. Explain why the gradient $\nabla F(P_0)$ is normal to the surface $F(P) = 0$ at the point P_0.

Answer: Let $(x(t), y(t), z(t))$ be any parametric curve which lies in the surface and passes through the point $P_0(x_0, y_0, z_0)$ at $t = 0$. Then the velocity vector

$$\mathbf{v} = \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}\right)_{t=0}$$

is tangent to the surface at the point P_0. Differentiate the equation

$$F(x(t), y(t), z(t)) = 0$$

and evaluate at zero; the chain rule gives

$$\nabla F(P_0) \cdot \mathbf{v} = 0,$$

i.e. $\nabla F(P_0)$ is perpendicular to $\mathbf{v}$. We have shown that $\nabla F(P_0)$ is perpendicular to (the tangent vector to) every curve through P_0 which lies in the surface $F(P) = 0$, i.e. the vector $\nabla F(P_0)$ is perpendicular to the surface $\nabla F(P_0)$ at P_0. □

12. Find an equation for the tangent plane to the surface

$$x^3z + y^2x^2 + \sin(yz) + 54 = 0$$

at the point $P_0 = (3, 0, -2)$.

Answer: The gradient is

$$\nabla F = (3x^2z + 2y^2x)\mathbf{i} + (2yx^2 + z\cos yz)\mathbf{j} + (x^3 + y\cos yz)\mathbf{k},$$

so the vector

$$\nabla f|_{(3,0,-2)} = -54\mathbf{i} - 2\mathbf{j} + 27\mathbf{k},$$

is normal to the surface at the point $(3, 0, -2)$ and and hence an equation for tangent plane at this point is is

$$-54(x - 3) - 2(y - 0) + 27(z + 2) = 0.$$

□

13. Find an equation for the tangent plane to the surface $x^2 - y^2 + z^2 = 4$ at the point $(2, -3, 3)$.

14. Find an equation for the tangent plane to the surface $z = 1 + x^2 + y^3$ at the point $(x, y, z) = (2, 1, 6)$.

15. Find the plane tangent to the surface $F(x, y, z) = 0$ at the given point P_0. Also find the line through P_0 normal to the surface.

(a) $F(x, y, z) = x^2 + y^2 + z^2 - 14$, $P_0 = (1, 2, 3)$.

(b) $F(x, y, z) = x^2 + y^2 + z^2 - 1$, $P_0 = (x_0, y_0, z_0)$.

(c) $F(x, y, z) = 5x^2 + 4y^2 + 7z^2 - 28$, $P_0 = (1, 2, 1)$.

(d) $F(x, y, z) = xyz - 30$, $P_0 = (2, 3, 5)$.

16. Find the plane tangent to the surface $z = f(x, y)$ at the given point P_0. Also find the line through P_0 normal to the surface.

(a) $z = x^2 + y^2$, $P_0 = (3, 4, 25)$.

(b) $z = \sqrt{9 - x^2 - y^2}$, $P_0 = (1, -2, 2)$.

(c) $z = x^2 - xy - y^2$, $P_0 = (1, 1, -1)$.

(d) $z = \tan^{-1}(y/x)$, $P_0 = (1, 1, \pi/4)$.

17. Write a formula for a normal vector to the graph $z = f(x, y)$ at a point $P_0 = (x_0, y_0, f(x_0, y_0))$ on that graph.

Answer: In general, a graph $z = f(x, y)$ is a special case of $F(x, y, z) = 0$ via

$$F(x, y, z) = f(x, y) - z$$

so a normal vector to the surface $z = f(x, y)$ at $P_0 = (x_0, y_0, f(x_0, y_0))$ is

$$\nabla F(P_0) = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} - \mathbf{k}.$$

□

18. Show that the tangent plane to the graph $z = f(x, y)$ at the point

$$P_0 = (x_0, y_0, z_0), \qquad z_0 = f(x_0, y_0)$$

is the graph of the linear approximation.

Answer: The graph $z = f(x, y)$ can be written as $F(x, y, z) = 0$ where

$$F(x, y, z) = f(x, y) - z.$$

The equation for the tangent plane is

$$\nabla F(P_0) \cdot \overrightarrow{P_0P} = 0$$

where

$$\nabla F(P_0) = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} - \mathbf{k},$$
$$\overrightarrow{P_0P} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}.$$

Since

$$\nabla F(P_0) \cdot \overrightarrow{P_0P} = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0)$$

and $z_0 = f(x_0, y_0)$, the equation $\nabla F(P_0) \cdot \overrightarrow{P_0P} = 0$ is the same as

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

□

19. Find an equation for the plane that is tangent to the surface $z = f(x, y)$ at the point $P_0 = (x_0, y_0, f(x_0, y_0))$.

Answer: The equation for the surface has form $z = f(x, y)$ so the answer is given by the linear approximation

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

i.e.

$$z = -1 + (x - 1) - 3(y - 1).$$

Alternatively the surface has equation $F(x, y, z) = 0$ where $F(x, y, z) = z - x^2 + xy + y^2$ so the equation for the tangent plane is

$$0 = \nabla F(x_0, y_0, z_0) \cdot \overrightarrow{P_0P} = -(x - 1) + 3(y - 1) + (z + 1).$$

(This is Problem 3 on page 583 of [TF].) □

11 Implicit differentiation

1. A function $z = z(x, y)$ satisfies the identity $3x^2z + y^3 - xyz^3 = 0$. Find $\partial z/\partial x$ and $\partial z/\partial y$.

2. The function $f(x, y)$ is defined by

$$f(x, y) = uv, \qquad x = u, \quad y = v + u^2.$$

Find the gradient ∇f at the point $(x_0, y_0) = (3, 13)$. Express your answer in the form $a\mathbf{i} + b\mathbf{j}$.

Answer: We can use implicit differentiation, but it is a little easier to express f in terms of x and y directly using $u = x$ and $v = y - u^2 = y - x^2$ so

$$f(x, y) = x(y - x^2) = xy - x^3.$$

Hence

$$\nabla f = (y - 3x^2)\mathbf{i} + x\mathbf{j}.$$

□

3. (a) Find an equation for the tangent plane to the surface $z = -11 + x^2 + y^3$ at the point $(x, y, z) = (4, 0, 5)$.

(b) Find an equation for the tangent plane to the surface $x+2y+3z-\cos(xyz) = 18$ at the point $(x, y, z) = (4, 0, 5)$.

(c) A curve is given parametrically by the equations

$$x = 4 + t, \qquad y = u(t), \qquad z = v(t).$$

The curve lies in both surfaces and passes through the point $P_0 = (4, 0, 5)$ at $t = 0$. Find the velocity vector of this curve at $t = 0$.

4. Suppose that $f(u)$ is a function of one variable so that $W(x, y) = f(x + y)$ is a function of two variable x, y. Is it always true that $\partial W/\partial x = \partial W/\partial y$?

Answer: By the chain rule

$$\frac{\partial W}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x}, \qquad \frac{\partial W}{\partial y} = \frac{df}{du}\frac{\partial u}{\partial y}$$

where $u = x + y$. But

$$\frac{\partial u}{\partial x} = 1 = \frac{\partial u}{\partial y}$$

so

$$\frac{\partial W}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} = \frac{df}{du}\cdot 1 = \frac{df}{du}\frac{\partial u}{\partial y} = \frac{\partial W}{\partial y}.$$

□

5. Suppose that x and y are given as functions of (s, t) via the equations

$$x = s + t, \qquad y = s - t.$$

If we regard s and t as functions of (x, y), what is $\dfrac{\partial s}{\partial x}$?

Answer: To calculate $\partial s/\partial x$, we must write s, t as functions of x, y at first. so by adding two equations we get $x+y = 2s$ so $s = (x+y)/2$. By taking partial derivative, we get $\partial s/\partial x = \frac{1}{2}$. □

6. Let $\theta = \theta(x, y)$ and $r = r(x, y)$ be defined implicitly by

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

(a) Differentiate these equations with respect to x.

(b) Solve the resulting two equations for $\partial r/\partial x$ and $\partial\theta/\partial x$ in terms of r, $\cos\theta$ and $\sin\theta$.

(c) Using part (b) find $\partial r/\partial x$ and $\partial\theta/\partial x$ in terms of x and y.

(d) Repeat steps (a-c) with y instead of x.

(e) Compare your answers with the answers you got for Problem 6 of section 7.

7. Do Problem 7 of section 7 using the method of Problem 6.

8. The equation $G(x, y, z) = 0$ implicity defines three functions f, g, and h of two variables by the formulas

$$G(x, y, f(x, y)) = 0, \qquad G(x, g(x, z), z) = 0, \qquad G(h(y, z), y, z) = 0.$$

Given a function $F = F(x, y, z)$ define two new functions $P = P(x, y)$ and $Q = Q(x, z)$ by

$$P(x, y) = F(x, y, f(x, y)), \qquad Q(x, z) = F(x, g(x, z), z).$$

Assume (a, b, c) is a point on the surface $G(x, y, z) = 0$, i.e. $G(a, b, c) = 0$. Find formulas for $P_x(a, b)$ and $Q_x(a, c)$ in terms of the six quantities

$$F_x(a, b, c), \quad F_y(a, b, c), \quad F_z(a, b, c), \quad G_x(a, b, c), \quad G_y(a, b, c), \quad G_z(a, b, c).$$

Answer: Since $G(x, y, f(x, y))$ is identically zero, the function $f(x, y)$ satisfies

$$\frac{\partial}{\partial x} G(x, y, f(x, y)) = G_x(x, y, f(x, y)) + G_z(x, y, f(x, y)) f_x(x, y) = 0.$$

Since $G(a, b, c) = 0$ we have $c = f(a, b)$ so $F_x(a, b, c) + F_z(a, b, c) f_x(a, b) = 0$, i.e.

$$f_x(a, b) = -\frac{G_x(a, b, c)}{G_z(a, b, c)}.$$

By the chain rule (again)

$$P_x(a, b) = \frac{\partial}{\partial x} F(x, y, f(x, y))\Big|_{(x,y)=(a,b)} = F_x(a, b, c) + F_z(a, b, c) f_x(a, b)$$

so

$$P_x(a, b) = F_x(a, b, c) - F_z(a, b, c) \frac{G_x(a, b, c)}{G_z(a, b, c)}.$$

Similarly

$$Q_x(a, c) = F_x(a, b, c) - F_y(a, b, c) \frac{G_x(a, b, c)}{G_y(a, b, c)}.$$

□

Remark 9. In a commonly used notation called **thermodynamic notation** the two derivatives in Problem 8 are written

$$\left(\frac{\partial F}{\partial x}\right)_y = P_x \qquad \text{and} \qquad \left(\frac{\partial F}{\partial x}\right)_z = Q_x.$$

The subscript indicates which variable is held constant. In evaluating the derivative on the left y is held constant and z is implicitly defined as a function of x and y via the equation $G(x, y, z) = 0$ whereas in evaluating the derivative on the left z is held constant and y is implicitly defined as a function of x and z by this same equation. In the former case x and y are the independent variables while in the latter case x and z are.

10. Suppose that

$$w = x^2 - y^2 + 4z, \quad \text{and} \quad 3x + y + 4z = 25.$$

Show that the equations

$$\frac{\partial w}{\partial x} = 2x - 3 \qquad \frac{\partial w}{\partial x} = 2x + 6$$

each give $\partial w/\partial x$ depending on which variables are chosen to be dependent and which are chosen to be independent. Identify (using thermodynamic notation) the independent variables in each case.

Answer: First assume that the given equations define w and z as functions of x and y. Computing the partial derivative with respect to x and holding y constant gives

$$\left(\frac{\partial w}{\partial x}\right)_y = 2x + 4\left(\frac{\partial z}{\partial x}\right)_y, \quad \text{and} \quad 3 + 4\left(\frac{\partial z}{\partial x}\right)_y = 0.$$

Substituting the second equation in the first gives

$$\left(\frac{\partial w}{\partial x}\right)_y = 2x - 3.$$

Now assume that the given equations define w and y as functions of x and z. Computing the partial derivative with respect to x and holding z constant gives

$$\left(\frac{\partial w}{\partial x}\right)_z = 2x - 2\left(\frac{\partial y}{\partial x}\right)_z, \quad \text{and} \quad 3 + \left(\frac{\partial y}{\partial x}\right)_z = 0.$$

Substituting the second equation in the first gives

$$\left(\frac{\partial w}{\partial x}\right)_z = 2x + 6.$$

□

11. The function $w = w(x, y)$ is defined implicitly by the equations

$$4x + 5y + 6z = 32 + w \qquad \text{and} \qquad 7x^2 + 8y^2 + 9z^2 = w + 120e^w.$$

Find $\dfrac{\partial w}{\partial y}$ at $(w, x, y, z) = (0, 1, 2, 3)$.

12. (a) The equations

$$w = x^2y^2 + yz - z^3, \qquad x^2 + y^2 + z^2 = 6,$$

define $w = W_1(x, y)$ and $z = Z(x, y)$ implicitly as functions of x and y. Find the value of the partial derivative $\left(\frac{\partial w}{\partial y}\right)_x$ of the function $w = W_1(x, y)$ at the point $(w, x, y, z) = (4, 2, 1, -1)$.

(b) These same equations also define $w = W_2(z, y)$ and $x = X(z, y)$ implicitly as functions of z and y. Find the value of the partial derivative $\left(\frac{\partial w}{\partial y}\right)_z$ of the function $w = W_2(z, y)$ at the point $(w, x, y, z) = (4, 2, 1, -1)$.

12 Approximation

1. (a) Let $f(x, y) = 2 + 3x + 5y + 7x^2 + 11xy + 13y^2$. Find $f(0, 0)$, $f_x(0, 0)$, $f_y(0, 0)$, $f_{xx}(0, 0)$, $f_{xy}(0, 0)$, $f_{yy}(0, 0)$.

(b) Find the quadratic polynomial $f(x, y)$ such that $f(0, 0) = 2$, $f_x(0, 0) = 3$, $f_y(0, 0) = 5$, $f_{xx}(0, 0) = 22$, $f_{xy}(0, 0) = 11$, and $f_{yy}(0, 0) = 26$.

(c) Let $f(x, y)$ be a general quadratic polynomial, i.e. $f(x, y)$ has the form

$$f(x, y) = F + Dx + Ey + Ax^2 + Bxy + Cy^2.$$

Find $f(0, 0)$, $f_x(0, 0)$, $f_y(0, 0)$, $f_{xx}(0, 0)$, $f_{xy}(0, 0)$, $f_{yy}(0, 0)$.

(d) Find the quadratic polynomial $f(x, y)$ such that $f(0, 0) = F$, $f_x(0, 0) = D$, $f_y(0, 0) = E$, $f_{xx}(0, 0) = 2A$, $f_{xy}(0, 0) = B$, and $f_{yy}(0, 0) = 2C$.

2. (a) Let $f(x, y) = 2+3(x-1)+5(y-4)+7(x-1)^2+11(x-1)(y-4)+13(y-4)^2$. Find $f(1, 4)$, $f_x(1, 4)$, $f_y(1, 4)$, $f_{xx}(1, 4)$, $f_{xy}(1, 4)$, $f_{yy}(1, 4)$.

(b) Find the quadratic polynomial $f(x, y)$ such that $f(1, 4) = 2$, $f_x(1, 4) = 3$, $f_y(1, 4) = 5$, $f_{xx}(1, 4) = 22$, $f_{xy}(1, 4) = 11$, and $f_{yy}(1, 4) = 26$.

(c) Let $f(x, y)$ be a quadratic polynomial in the form

$$\begin{aligned} f(x, y) &= F + D(x - x_0) + E(y - y_0) + \\ &\quad A(x - x_0)^2 + B(x - x_0)(y - y_0) + C(y - y_0)^2 \end{aligned}$$

where x_0 and y_0 are constants. Find $f(x_0, y_0)$, $f_x(x_0, y_0)$, $f_y(x_0, y_0)$, $f_{xx}(x_0, y_0)$, $f_{xy}(x_0, y_0)$, $f_{yy}(x_0, y_0)$.

(d) Find the quadratic polynomial $f(x, y)$ such that $f(x_0, y_0) = F$, $f_x(x_0, y_0) = D$, $f_y(x_0, y_0) = E$, $f_{xx}(x_0, y_0) = 2A$, $f_{xy}(x_0, y_0) = B$, and $f_{yy}(x_0, y_0) = 2C$.

(e) Do part (a) for the function $f(x, y) = 238 - 55x - 110y + 7x^2 + 11xy + 13y^2$.

(f) What is the relation between the functions of part (a) and part (b)?

3. Find an equation for the plane that is tangent to the surface $z = x^2 - xy - y^2$ at the point $P_0 = (1, 1, -1)$.

Answer: The equation for the surface has form $z = f(x, y)$ so the answer is given by the linear approximation

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

i.e.

$$z = -1 + (x - 1) - 3(y - 1).$$

Alternatively the surface has equation $F(x, y, z) = 0$ where $F(x, y, z) = z - x^2 + xy + y^2$ so the equation for the tangent plane is

$$0 = \nabla F(x_0, y_0, z_0) \cdot \overrightarrow{P_0 P} = -(x - 1) + 3(y - 1) + (z + 1).$$

(This is Problem 3 on page 583 of [TF].) □

4. Find the linear polynomial $L(x, y)$ which best approximates the function

$$f(x, y) = \frac{1}{1 + x - y}$$

near the point $(2, 1)$.

Answer: This is the linear approximation

$$L(x, y) = f(2, 1) + f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1),$$

i.e.

$$L(x, y) = \frac{1}{2} - \frac{x - 2}{4} + \frac{y - 1}{4}.$$

(This is Example 3 on page 589 of [TF].) □

5. Find the linear polynomial which best approximates the function $f(x, y) = \sqrt{x^2 + y^2}$ near (a) $(1, 0)$; (b) $(0, 1)$; (c) $(1, 1)$.

6. Find the linear polynomial which best approximates the function $f(x, y) = (\sin x)/y$ near (a) $(\pi/2, 1)$; (b) $(0, 1)$.

7. Find the linear polynomial which best approximates the function $f(x, y) = e^x \cos y$ near (a) $(0, 0)$; (b) $(0, \pi/2)$.

8. Define the *quadratic approximation* to a function $f(x, y)$ of two variables.

Answer: The **quadratic approximation** to the function $f(x, y)$ at the point (x_0, y_0) is the quadratic polynomial

$$\begin{aligned} Q(x, y) &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + \\ &\quad + \tfrac{1}{2} f_{xx}(x_0, y_0)(x - x_0)^2 + f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + \tfrac{1}{2} f_{yy}(x_0, y_0)(y - y_0)^2. \end{aligned}$$

It is the unique quadratic polynomial having the same value, first derivatives, and second derivatives as f at the point (x_0, y_0), □

9. Find the quadratic polynomial $Q(x, y)$ which best approximates the function

$$f(x, y) = \frac{1}{1 + x - y}$$

near the point $(2, 1)$.

Answer: This is the quadratic approximation

$$Q(x, y) = L(x, y) + \frac{f_{xx}(2, 1)(x - 2)^2 + 2f_{xy}(2, 1)(x - 2)(y - 1) + f_{yy}(2, 1)(y - 1)^2}{2},$$

i.e.

$$Q(x, y) = \frac{1}{2} - \frac{x - 2}{4} + \frac{y - 1}{4} + \frac{(x - 2)^2}{8} - \frac{(x - 2)(y - 1)}{4} + \frac{(y - 1)^2}{8}.$$

(This was explained in the lecture and in section 16-11.) □

10. Find the quadratic polynomial $Q(x, y)$ which best approximates the function $f(x, y) = x^2y^3$ near the point $(x_0, y_0) = (1, -1)$.

Answer:

$$\begin{aligned}
f(x, y) &= x^2y^3 & f(1, -1) &= -1 \\
f_x(x, y) &= 2xy^3 & f_x(1, -1) &= -2 \\
f_y(x, y) &= 3x^2y^2 & f_y(1, -1) &= 3 \\
f_{xx}(x, y) &= 2y^3 & f_{xx}(1, -1) &= -2 \\
f_{xy}(x, y) &= 6xy^2 & f_y(1, -1) &= 6 \\
f_{yy}(x, y) &= 6x^2y & f_{yy}(1, -1) &= -6
\end{aligned}$$

so the quadratic approximation is

$$Q(x, y) = -1 - 2(x - 1) + 3(y + 1) - (x - 1)^2 + 6(x - 1)(y + 1) - 3(y + 1)^2.$$

□

11. Find the quadratic polynomial which best approximates the function $f(x, y) = x^3 + x^2y + y^3$ near the point $(1, 2)$.

12. In what sense is the linear approximation the linear polynomial which best approximates $f(x, y)$ near (x_0, y_0)?

Answer: It is the only linear polynomial $L(x, y)$ such that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - L(x, y))}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.$$

□

13. In what sense is the quadratic approximation the quadratic polynomial which best approximates $f(x, y)$ near (x_0, y_0)?

Answer: It is the only quadratic polynomial $Q(x, y)$ such that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - Q(x, y))}{(x - x_0)^2 + (y - y_0)^2} = 0.$$

□

13 Max and min

1. True or False?[2] If gradient of $f(x, y, z)$ at (x_0, y_0, z_0) is zero , then $f(x, y, z)$ has a local maximum or minimum at (x_0, y_0, z_0).

Answer: The condition $\nabla f = 0$ is a necessary condition for an interior local maximum or local minimum, but it can happen that $\nabla f(P_0) = 0$ even though f doesn't have a local maximum or minimum at P_0. For example , $f = x^2 - y^2$ has $\nabla f = 0$ at $(0, 0)$ but $f(x, 0) = x^2$ which is positive,$f(0, y) = -y^2$ which is negative. so $f(0, 0) = 0$ isn't a minimum or a maximum. □

2. Prove that if $f(x, y)$ has a local extremum[3] at an interior point (x_0, y_0) of its domain, then $\nabla f(x_0, y_0) = \mathbf{0}$.

Answer: The function $g(x) = f(x, y_0)$ is a function of one variable having an interior extremum at $x = x_0$ so by Calculus 221, $0 = g'(x_0) = f_x(x_0, y_0)$. Similarly, the function $h(y) = f(x_0, y)$ has an interior extremum at $y = y_0$ so by calculus 221, $0 = g'(x_0) = f_x(x_0, y_0)$. Hence $\nabla f(x_0, y_0) = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} = \mathbf{0}$. □

3. For the function $h(x, y) = x^3 + y^3 - 9xy$:

(i) Find all points $P = (x, y)$ where the gradient ∇h is zero.

(ii) For each point in part (i) say whether it is a local minimum, a local maximum, or a saddle.

Answer: $h_x = 3x^2 - 9y$, $h_y = 3y^2 - 9x$, so $h_x = h_y = 0$ when $x^2 = 3y$ and $y^2 = 3x$. So $x^4 = 9y^2 = 27x$ so $x = 0$ (and $y = 0$) or $x = 3$ (and $y = 3$). At $(x, y) = (0, 0)$ we have $h_{xx}h_{yy} - h_{xy}^2 = -81 < 0$ so we have a saddle. At $(x, y) = (3, 3)$ we have $h_{xx}h_{yy} - h_{xy}^2 = (6x)(6y) - (-9)^2 = 18^2 - 9^2 > 0$ and $h_{xx} = 18 > 0$ so we have a (local) minimum. □

4. (a) Find the unique critical point of the function

$$f(x, y) = x^2 + 3xy + 2y^2 - 8x - 11y + 30.$$

(b) Is this critical point a minimum, maximum, or saddle? (c) Does the function $f(x, y)$ take negative values? (I.e. is there a point (x, y) where $f(x, y) < 0$?).

5. Find the quadratic polynomial which best approximates

$$f(x, y) = \sin(xy)$$

near $(x, y) = (1, \pi)$.

[2]In the context of a true-false question, *true* means *always true* and *false* means *sometimes false*.

[3]i.e. a maximum or minimum.

6. Consider the function $f(x,y) = x^2 + 3xy + 2y^2$. True or false?

(a) The level curves of f are ellipses.

(b) The level curves of f are hyperbolas.

(c) The function f takes on only values which are greater than or equal to zero, i.e. $f(x,y) \geq 0$ for all (x,y).

(d) The function f takes on all values, i.e. for every real number z there is a point (x,y) such that $z = f(x,y)$.

(e) The function has a minimum at the origin.

(e) The function has a maximum at the origin.

7. Find the absolute maximum and the absolute minimum of

$$f(x,y) = (x-1)(y-2)$$

in the closed triangle $0 \leq x$, $0 \leq y$, $x + y \leq 7$ bounded by the x-axis, the y-axis, and the line $x + y = 7$.

8. Let $f(x,y) = 2x^2 + 2xy + y^2 - 8x - 6y$.

(a) What is the smallest value $f(x,y)$ can take?

(b) What is the largest value $f(x,y)$ can take?

9. Minimize the function $f(x,y) = x^2 - 5xy + y^2$ on the square $-1 \leq x \leq 1$, $-1 \leq y \leq 1$.

10. (a) Find the unique critical point of the quadratic polynomial

$$f(x,y) = 2035 - 107\,x - 321\,y + 3\,x^2 + 7\,xy + 13\,y^2$$

(b) Find the unique critical point of the quadratic polynomial

$$f(x,y) = 2 + 3(x-5)^2 + 7(x-5)(y-11) + 13(y-11)^2.$$

(c) Assume that $B^2 - 4AC \neq 0$. Find the unique critical point of the quadratic polynomial

$$f(x,y) = f(x_0,y_0) + A(x-x_0)^2 + B(x-x_0)(y-y_0) + C(y-y_0)^2.$$

(d) Assume that $B^2 - 4AC \neq 0$ and let $f(x,y)$ be the quadratic polynomial

$$f(x,y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F.$$

Explain how to find (x_0,y_0) so that

$$f(x,y) = f(x_0,y_0) + A(x-x_0)^2 + B(x-x_0)(y-y_0) + C(y-y_0)^2.$$

(e) Write the quadratic polynomial

$$f(x,y) = x^2 + 2xy + 3y^2 + 4x + 5y$$

in the form

$$f(x,y) = f(x_0,y_0) + A(x-x_0)^2 + B(x-x_0)(y-y_0) + C(y-y_0)^2.$$

14 Lagrange multipliers

1. Find the maximum and minimum values of $x^2 + y^2$ subject to the constraint $x^2 - 2x + y^2 - 4y = 0$.

Answer: The equations $f_x = \lambda g_x$ and $f_y = \lambda g_y$ are

$$2x = \lambda(2x-2), \qquad 2y = \lambda(2y-4).$$

Dividing gives

$$\frac{x}{y} = \frac{2x-2}{2y-4} \qquad \text{or} \qquad 2xy - 4x = 2xy - 2y$$

so $y = 2x$. Substitute into the constraint:

$$0 = x^2 - 2x + 4x^2 - 8x = 5x(x-2)$$

so $x = y = 0$ or $x = 2$ and $y = 4$. Thus $f(0,0) = 0$ is the minimum and $f(2,4) = 20$ is the maximum. □

2. Find the extreme values of the function $f(x,y) = x^2 + 3y^2 + 2y$ on the unit disk $x^2 + y^2 \le 1$. Hint: On the boundary $x^2 = 1 - y^2$.

Answer: The critical points are given by

$$0 = \frac{\partial f}{\partial x} = 2x, \qquad 0 = \frac{\partial f}{\partial y} = 6y + 2,$$

so $x = 0$ and $y = -1/3$. The point $(x,y) = (0,-1/3)$ lies in the region $x^2 + y^2 \le 1$ so (by the first derivative test) this is a candidate for an extremum. At this point the discriminant is

$$f_{xx}f_{yy} - f_{xy}^2 = 1 > 0$$

and f_{xx} and f_{yy} are both positive so the point is a local minimum by the second derivative test. The value at the critical point is $f(0,-1/3) = 0 + 1/3 - 2/3 = -1/3 < 0$. The maximum occurs on the boundary. We can find the maximum using Lagrange multipliers (Maximize $x^2 + 3y^2 + 2y$ subject to $x^2 + y^2 = 1$), *or* by maximizing (using Calc 221)

$$f(\cos\theta, \sin\theta) = \cos^2\theta + 3\sin^2\theta + 2\sin\theta,$$

or by using the hint.

Here is how to finish the problem using the hint. On the boundary $x^2 = y^2 - 1$, $-1 \le y \le 1$, and $f = (1-y^2) + 3y^2 + 2y = 2y^2 + 2y + 1 = F(y)$. We must maximize

$F(y)$ on the interval $-1 \le y \le 1$. This is a calculus 221 problem. The critical point occurs at $F'(y) = 4y + 2 = 0$ so $y = -1/2$ and $F(-1/2) = 1/2 - 1 + 1 = 1/2$. At the endpoints $F(-1) = 1$ and $F(1) = 5$.

In summary: the minimum value $f(0, -1/3) = -1/3$ occurs at at the interior point $(x, y) = (0, -1/3)$, and the maximum value $f(0, 1) = F(1) = 5$ occurs at the boundary point $(x, y) = (0, 1)$. □

3. Find the point on the ellipse $2x^2 + 3y^2 = 11$ where the function $f(x, y) = 8x - 6y$ achieves its maximum.

Answer: We can do this problem in either of two ways: we can either (a) use the method of Lagrange multipliers. or else (b) parameterize the ellipse as in Math 222 and then use Math 221, Method (a) is easier.

(a) We are maximizing $f(x, y) = 8x - 6y$ subject to the constraint $g(x, y) = 2x^2 + 3y^2 - 11 = 0$. The maximum occurs at a point where

$$g(x, y) = 0, \qquad \nabla f = \lambda \nabla \mathbf{g},$$

which say that we seek points on the ellipse $g(x, y) = 0$ where the gradient is parallel to ∇f. These equations are

$$g(x, y) = 0, \qquad f_x(x, y) = \lambda g_x(x, y), \qquad f_y(x, y) = \lambda g_y(x, y)$$

or

$$2x^2 + 3y^2 - 11 = 0, \qquad 4x = 8\lambda, \qquad 6y = -6\lambda.$$

Divide the last two equations to eliminate λ:

$$x = -2y.$$

Plug in to the equation $g(x, y) = 0$:

$$2(-2y)^2 + 3y^2 - 11 = 0 \Longrightarrow y = \pm 1, \quad x = \mp 2.$$

In the second quadrant $f(x, y)$ is negative, in the fourth it is positive, so the maximum occurs at

$$x = 2, \quad y = -1.$$

(b) Now here's the hard way. The ellipse is given parametrically by

$$x = \frac{\sqrt{11}\cos\theta}{\sqrt{2}}, \qquad y = \frac{\sqrt{11}\sin\theta}{\sqrt{3}},$$

and in terms of θ the function is

$$f = 8\frac{\sqrt{11}\cos\theta}{\sqrt{2}} - 6\frac{\sqrt{11}\sin\theta}{\sqrt{3}}.$$

The derivative is

$$\frac{df}{d\theta} = -8\frac{\sqrt{11}\sin\theta}{\sqrt{2}} - 6\frac{\sqrt{11}\cos\theta}{\sqrt{3}}$$

which vanishes when

$$\tan\theta = -\frac{6\sqrt{2}}{8\sqrt{3}}.$$

There are two such values in the range $0 \le \theta \le 2\pi$:

$$\theta_1 = \tan^{-1}\left(-\frac{6\sqrt{2}}{8\sqrt{3}}\right), \qquad \theta_2 = \theta_1 + \pi.$$

The corresponding points (x_1, y_1) and (x_2, y_2) lie in the fourth quadrant $(x > 0 > y)$ and the second quadrant $(x < 0 < y)$ respectively. In the fourth quadrant the function $f(x, y)$ is positive and in the second quadrant it is negative. Hence the maximum occurs at (x_1, y_1). □

4. Find the point on the curve $xy^2 = 54$, $(y > 0)$ which is nearest the origin.

5. Let $T = f(x, y)$ be the temperature at the point (x, y) on the circle

$$x = \cos\theta, \qquad y = \sin\theta,$$

and suppose that

$$\frac{\partial T}{\partial x} = 2x - y, \qquad \frac{\partial T}{\partial y} = 2y - x.$$

Find where the maximum temperature on the circle occurs.

Answer: By the chain rule

$$\frac{dT}{d\theta} = \frac{\partial T}{\partial x}\frac{dx}{d\theta} + \frac{\partial T}{\partial y}\frac{dy}{d\theta}.$$

In terms of θ we have

$$\frac{\partial T}{\partial x} = 2x - y = 2\cos\theta - \sin\theta, \qquad \frac{\partial T}{\partial y} = 2y - x = 2\sin\theta - \cos\theta,$$

and $dx/d\theta = -\sin\theta$, $dy/d\theta = \cos\theta$ so

$$\frac{dT}{d\theta} = \sin^2\theta - \cos^2\theta = -\cos(2\theta). \qquad (\spadesuit)$$

The critical points are at $\theta = \pm\pi/4, \pi \pm \pi/4$. Integrating $(\spadesuit)$ gives

$$T(\theta) = \frac{-\sin(2\theta)}{2} + C$$

so $T(\theta) < C$ when $\sin(2\theta) > 0$ (i.e. in the first and third quadrants) and $T(\theta) > C$ when $\sin(2\theta) < 0$ (i.e. in the second and fourth quadrants). The maximum $C + 1/2$ occurs at both points $\theta = -\pi/4$ and $\theta = \pi - pi/4$. □

6. Let $f(x, y) = x(y - 4)$.

(a) Find the point (or points) on the circle $x^2 + y^2 = 36$ where $f(x, y)$ is smallest.

(b) Find the point (or points) on the disk $x^2 + y^2 \le 36$ where $f(x, y)$ is smallest.

7. Consider the function $f(x, y) = (x-3)^2 + (x-3)(y-2) + (y-2)^2$ on the square $0 \leq x \leq 1$, $0 \leq y \leq 1$.

(a) At what point in the square is the function smallest?

(b) At what point in the boundary of the square is the function smallest?

8. Find the minimum distance from the origin to the plane $2x + y - z = 5$. (This is Example 1 on page 618 of [TF].)

9. Find the minimum distance from the origin to the plane $2x - 3y + 5z = 19$. (This is Example 3 on page 620 of [TF].)

10. Find the closest point to the origin on the plane $x + 3y - 2z = 4$.

11. Find the minimum distance from the origin to surface $x^2 - z^2 = 1$. (This is Example 2 on page 618 of [TF].)

12. The cone $z^2 = x^2 + y^2$ is cut by the plane $z = 1 + x + y$ in an ellipse C. Find the points on C that are nearest to and farthest from the origin. (This is Example 4 on page 624 of [TF].)

13. Find the extreme values of the function $f(x, y) = xy$ on the ellipse $x^2/8 + y^2/2 - 1 = 0$. (This is Example 5 on page 625 of [TF].)

14. Find the points closest to and farthest from the origin on the ellipse $x^2 + 2xy + 3y^2 = 9$.

15. Find the points on the unit sphere $x^2 + y^2 + z^2 = 1$ where the function $f(x, y, z) = x + 2y + 3z$ is smallest and largest.

16. Find the points on the unit sphere $x^2 + y^2 + z^2 = 1$ where the function $f(x, y, z) = ax + by + cz$ is smallest and largest.

17. What is the greatest area that a rectangle can have if the length of its diagonal is 2? (This is Example 1 on page 678 of [VPR].)

18. Find the maximum and minimum values of $y^2 - x^2$ on the ellipse $x^2/4 + y^2 = 1$. (This is Example 2 on page 678 of [VPR].)

19. Find the minimum of $f(x, y, z) = 3x + 2y + z + 5$ subject to the constraint $g(x, y, z) = 9x^2 + 4y^2 - z = 0$. (This is Example 3 on page 679 of [VPR].)

20. Find the maximum and minimum values of $f(x, y, z) = x + 2y + 3z$ on the ellipse that is the intersection of the cylinder $x^2 + y^2 = 2$ and the plane $y + z = 1$. (This is Example 4 on page 680 of [VPR].)

Part II

Multiple Integrals

15 Double integrals and iterated integrals

1. Write the definition of a Riemann sum S for a function $f(x,y)$ defined on a region A in the xy-plane. Then write the definition of the integral $\iint\limits_A f(x,y)\,dA$.

Answer: A **rectangular approximation** to a plane region A is a finite collection $A_1,\ldots,A_n$ of rectangles such that

(1) every point of A lies in some rectangle of the approximation,

(2) each rectangle intersects A (this includes the case where the rectangle lies completely inside A),

(3) any two of these rectangles overlap only in their common boundary (if at all).

The **norm** of the rectangular approximation $A_1,\ldots,A_n$ is the maximum of the lengths of the diagonals of the rectangles $A_1,\ldots,A_n$. A **Riemann sum** for a function f defined on a plane region A is a finite sum of form

$$S = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

where $A_1,\ldots,A_n$ is a rectangular approximation to the region A, ΔA_k is the area of the rectangle A_k, and (x_k, y_k) is a point in this rectangle. The double integral is defined as the limit

$$\iint\limits_A f(x,y)\,dA = \lim_{\text{norm}\to 0} S$$

where the notation on the right means that the Riemann sum S is close to the double integral when the norm of the rectangular approximation used to define S is sufficiently small. There are functions for which the limit on the right does not exist; when the limit exists we say that the function f is **integrable** on the region A. □

2. Evaluate the integral

$$\int_0^2 \int_{x^2}^4 (x+y)\,dy\,dx$$

in three steps:

(a) Sketch the region A over which the integration extends.

(b) Write an equivalent double integral with the order of integration reversed.

(c) Evaluate both integrals. (You should get the same answer.)

Answer: (a) The area over which we are integrating is bounded above by the line $y = 4$ and below by the curve $y = x^2$, for x-values between 0 and 2.

(b) When we switch the order of integration, the same region is bounded from the left by $x = 0$ and bounded on the right by the curve $x = \sqrt{y}$, for y-values between 0 and 4. Thus the new integral is

$$\int_0^4 \int_0^{\sqrt{y}} (x+y)\, dx\, dy$$

(c) The given iterated integral is

$$\int_0^2 \int_{x^2}^4 (x+y)\, dy\, dx \;=\; \int_0^2 \left(xy + \frac{y^2}{2} \right)\Bigg|_{y=x^2}^4 dy \;=\; \int_0^2 \left(4x + 8 - x^3 - \frac{x^4}{2} \right) dy \;=$$

$$= 2x^2 + 8x - \frac{x^4}{4} - \frac{x^5}{10}\Bigg|_0^2 == 8 + 16 - 4 - \frac{16}{5} = \frac{100-16}{5} = \frac{84}{5}.$$

The iterated integral in part (b) is

$$\int_0^4 \int_0^{\sqrt{y}} (x+y)\, dx\, dy \;=\; \int_0^4 \left(\frac{x^2}{2} + xy \right)\Bigg|_{x=0}^{x=\sqrt{y}} dy \;=$$

$$\int_0^4 \left(\left(\frac{y}{2} + y^{3/2} \right) - 0 \right) dy = \frac{y^2}{4} + \frac{2}{5} y^{5/2} \Bigg|_0^4 = 4 + \frac{64}{5} = \frac{84}{5}$$

□

3. Evaluate the iterated integral $\displaystyle\int_3^5 \int_{-x}^{x^2} (4x + 10y)\, dy\, dx$.

Answer: See [VPR] page 697. □

4. Evaluate the iterated integral $\displaystyle\int_0^1 \int_0^{y^2} 2ye^x\, dx\, dy$.

Answer: See [VPR] page 697. □

5. Find the volume of the solid in the first octant ($x \geq 0,\ y \geq 0,\ z \geq 0$) bounded by the circular paraboloid $z = x^2 + y^2$, the cylinder $x^2 + y^2 = 4$, and the coordinate planes.

Answer: See [VPR] page 698. □

6. Evaluate the integral $\displaystyle\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy$ by reversing the order of integration.

7. Evaluate the integral $\displaystyle\int_0^4 \int_{x/2}^2 e^{y^2}\, dy\, dx$ by reversing the order of integration.

Answer: See [VPR] page 698. ☐

8. Evaluate the integral $\iint\limits_A \frac{\sin x}{x}\,dx\,dy$ where A is the triangle in the xy plane bounded by the line $y = 0$, the line $y = x$, and the line $x = 1$.

9. Find the area bounded by the parabola $y = x^2$ and the line $y = x + 2$.

Answer: See [TF] page 653, ☐

10. Find the volume of the solid whose base is in the xy-plane and is the triangle bounded by the line $y = 0$, the line $y = x$, and the line $x = 1$, while the top of the solid is the plane $z = 3 - x - y$. (The solid has five faces: the base, the top, and three other vertical faces.)

Answer: See [TF] page 650. ☐

16 Integrals over nonrectangular regions

1. Let the domain D be defined by the inequalities

$$0 \le x, \qquad 5 - y^2 \le x \le 11 - y^2$$

and let $f(x, y)$ be a function defined on D.

(a) Draw D.

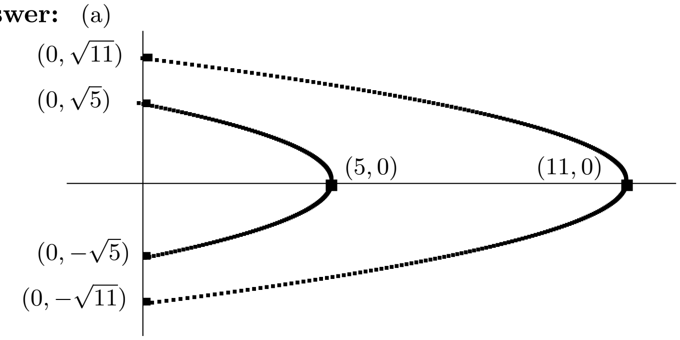
(b) Express the integral of f over D as a sum of iterated integrals of form

$$\int_{y=a_1}^{a_2} \left\{ \int_{x=b_1(y)}^{b_2(y)} f(x, y)\,dx \right\} dy$$

(c) Express the integral of f over D as a sum of iterated integrals of form

$$\int_{x=c_1}^{c_2} \left\{ \int_{y=d_1(x)}^{d_2(x)} f(x, y)\,dy \right\} dx$$

Answer: (a)



(b)

$$\iint_D f(x,y)\,dx\,dy = \int_{y=-\sqrt{11}}^{-\sqrt{5}} \left\{ \int_{x=0}^{11-y^2} f(x,y)\,dx \right\} dy$$
$$+ \int_{y=-\sqrt{5}}^{\sqrt{5}} \left\{ \int_{x=5-y^2}^{11-y^2} f(x,y)\,dx \right\} dy$$
$$+ \int_{y=\sqrt{5}}^{\sqrt{11}} \left\{ \int_{x=0}^{11-y^2} f(x,y)\,dx \right\} dy$$

(c)

$$\iint_D f(x,y)\,dx\,dy = \int_{x=0}^{5} \left\{ \int_{y=-\sqrt{11-x}}^{-\sqrt{5-x}} f(x,y)\,dy \right\} dx$$
$$+ \int_{x=0}^{5} \left\{ \int_{y=\sqrt{5-x}}^{\sqrt{11-x}} f(x,y)\,dy \right\} dx$$
$$+ \int_{x=5}^{11} \left\{ \int_{y=-\sqrt{11-x}}^{\sqrt{11-x}} f(x,y)\,dy \right\} dx$$

□

2. Find the volume of the tetrahedron bounded by the coordinate planes and the plane $3x + 6y + 4z = 12$.

Answer: See [VPR] page 697. □

3. Let $f(x,y) = (x^2 - y^2)/(x^2 + y^2)^2$. Find the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ along the line $y = mx$. Find the iterated integrals

$$\int_0^1 \left\{ \int_0^1 f(x,y)\,dx \right\} dy \text{ and } \int_0^1 \left\{ \int_0^1 f(x,y)\,dy \right\} dx,$$

4. Evaluate the integral $\iint_R x\,dx\,dy$ where R is the triangle with vertices $(1,2)$, $(3,3)$, $(4,5)$.

17 Polar coordinates

1. Write the integral $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ in polar coordinates and then evaluate it.

Answer: Converting to polar coordinates gives

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r dr\,d\theta = \frac{1}{2}\int_0^{\pi/2}\int_0^\infty e^{-u}\,du\,d\theta$$

where we used the change of variables $u = r^2$, so $d\,u = 2r\,dr$, $r = 0$ when $u = 0$ and $r = \infty$ when $u = \infty$. The integral is

$$\frac{1}{2}\int_0^{\pi/2} -e^{-u}\Big|_0^\infty\,d\theta = \frac{1}{2}\int_0^{\pi/2} 1\,d\theta = \pi/4.$$

□

Change each of the following double integrals to an equivalent double integral in polar coordinates. Sketch the region of integration in both (r,θ)-space and (x,y)-space. (The transformation $x = r\cos\theta$, $y = r\sin\theta$ should map the former region one-one onto the latter.) Evaluate both integrals. (You should get the same answer.)

2. $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$

3. $\displaystyle\int_0^{a/\sqrt{2}}\int_y^{\sqrt{a^2-y^2}} x\,dx\,dy$

4. $\displaystyle\int_0^2\int_0^x y\,dy\,dx$

5. $\displaystyle\int_0^a\int_0^{\sqrt{a^2-x^2}} (x^2+y^2)\,dy\,dx$

6. Find the area that lies inside the cardioid $r = a(1+\cos\theta)$ and outside the circle $r = a$.

7. Find the area enclosed by the lemniscate $r^2 = 2a^2\cos 2\theta$.

8. The population density of a city is $\rho = 50000e^{-r^2}$ people per square mile where r is the distance in miles to the center of the city. How many people live within 2 miles of the center?

Answer: $dN = \rho\,dA$ so

$$N = \iint_{r\le 2} \rho\,dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} 50000e^{-r^2}\,r\,dr\,d\theta = 50000\pi\left(e^0 - e^{-4}\right)$$

□

9. (a) Express the distance from the point $x = r\cos\theta$, $y = r\sin\theta$ to the line $x = y$ in terms of r and θ.

(b) The moment of a plane figure D of mass density 1 about a line L is defined to be $\iint_D h(x,y)\,dx\,dy$ where $h(x,y)$ is the signed distance[4] from the point (x,y) to L. Use the result of part (a) to find the moment of the semicircle $x^2+y^2 \le b^2$, $x \ge 0$ about the line $x = y$.

[4] The phrase *signed distance* means that $h(x,y)$ is positive on one side of the line and negative on the other. Thus the answer to this problem is determined only up to a sign.

Answer: (a) The line has polar equation $\theta = \pi/4$. The distance $h(x, y)$ is obtained by trigonometry as $h(x, y) = r\sin(\theta - \pi/4)$.

(b)

$$\int\int_D h(x,y)\,dx\,dy = \int_{r=0}^{b}\int_{\theta=-\pi/2}^{\pi/2} r\sin(\theta-\pi/4)\,r\,d\theta\,dr =$$

$$= \frac{b^3}{3}\big(\cos(\pi/2-\pi/4) - \cos(-\pi/2-\pi/4)\big)$$

□

18 Triple integrals

1. Evaluate $\displaystyle\int_{x=0}^{1}\int_{y=2x}^{1}\int_{z=x^3+y}^{x^2+2y} y\,dz\,dy\,dx$.

Answer:

$$\begin{aligned}
\int_{x=0}^{1}\int_{y=2x}^{1}\int_{z=x^3+y}^{x^2+2y} y\,dz\,dy\,dx &= \int_{x=0}^{1}\int_{y=2x}^{1} y(x^2+2y) - y(x^3+y)\,dy\,dx \\
&= \int_{x=0}^{1}\int_{y=2x}^{1} (y^2 + yx^2 - yx^3)\,dy\,dx \\
&= \int_{x=0}^{1}\left(\frac{y^3}{3} + \frac{y^2x^2 - y^2x^3}{2}\right)\Bigg|_{y=2x}^{1} dx \\
&= \int_{x=0}^{1}\left(\frac{1}{3} + \frac{x^2-x^3}{2} - \frac{8x^3}{3} - \frac{4x^4-4x^5}{2}\right) dx \\
&= \frac{1}{3} + \frac{1}{6} - \frac{1}{8} - \frac{8}{12} - \frac{2}{5} + \frac{2}{6}.
\end{aligned}$$

□

2. Find the volume of the solid in the first octant bounded below by the xy plane, above by the plane $z = y$, and laterally by the cylinder $y^2 = x$ and the plane $x = 1$.

Answer: There are six ways of ordering the variables (x, y, z) and hence six ways of doing the integral. We do it two of these six ways. The first octant is defined by the inequalities $0 \le x$, $0 \le y$, $0 \le z$ and the solid is defined by the additional inequalities $z \le y$ and $y^2 \le x \le 1$. The volume is

$$\int_{y=0}^{1}\int_{x=y^2}^{1}\int_{z=0}^{y} dz\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{\sqrt{x}}\int_{z=0}^{y} dz\,dy\,dx$$

□

3. Evaluate $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$. (Hint: what is the region of integration?)

Answer: The integral represents the volume of a quarter of the unit sphere, that is, $1/4(4\pi/3)$. □

4. Evaluate the iterated integral

$$\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} dz\,dy\,dx = \frac{\pi}{6}.$$

5. Find the volume below the plane $z = 3$ and above the paraboloid $2z = x^2+y^2$.

6. Find the volume of the solid region R bounded by the planes

$$x = 1, \quad x = y, \quad z = x + y, \quad z = x + 2.$$

Answer: Four planes bound a tetrahedron. Each three of the four planes determine a vertex as follows:

three planes			vertex
$x = 1,$	$x = y,$	$z = x + 2$	$(1, 1, 3)$
$x = 1,$	$x = y,$	$z = x + y$	$(1, 1, 2)$
$x = 1,$	$z = x + y,$	$z = x + 2$	$(1, 2, 3)$
$x = y,$	$z = x + y,$	$z = x + 2$	$(2, 2, 4)$

On the tetrahedron x takes values between 1 and 2, $x = 1$ is one of the faces of the tetrahedron, and each plane $x = x_0$ cuts the tetrahedron in the triangle

$$x_0 \le y \le 2, \qquad x_0 + y \le z \le x_0 + 2.$$

Hence the volume is

$$V = \int_{x=1}^{2} \int_{y=x}^{2} \int_{z=x+y}^{x+2} dz\,dy\,dx.$$

The value of this integral is

$$V = \int_{x=1}^{2} \int_{y=x}^{2} (2 - y)\,dy\,dx = \int_{1}^{2} \left(\left(2(2) - \frac{2^2}{2} \right) - \left(2x - \frac{x^2}{2} \right) \right) dx = \frac{1}{6}.$$

□

19 Center of mass

1. Define the terms center of mass, center of gravity, and centroid.

Answer: The **center of mass** of a mass distribution δ is the point whose coordinates are the average values of the coordinate functions with respect to that mass distribution. In one dimension the center of mass is the point $\bar{x}$ given by

$$\bar{x} = \frac{\int x\,dm}{\int dm}, \qquad dm = \delta(x)\,dx.$$

In two dimensions the center of mass is the point $(\bar{x}, \bar{y})$ given by

$$\bar{x} = \frac{\iint x\,dm}{\iint dm}, \quad \bar{y} = \frac{\iint y\,dm}{\iint dm}, \qquad dm = \delta(x,y)\,dx\,dy.$$

In three dimensions the center of mass is the point $(\bar{x}, \bar{y}, \bar{z})$ given by

$$\bar{x} = \frac{\iiint x\,dm}{\iiint dm}, \quad \bar{y} = \frac{\iiint y\,dm}{\iiint dm}, \quad \bar{z} = \frac{\iiint z\,dm}{\iiint dm}, \qquad dm = \delta(x,y,z)\,dx\,dy\,dz.$$

The **centroid** of a body is the center of mass of a uniform mass distribution on the body, i.e. δ is a constant. (The centroid is independent of the value of this constant as it cancels.) The centroid is also called the **center of gravity**. □

2. Find the centroid of the square $0 \le x \le 1$, $0 \le y \le 1$.

3. Find the center of mass of the square $0 \le x \le 1$, $0 \le y \le 1$ where the mass density is $\delta(x,y) = xy^2$.

4. Find the centroid of the region $x^2 + y^2 \le 1, \quad x, y \ge 0$.

5. Find the centroid of the quarter circle defined in polar coordinates by the inequalities $0 \le r \le 1, \;\; 0 \le \theta \le \pi/4$.

6. Find the center of mass of the first quadrant of the circle $x^2 + y^2 \le 1$ where the mass density is $\delta(x,y) = r^2$, $r = \sqrt{x^2+y^2}$.

7. A region A is defined in polar coordinates by inequalities $0 \le r \le f(\theta)$ where $f(\theta) > 0$ and $f(\theta + 2\pi) = f(\theta)$.

(1) True or false? The area is $\displaystyle\int_0^{2\pi}\int_0^{f(\theta)} dA$ where $dA = r\,dr\,d\theta$.

(2) True or false? The centroid is given in polar coordinates by

$$\bar{\theta} = \frac{\displaystyle\int_0^{2\pi}\int_0^{f(\theta)} \theta\,dA}{\displaystyle\int_0^{2\pi}\int_0^{f(\theta)} dA}, \qquad \bar{r} = \frac{\displaystyle\int_0^{2\pi}\int_0^{f(\theta)} r\,dA}{\displaystyle\int_0^{2\pi}\int_0^{f(\theta)} dA},$$

where $dA = r\,dr\,d\theta$.

8. Find the centroid of the triangle with vertices $(0, 0)$, $(a, 0)$, $(0, b)$.

9. Find the centroid of the triangle with vertices $(0, 0)$, $(1, 0)$, $(3, 2)$.

10. Find the centroid of the triangle with vertices $(0, 0)$, $(1, 0)$, (a, b).

11. Show that the centroid of a triangle is the intersection of its medians. (A median of a triangle is the line joining a vertex to the midpoint of the opposite side. It is a theorem of high school geometry that the thee medians intersect in a common point.)

12. Find the mass in the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y + 2z = 2$ if the mass density at the point (x, y, z) is $\delta(x, y, z) = 1 - z$.

Answer: The mass dm in a tiny volume $dV = dx\,dy\,dz$ at the point (x, y, z) is

$$dm = \delta(x, y, z)\,dV = (1 - z)\,dx\,dy\,dz.$$

The total mass m is thus

$$m = \iiint dm = \int_0^1 \int_0^{2-2z} \int_0^{2-2z-y} (1 - z)\,dx\,dy\,dz.$$

□

13. Find the volume and centroid of the hemisphere

$$x^2 + y^2 + z^2 \leq 1, \qquad z \geq 0.$$

Answer: The volume is $V = 2\pi/3$, half the volume of the sphere. using cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ we get

$$\bar{x} = \frac{1}{V}\iiint x\,dV = \frac{1}{V}\int_{r=0}^{1}\int_{z=0}^{\sqrt{1-r^2}}\int_{\theta=0}^{2\pi} r^2\cos\theta\,d\theta\,dz\,dr = 0$$

since $\int_0^{2\pi}\cos\theta\,d\theta = 0$. Similarly $\bar{y} = 0$. Now

$$\bar{z} = \frac{1}{V}\iiint z\,dV = \frac{1}{V}\int_{r=0}^{1}\int_{z=0}^{\sqrt{1-r^2}}\int_{\theta=0}^{2\pi} rz\,d\theta\,dz\,dr = \frac{2\pi}{V}\int_{r=0}^{1}\frac{1-r^2}{2}\,r\,dr$$

so

$$\bar{z} = \frac{2\pi}{V}\left(\frac{r^2}{4} - \frac{r^4}{8}\right)\Bigg|_{r=0}^{1} = \frac{\pi}{4V} = \frac{3}{8}.$$

□

14. Calculate the centroid $(\bar{x}, \bar{y}, \bar{z})$ of the body B defined by

$$B: \qquad x^2 + y^2 + z^2 \leq 1, \qquad 0 \leq x, \quad 0 \leq y, \quad 0 \leq z.$$

Answer: Since we are asked to find the centroid we are to assume a uniform (constant) mass density $c = \delta(x, y, z)$ and since the constant c will cancel, we may as well assume that $c = 1$; i.e. that the mass dm in a tiny cube is the same as the volume dV of that cube. We must evaluate the ratios

$$\bar{x} = \frac{\displaystyle\iiint_B x\,dV}{\displaystyle\iiint_B dV}, \quad \bar{y} = \frac{\displaystyle\iiint_B y\,dV}{\displaystyle\iiint_B dV}, \quad \bar{z} = \frac{\displaystyle\iiint_B z\,dV}{\displaystyle\iiint_B dV}.$$

Now it is obvious that

$$\iiint\limits_{\substack{x^2+y^2+z^2\le 1\\ 0\le x,\,0\le y,\,0\le z}} x\,dx\,dy\,dz = \iiint\limits_{\substack{x^2+y^2+z^2\le 1\\ 0\le x,\,0\le y,\,0\le z}} y\,dx\,dy\,dz$$

because $dx\,dy\,dz = dy\,dz\,dx$ and we can replace (x, y, z) by (y, z, x) (this replaces x by y) to transform the left side to the right. Similar remarks apply for the integral involving z. Thus

$$\bar{x} = \bar{y} = \bar{z}$$

and we need only do two integrals.

The denominator in the formula for $\bar{x}$ is the volume of the body B. The body B is intersection of a ball with the positive octant so its volume is one eighth the volume of a ball:

$$\iiint_B dV = \frac{1}{8} \cdot \frac{4\pi}{3} = \frac{\pi}{6}.$$

We can write the numerator of $\bar{x}$ as an iterated integral in six different ways corresponding to the six permutations of x, y, z. Any of these six will give the correct answer, but which order will make the integral easiest to evaluate? My impulse is to make the x-integral innermost. That way we will get $x^2/2$ which we will have to evaluate at 0 and $\sqrt{1 - y^2 - z^2}$ and we will avoid a $\sqrt{\ }$ in the integrand of the middle integral. (Another order might be easier but the only way to find out would be to try it.)

$$\iiint_B x\,dV = \int_{y=0}^{1} \int_{z=0}^{\sqrt{1-y^2}} \int_{x=0}^{\sqrt{1-y^2-z^2}} x\,dx\,dz\,dy.$$

Evaluate the x-integral:

$$\int_{y=0}^{1} \int_{z=0}^{\sqrt{1-y^2}} \int_{x=0}^{\sqrt{1-y^2-z^2}} x\,dx\,dz\,dy = \int_{y=0}^{1} \int_{z=0}^{\sqrt{1-y^2}} \left.\frac{x^2}{2}\right|_{x=0}^{\sqrt{1-y^2-z^2}} dz\,dy$$

$$= \int_{y=0}^{1} \int_{z=0}^{\sqrt{1-y^2}} \frac{1 - y^2 - z^2}{2}\,dz\,dy$$

Evaluate the z-integral:

$$\int_{y=0}^{1}\int_{z=0}^{\sqrt{1-y^2}} \frac{1-y^2-z^2}{2}\,dz\,dy = \int_{y=0}^{1} \left.\frac{3(1-y^2)z-z^3}{6}\right|_{z=0}^{\sqrt{1-y^2}} dy$$

$$= \int_{y=0}^{1} \frac{3(1-y^2)\sqrt{1-y^2}-\left(\sqrt{1-y^2}\right)^3}{6}\,dy = \frac{1}{3}\int_{y=0}^{1}\left(\sqrt{1-y^2}\right)^3 dy$$

For the y integral use the substitutions

$$y=\sin\theta, \qquad dy=\cos\theta\,d\theta, \qquad \theta=0 \Rightarrow y=0, \qquad \theta=\frac{\pi}{2} \Rightarrow y=1.$$

$$\frac{1}{3}\int_{y=0}^{1}\left(\sqrt{1-y^2}\right)^3 dy = \frac{1}{3}\int_{\theta=0}^{\pi/2}\left(\sqrt{1-\sin^2\theta}\right)^3\cos\theta\,d\theta = \frac{1}{3}\int_{\theta=0}^{\pi/2}\cos^4\theta\,d\theta.$$

To do the even power of the cosine we use the half angle formulas since $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $1=\cos^2\theta+\sin^2\theta$ we add and get $1+\cos 2\theta = 2\cos^2\theta$ so

$$\cos^4\theta = \left(\frac{1+\cos 2\theta}{2}\right)^2 = \frac{1+2\cos 2\theta+\cos^2 2\theta}{4}.$$

Now read 2θ for θ so

$$\cos^2 2\theta = \frac{1+\cos 4\theta}{2}.$$

Substitute to get

$$\cos^4\theta = \frac{3+4\cos 2\theta+\cos 4\theta}{8}.$$

But

$$\int_0^{\pi/2}\cos n\theta\,d\theta = \left.\frac{\sin n\theta}{n}\right|_0^{\pi/2}$$

and this is zero if n is even. Hence

$$\iiint_B x\,dV = \frac{1}{3}\int_{\theta=0}^{\pi/2}\cos^4\theta\,d\theta = \frac{1}{3}\int_{\theta=0}^{\pi/2}\frac{3}{8}\,d\theta = \frac{\pi}{16}.$$

Our final answer is

$$\bar{x}=\bar{y}=\bar{z}=\frac{\pi/16}{\pi/6}=\frac{3}{8}.$$

□

15. A plane region is bounded by the polar curve $r=f(\theta)$ where $f(\theta+2\pi)=f(\theta)$. Give a formula for the centroid and use it to find the centroid of the cardioid $r=1+\cos\theta$.

20 Surface area and change of variables

1. Derive the formula for the surface area of a surface given by the parametric equations

$$x=x(u,v), \qquad y=y(u,v), \qquad z=z(u,v).$$

Answer: The tiny rectangle with vertices (u,v), $(u+du,v)$, $(u,v+dv)$ is mapped approximately to the parallelogram with vertices

$$(x,y,z), \qquad (x+x_u\,du, y+y_u\,du, z+z_u\,du), \qquad (x+x_v\,dv, y+y_v\,dv, z+z_v\,dv).$$

(The fourth vertex of a parallelogram is determined by the other three.) The edge vectors of this parallelogram are

$$\mathbf{R}_u\,du = (x_u\mathbf{i} + y_u\mathbf{j} + z_u\mathbf{k})\,du$$

and

$$\mathbf{R}_v\,dv = (x_v\mathbf{i} + y_v\mathbf{j} + z_v\mathbf{k})\,dv.$$

The area of this infinitessimal parallelogram is

$$dA = |\mathbf{R}_u \times \mathbf{R}_v|\;du\,dv$$

so the area of the surface is

$$\iint dA = \iint |\mathbf{R}_u \times \mathbf{R}_v|\;du\,dv$$

where the integral is over a region in uv space chosen so that the parameterization is one to one and onto the surface. □

2. Derive the formula for the surface area of the graph $z = f(x,y)$.

Answer: This is a special case of the problem 1 with

$$x = u, \qquad y = v, \qquad z = f(u,v) = f(x,y).$$

Thus

$$\mathbf{R}_x\,dx = (\mathbf{i} + f_x\mathbf{k})\,dx, \qquad \mathbf{R}_y\,dy = (\mathbf{j} + f_y\mathbf{k})\,dy$$

so

$$dA = \sqrt{1 + f_x^2 + f_y^2}\,dx\,dy$$

and the area is

$$\iint dA = \iint \sqrt{1 + f_x^2 + f_y^2}\,dx\,dy$$

where the integral is over the region in the xy-plane beneath the graph. □

3. A transformation

$$x = x(u,v), \qquad y = y(u,v)$$

transforms the region G in the uv-plane one to one onto the region R in the xy plane. Derive the formula for the area of R as an integral over the region G.

Answer: This is a special case of the problem 1 with $z = 0$. Thus

$$\mathbf{R}_u = x_u\mathbf{i} + y_u\mathbf{j}, \qquad \mathbf{R}_v = x_v\mathbf{i} + y_v\mathbf{j}$$

so

$$\mathbf{R}_u \times \mathbf{R}_v = (x_u y_v - x_v y_u)\mathbf{k}$$

and hence

$$dA = |x_u y_v - x_v y_u|\, du\, dv$$

so

$$\iint_R dA = \iint_G |x_u y_v - x_v y_u|\, du\, dv.$$

□

4. Derive the formula for the area element in polar coordinates from problem 3.

Answer: From $x = r\cos\theta$ and $y = r\sin\theta$ we get

$$\mathbf{R}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \qquad \mathbf{R}_\theta = -r\sin\theta\mathbf{i} + r\cos\theta\,\mathbf{j}$$

so

$$dA = |\mathbf{R}_r \times \mathbf{R}_\theta|\, dr\, f\theta = r\, dr\, d\theta.$$

□

5. Derive the formula for the surface area element on the unit sphere using problem 1 and spherical coordinates.

Answer: The parametric equations are

$$x = \sin\phi\cos\theta, \qquad y = \sin\phi\sin\theta, \qquad z = \cos\phi$$

and (except for the north and south pole) every point of the sphere is covered exactly once in the range $0 < \phi < \pi$, $0 \le \theta < 2\pi$. Now

$$\mathbf{R}_\phi = \cos\phi\cos\theta\,\mathbf{i} + \cos\phi\sin\theta\,\mathbf{j} - \sin\phi\,\mathbf{k}, \qquad \mathbf{R}_\theta = -\sin\phi\sin\theta\,\mathbf{i} + \sin\phi\cos\,\theta\mathbf{j}$$

so

$$\mathbf{R}_\phi \times \mathbf{R}_\theta = \sin^2\phi\cos\theta\,\mathbf{i} + \sin^2\phi\sin\theta\,\mathbf{j} + \cos\phi\sin\phi\,\mathbf{k},$$

so

$$dA = |\mathbf{R}_\phi \times \mathbf{R}_\theta|\; d\phi\, d\theta = \sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi}\, d\phi\, d\theta = \sin\phi\, d\phi\, d\theta.$$

□

6. Find the area cut from the plane $x + y + z = 1$ by the cylinder $x^2 + y^2 = 1$.

Answer: We are finding the surface area of the plane $z = f(x, y) = -x - y + 1$ over the unit disk $x^2 + y^2 = 1$. So using the formula

$$S = \iint \sqrt{(f_x)^2 + (f_y)^2 + 1}dx\, dy$$

where $f_x = -1$, $f_y = -1$ gives

$$S = \iint_{x^2+y^2\le 1} \sqrt{1+1+1}\, dy\, dx = \int_0^{2\pi}\int_0^1 \sqrt{3}r\, dr\, d\theta = \sqrt{3}\pi.$$

□

7. Let $h(x, y, z) = x + y + z$ and let S be the portion of the surface $z = x^3 + yx$, for which $0 \le x \le 2$ and $0 \le y \le 1$. Express $\iint_S h\,dA$ as a double integral in terms of x and y. You need not evaluate the integral.

Answer: The surface is described in the form $z = f(x, y)$ so the area element is given by

$$dA = \sqrt{1 + f_x^2 + f_y^2}\,dx\,dy = \sqrt{1 + (3x^2 + y)^2 + (x)^2}\,dx\,dy$$

At a point $(x, y, z) = (x, y, f(x, y))$ on the surface the value of the function h is $h(x, y) = x + y + (x^3 + yx)$. Thus

$$\iint_S h\,dA = \int_0^1 \int_0^2 (x + y + (x^3 + yx))\sqrt{1 + (3x^2 + y)^2 + (x)^2}\,dx\,dy.$$

□

8. Redo Problems 11 and 12 from section 3 Part I. (These problems deal with finding the arclength of a parameterized curve; you should compare the reasoning there with the reasoning used in doing problems 9 and 10 which follow.)

9. Use the parameterization

$$x = \cos u \sin v, \qquad y = \sin u \sin v, \qquad z = \cos v$$

from Problem 9 in Part I to find the area of that portion S of the unit sphere in the first octant. The main problem here is to choose the limits of integration in uv-space so that S is covered exactly once. (The answer should come out to $4\pi/8$.)

10. Use the parameterization

$$x = v \cos u - \sin u, \qquad y = v \sin u + \cos u, \qquad z = v$$

from Problem 10 in Part I to set up an integral for the area of that portion S of hyperboloid $x^2 + y^2 = z^2 + 1$ between the planes $z = 0$ and $z = 1$. The main problem here is to choose the limits of integration in uv-space so that S is covered exactly once.

11. Consider the function

$$f(x, y) = \frac{194\,x^2}{169} + \frac{120\,xy}{169} + \frac{313\,y^2}{169} - \frac{628\,x}{169} - \frac{1372\,y}{169} + \frac{1686}{169}$$

and the transformation $(x, y) = T(u, v)$ defined by

$$x = \frac{5\,u}{13} - \frac{12\,v}{13} + 1, \qquad y = \frac{12\,u}{13} + \frac{5\,v}{13} + 2.$$

Let $g = f \circ T$ be the composition, i.e.

$$g(u, v) = f\left(\frac{5\,u}{13} - \frac{12\,v}{13} + 1, \frac{12\,u}{13} + \frac{5\,v}{13} + 2\right).$$

(a) Simplify the expression for g.

(b) Draw the lines $u = 0$ and $v = 0$ in the (x, y) plane. Where do they intersect? What is the angle between them? Draw the curve $g(u, v) = 4$ in the (u, v) plane and the curve $f(x, y) = 4$ in the (x, y) plane.

(c) Minimize $g(u, v)$. What is the discriminant $g_{uu}g_{vv} - g_{uv}^2$?

(d) Minimize $f(x, y)$. What is the discriminant $f_{xx}f_{yy} - f_{xy}^2$?

(e) What is the Jacobian of the transformation T?

(f) Find the area of the ellipse $f(x, y) \le 4$.

(g) The point of this problem is that the transformation T converts a messy problem in (x, y) into a simple problem in (u, v). Had you not been given the formula for T, how would you find it?

12. Find the area of the surface $y^2 + z^2 = 2x$ cut off by the plane $x = 1$.

13. The triangle $0 \le u \le v \le 1$ in the uv-plane is transformed one-to-one onto the region R in the xy-plane by the equations

$$x = v, \qquad y = u + v^2.$$

(a) Sketch the region R and find equations for its boundary curves.

(b) Evaluate the integral $\displaystyle\iint_R x\,dx\,dy$.

14. The transformation $(x, y) = T(u, v)$ is given by

$$x = 1 + 2v + u^2, \qquad y = \frac{u}{3}.$$

It transforms the triangle

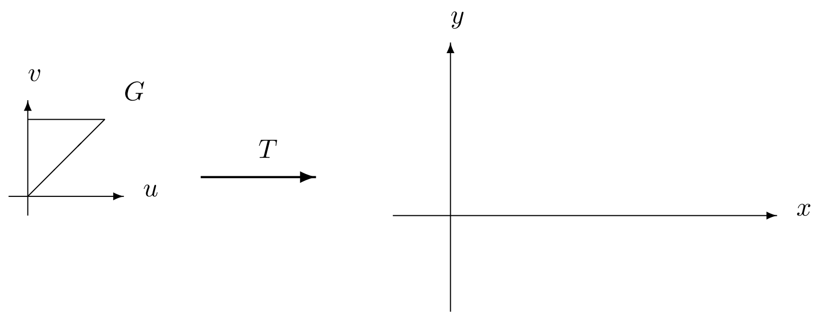
$$G: \qquad 0 \le u \le v \le 1$$

in (u, v) space to a region R in (x, y) space. The region R is bounded by three curves.

(a) Find parametric equations for each of the boundary curves.

(b) Find equations in (x, y) for each of the boundary curves.

(c) Sketch the region R. Indicate clearly which points of the boundary of R correspond to the vertices $(0,0)$, $(0,1)$, $(1,1)$ of the triangle G and boundary curve of R corresponds to which equation in part (b).



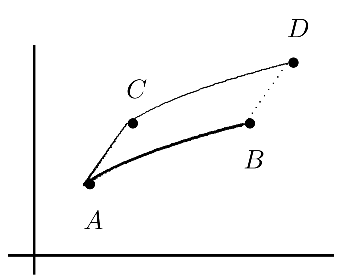
(d) Evaluate $\iint\limits_R (x+y^2)\, dx\, dy$ by using the change of variables formula to convert it into and integral over the triangle G.

15. The transformation $(x, y) = T(u, v)$ is defined by

$$x = e^{2u} + e^v, \qquad y = e^u + e^v$$

carries the unit square $0 \le u \le 1$, $0 \le v \le 1$ in the (u, v) plane to a region R in the (x, y) plane shown in the diagram.

(a) Complete the table to give the coordinates of the vertices of R.



P	(x, y)
A	
B	
C	
D	

(b) Which of the four sides of R are straight line segments?

(c) Find the area of R.

16. Problems 14 and 15 illustrate the change of variables formula from problem 3. This formula says that if the equations

$$x = x(u, v), \qquad y = y(u, v),$$

transform a region G in the uv plane one to one onto a region R in the xy plane then

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

Note that the integrand contains the absolute value of the determinant

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - y_u x_v.$$

There is an analogous formula for equations

$$x = x(u,v,w), \qquad y = y(u,v,w), \qquad z = z(u,v,w)$$

which transform a region H in uvw-space onto a region B in xyz-space. It is

$$\begin{aligned}&\iiint_B f(x,y,z)\,dx\,d\,dz\\ &= \iiint_H f(x(u,v,w),y(u,v,w),z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw.\end{aligned}$$

where the factor on the right is the determinant

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}.$$

The proofs are analogous: for the two dimensional formula we use the fact that the length of the cross product is the area of the corresponding parallelogram; for the three dimensional formula we use the fact that the triple product is the volume of the corresponding parallelopiped. Use the three dimensional formula to find the appropriate factors in the formulas

$$\iiint_{\substack{x^2+y^2\le a^2\\ b\le z\le c}} f(x,y,z)\,dx\,dy\,dz = \int_b^c \int_0^a \int_0^{2\pi} f(r\cos\theta, r\sin\theta, z) \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| d\theta\,dr\,dz$$

and

$$\begin{aligned}&\iiint_{x^2+y^2+z^2\le a^2} f(x,y,z)\,dx\,dy\,dz\\ &= \int_0^a \int_0^\pi \int_0^{2\pi} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right| d\theta\,d\phi\,d\rho.\end{aligned}$$

17. Show that the area of the portion of the sphere $x^2+y^2+z^2=a^2$ between the planes $z=h_1$ and $z=h_2$ is the same as the area of the portion of the cylinder $x^2+y^2=a^2$ between these two planes.[5]

[5]Cicero said he saw this fact inscribed on Archimedes' tomb in 75 B.C.; Archimedes died 137 years earlier.

Answer: See [VPR] page 713. ☐

18. Find the center of mass of the sphere $x^2 + y^2 + z^2 = a^2$ between the plane $z = h_1$ and $z = h_2$.

Answer: See [VPR] problem 15 page 715. ☐

19. Sketch the polar cap of the sphere $\rho = a$ defined by the inequalities $0 \leq \phi \leq \phi_0$ in spherical coordinates. Then show that its area is $2\pi a^2(1 - \cos\phi_0)$.

Answer: See [VPR] problem 16 page 715. ☐

20. The circle $z^2 + (y-5)^2 = 1$, $x = 0$ in the (y, z) plane is revolved about the z-axis. Find the volume of the torus (donut shaped figure) which is swept out.

Answer: In cylindrical coordinates the torus has equation $z^2 + (r-5)^2 = 1$. Hence the volume is

$$V = \int dV = \int_{r=4}^{6} \int_{z=-\sqrt{1-(r-5)^2}}^{\sqrt{1-(r-5)^2}} \int_{\theta=0}^{2\pi} r\, d\theta\, dz\, dr$$

The inner integrals are constant so

$$V = 4\pi \int_4^6 \sqrt{1-(r-5)^2}\, r\, dr$$

Rewrite as

$$V = 4\pi \int_4^6 \sqrt{1-(r-5)^2}\, (r-5)\, dr + 4\pi \int_4^6 \sqrt{1-(r-5)^2}\, 5\, dr$$

In the first integral take $u = 1 - (r-5)^2$ so $(r-5)r\, dr = \frac{-du}{2}$ and $u = 0$ when $r = 4$ or $r = 6$ so this integral is zero. In the second integral take $v = (r-5)$ so $dv = dr$ and $v = -1$ when $r = 4$ and $v = 1$ when $r = 6$. The integral is

$$V = 20\pi \int_{v=-1}^{1} \sqrt{1-v^2}\, dv = 10\pi^2.$$

☐

21. **(Pappus's Theorem for regions)** A region R in the rz plane is rotated about the z axis to obtain a body B. Show that the resulting volume is

$$\iiint_B dV = 2\pi\bar{r} \iint_R dA, \qquad \bar{r} = \frac{\displaystyle\iint_R r\, dA}{\displaystyle\iint_R dA},$$

i.e. it is equal to the area of R multiplied by the circumference of the circle swept out by the center of gravity of the region R. Hint: Use cylindrical coordinates.

22. **(Pappus's Theorem for curves)** A curve C in the rz plane is rotated about the z axis to obtain a surface S. Show that the resulting area is

$$\iint_S dA = 2\pi\bar{r}\int_C ds, \qquad \bar{r} = \frac{\displaystyle\int_C r\,ds}{\displaystyle\int_C ds},$$

i.e. it is equal to the length of C multiplied by the circumference of the circle swept out by the center of gravity of the curve C. Hint: Use cylindrical coordinates.

23. The volume above the cone $z = \sqrt{x^2+y^2}$ is removed from the sphere $x^2+y^2+z^2 = 1$. Find the volume that remains.

24. Spherical coordinates (ρ, ϕ, θ) are related to cartesian coordinates via the equation

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi.$$

Find the remaining volume of the sphere $\rho = 2$ if we take out the cone $\phi = \frac{\pi}{3}$. (Hint: the part you remove resembles a filled ice cream cone.)

Answer: In spherical coordinates, the volume is given by

$$V = \int_0^{2\pi}\int_{\pi/3}^{\pi}\int_0^2 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Note that range of ϕ is between $\pi/3$ to π . So

$$V = \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\pi}\left(\frac{2^3}{3} - 0\right)\sin\phi\,d\phi\,d\theta = \int_0^{2\pi}\frac{8}{3}\cdot\left(-\cos\pi - \left(-\cos\frac{\pi}{3}\right)\right)\,d\theta = 4\cdot 2\pi = 8\pi.$$

□

25. The integral

$$W(z_0) = \frac{1}{4\pi}\iint_{x^2+y^2+z^2=1}\frac{dA}{\sqrt{x^2+y^2+(z-z_0)^2}}$$

gives the gravitational field at the point $(0,0,z_0)$ due a uniform mass distribution of total mass one on the unit sphere $x^2+y^2+z^2 = 1$. Evaluate this integral for $0 < z_0 < 1$ and then for $1 < z_0$. Graph $U(z_0)$ as a function of z_0. Hint: The quantity $\sqrt{x^2+y^2+(z-z_0)^2}$ is the distance from (x,y,z) to $(0,0,z_0)$. Use spherical coordinates and the law of cosines.

26. The gravitational potential at the point $(0, 0, z_0)$ due to a uniform mass distribution in the region

$$R: \qquad a \le \sqrt{x^2 + y^2 + z^2} \le b$$

is given by the triple integral

$$\iiint_R \frac{dx\,dy\,dz}{f(x, y, z)}$$

where $f(x, y, z)$ is the distance from the point (x, y, z) to the point $(0, 0, z_0)$.

(a) Show that $f(x, y, z) = \sqrt{\rho^2 + z_0^2 - 2\rho z_0 \cos\phi}$.

(b) Evaluate the potential as a function of z_0. Hint: $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$.

(c) Evaluate the integral in (b) when $z_0 > b$

(d) Evaluate the integral in (b) when $0 < z_0 < a$.

Part III

Vector Analysis

21 Vector fields

1. Define the **gradient** of the function $U = U(x, y, z)$ and the **divergence** and **curl** of the vector field

$$\mathbf{F} = M(x, y, z)\,\mathbf{i} + N(x, y, z)\,\mathbf{j} + P(x, y, z)\,\mathbf{k}.$$

Answer:

$$\begin{aligned}
\operatorname{grad} U &= \nabla U &&= U_x\,\mathbf{i} + U_y\,\mathbf{j} + U_z\,\mathbf{k}, \\
\operatorname{div} \mathbf{F} &= \nabla \cdot \mathbf{F} &&= M_x + N_y + P_z, \\
\operatorname{curl} \mathbf{F} &= \nabla \times \mathbf{F} &&= (P_y - N_z)\,\mathbf{i} + (M_z - P_x)\,\mathbf{j} + (N_x - M_y)\,\mathbf{k},
\end{aligned}$$

□

Remark 2. Here is a trick for remembering these formulas. Write the expression

$$\nabla = \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k}$$

and compute with it using the usual formulas for scalar multiplication, dot product, and cross product. Whenever you see a partial derivative symbol next to a function do the differentiation.

3. Is the vector field

$$\mathbf{F} = y\mathbf{i} + (x + z^3)\mathbf{j} + (2yz^2 + 1)\mathbf{k}$$

a gradient? If so find a function V with $\mathbf{F} = \nabla V$; if not, prove that no such V exists.

Answer: The gradient of V is

$$\nabla V = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}.$$

The vector equation $\mathbf{F} = \nabla V$ takes the form

$$V_x = y, \qquad V_y = x + z^3, \qquad V_z = 2yz^2 + 1.$$

But

$$V_y = x + z^3 \implies V_{yz} = 3z^2$$

and

$$V_z = 2yz^2 \implies V_{zy} = 2z^2.$$

There is no solution since $V_{yz} = V_{zy}$ but $3z^2 \neq 2z^2$. □

22 Line integrals

1. Let C be the line segment from $(0,0)$ to $(1,1)$ and $w = x + y^2$ and ds be the infinitesimal arclength.

(a) Evaluate the line integral $\int_C w\,ds$ for the parameterization

$$x = t, \qquad y = t, \qquad 0 \le t \le 1.$$

(b) Evaluate the line integral $\int_C w\,ds$ for the parameterization

$$x = \sin t, \qquad y = \sin t, \qquad 0 \le t \le \pi/2.$$

(See [TF] page 691 Example 1.)

2. Find the work done by the force $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} + z\mathbf{k}$, as the point of application moves from (1,1,3) to (3,9,1) along the curve

$$x = t, \qquad y = t^2, \qquad z = 4 - t.$$

Answer: By definition the work is

$$W = \int_C \mathbf{F} \cdot d\mathbf{R}.$$

Along the curve C we have

$$\mathbf{F} = 2y\,\mathbf{i} + 3x\,\mathbf{j} + z\,\mathbf{k} = 2t^2\,\mathbf{i} + 3t\,\mathbf{j} + (4-t)\,\mathbf{k}$$

and $dx = dt$, $dy = 2t\,dt$, $dz = -dt$ so

$$d\mathbf{R} = (\mathbf{i} + 2t\mathbf{j} - \mathbf{k})\,dt.$$

Hence

$$\begin{aligned} W = \int_{t=1}^{t=3} \mathbf{F} \cdot d\mathbf{R} &= \int_1^3 \Big((2t^2)(1) + (3t)(2t) + (4-t)(-1) \Big)\,dt = \\ &= \int_1^3 (8t^2 + t - 4)\,dt = \frac{8}{3}t^3 + \frac{1}{2}t^2 - 4t\Big|_1^3 = \\ &= \left(72 + \frac{9}{2} - 12\right) - \left(\frac{8}{3} + \frac{1}{2} - 4\right) = \frac{196}{3}. \end{aligned}$$

□

3. The force at the point (x, y) is

$$\mathbf{F}(x,y) = x^2 y\,\mathbf{i} + 2xy^2\,\mathbf{j}.$$

Find the work

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\,ds$$

done in moving a particle from $(0,0)$ to $(2,4)$ along the curve $y = x^2$. (Here $\mathbf{T}$ is the unit tangent vector and ds is the infinitesimal arc length.)

Answer: The answer is independent of how we parameterize the curve so we take the simplest parameterization: $x = t$ and $y = t^2$. Then the velocity vector is

$$\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} = (\mathbf{i} + 2t\mathbf{j})$$

so the unit tangent vector satisfies

$$\mathbf{T}\,ds = \mathbf{v}\,dt = (\mathbf{i} + 2t\mathbf{j})\,dt,$$

and along the curve the force is given by

$$\mathbf{F}(t, t^2) = t^4\mathbf{i} + 2t^5\mathbf{j},$$

so

$$\mathbf{F}\cdot\mathbf{T}\,ds = (t^4 + 4t^6)\,dt,$$

and

$$W = \int_{t=0}^{t=2} (t^4 + 4t^6)\,dt = \frac{2^5}{5} + \frac{4\cdot 2^7}{7}.$$

□

4. A particle moves from the origin to the point $(2, 4, 8)$ along the curve given parametrically by the equations

$$x = t, \qquad y = t^2, \qquad z = t^3.$$

Find the work done by the force field

$$\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}.$$

5. Let $\mathbf{F} = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$. Find the work done by $\mathbf{F}$ along a curve C lying on a sphere centered at the origin. Hint: What is special about $\mathbf{F}$ and C?

23 Independence of the path

In problems 1 to 6 find the the work done by the force $\mathbf{F}$ as the particle move from the point $P = (0, 0, 0)$ to the point $Q = (1, 1, 1)$

(a) along the straight line $x = y = z$;

(b) along the curve $x = t$, $y = t^2$, $z = t^4$;

(c) along a straight line from $(0, 0, 0)$ to $(1, 0, 0)$ then along a straight line to $(1, 1, 0)$, then along a straight line to $(1, 1, 1)$.

1. $\mathbf{F} = 2x\,\mathbf{i} + 3y\,\mathbf{j} + 4z\,\mathbf{k}$ 2. $\mathbf{F} = (y + z)\,\mathbf{i} + (z + x)\,\mathbf{j} + (x + y)\,\mathbf{k}$
3. $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$ 4. $\mathbf{F} = yz^2\,\mathbf{i} + zx^2\,\mathbf{j} + xy^2\,\mathbf{k}$
5. $\mathbf{F} = e^{y+2z}\,(\mathbf{i} + x\,\mathbf{j} + 2x\,\mathbf{k}$) 6. $\mathbf{F} = y\sin z\,\mathbf{i} + x\sin z\,\mathbf{j} + xy\cos z\,\mathbf{k}$

7. In problems 1 to 6 find the the work done by the force $\mathbf{F}$ as the particle move from the point $P = (0, 0, 0)$ to the point $Q = (x_0, y_0, z_0)$ along the straight line

$$x = tx_0, \qquad y = ty_0, \qquad z = tz_0, \qquad 0 \le t \le 1.$$

8. In problems 1 to 6 find a function $U(x, y, z)$ such that $\mathbf{F} = \nabla U$ or prove that no such function exists.

9. Find a function $w = f(x, y)$ whose first partial derivatives are

$$\frac{\partial w}{\partial x} = 1 + e^x \cos y, \qquad \frac{\partial w}{\partial y} = 2y - e^x \sin y$$

or prove that there is no such function.

10. Let $M = 3x^2y + 2xy^2$ and $N = x^3 + 2x^2y + 3y^2$.

(a) Is there a function $f(x, y)$ satisfying

$$\frac{\partial f}{\partial x} = M \qquad \text{and} \qquad \frac{\partial f}{\partial y} = N?$$

(If so, find it; if not, say why.)

(b) Evaluate $\int_C M\,dx + N\,dy$ where C is the curve given by the parameteric equations

$$x = e^{2t}\sin(3t), \qquad y = e^{2t}\cos(3t), \qquad (0 \le t \le \pi).$$

11. A force is given by $\mathbf{F} = (x^2 - y)\,\mathbf{i} + (y^2 - z)\,\mathbf{j} + (z^2 - x)\,\mathbf{k}$.

(a) Find the work done by the force as the particle moves from $(0, 0, 0)$ to $(1, 1, 1)$ along a straight line.

(b) Find the work done by the force as the particle moves from $(0, 0, 0)$ to $(1, 1, 1)$ along the curve

$$x = t, \qquad y = t^2, \qquad z = t^3, \qquad 0 \le t \le 1.$$

(c) Is $\mathbf{F}$ a gradient? (See [TF] page 694 Example 2.)

12. A force is given by $\mathbf{F} = \mathbf{i} + z\,\mathbf{j} + (y + 3z^2)\,\mathbf{k}$.

(a) Find the work done by the force as the particle moves from $(0, 0, 0)$ to $(1, 1, 1)$ along a straight line.

(b) Find the work done by the force as the particle moves from $(0, 0, 0)$ to $(1, 1, 1)$ along the curve

$$x = t, \qquad y = t^2, \qquad z = t^3, \qquad 0 \le t \le 1.$$

(c) Is $\mathbf{F}$ a gradient?

13. State the theorem which says when a line integral depends only on the endpoints and not on the curve connecting the endpoints.

Answer: A continuously differentiable vector field $\mathbf{F}$ is a gradient vector field if and only if the integral $\int_P^Q \mathbf{F} \cdot d\mathbf{R} := \int_C \mathbf{F} \cdot d\mathbf{R}$ is independent of the path C joining the point P to the point Q. When $\mathbf{F} = \nabla U$ the value if this integral is

$$\int_P^Q \mathbf{F} \cdot d\mathbf{R} = U(Q) - U(P).$$

□

14. State the theorem which says when a line integral around every closed curve is zero.

Answer: A continuously differentiable vector field $\mathbf{F}$ is a gradient vector field if and only if

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = 0$$

for every closed curve in the domain of $\mathbf{F}$. □

Answer: A continuously differentiable vector field $\mathbf{F}$ is a gradient vector field if and only if the integral $\int_A^B \mathbf{F} \cdot d\mathbf{R}$ is independent of the path joining the point A to the point B. □

15. **(a)** Find the gradient of the function $U(x, y) = \tan^{-1}(x/y)$.

(b) Find the line integral $\oint_C \mathbf{F} \cdot d\mathbf{R}$ where C is the curve

$$x = \cos t, \qquad y = \sin t, \qquad 0 \leq t \leq 2\pi$$

and the vector field $\mathbf{F}$ is defined by

$$\mathbf{F} = \frac{y}{x^2 + y^2}\,\mathbf{i} - \frac{x}{x^2 + y^2}\,\mathbf{j}.$$

(c) Why does this not contradict the answer to problem 13?

16. Let$P$ and Q be points in three dimensional space. Prove that

$$\int_P^Q z^2\,dx + 2y\,dy + 2xz\,dz$$

is independent of the path integration. What is the value of this integral if $P = (a_1, b_1, c_1)$ and $Q = (a_2, b_2, c_2)$?

17. **(a)** True or false? The line integral $\int_C \mathbf{F} \cdot d\mathbf{R}$ is independent of the parameterization.

(b) True or false? The line integral $\int_C \mathbf{F} \cdot d\mathbf{R}$ depends only on the endpoints of the curve C.

24 Surface Integrals

1. A curve is given parametrically by equations

$$x = x(t), \qquad y = y(t), \qquad z = z(t).$$

Write the formula for the infinitesimal arc length element ds at the point with parameter value t.

Answer: $ds = \left|\frac{d\mathbf{R}}{dt}\right| dt$ where $\mathbf{R} = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$. Thus

$$ds = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\; dt.$$

□

2. Write infinitesimal arc length element ds for a curve of form $y = f(x)$.

3. A surface is given parametrically by equations

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v).$$

Write the formula for the infinitesimal area element $d\sigma$ at the point with parameter value (u, v).

Answer: $d\sigma = |\mathbf{R}_u \times \mathbf{R}_v|\, du\, dv$ where $\mathbf{R} = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$ and

$$\mathbf{R}_u = x_u\,\mathbf{i} + y_u\,\mathbf{j} + z_u\,\mathbf{k}, \qquad \mathbf{R}_v = x_v\,\mathbf{i} + y_v\,\mathbf{j} + z_v\,\mathbf{k}.$$

Thus

$$d\sigma = \sqrt{(y_u z_v - y_v z_u)^2 + (z_u x_v - z_v x_u)^2 + (x_u y_v - x_v y_u)^2}\; du\, dv.$$

□

4. Write infinitesimal arc area element $d\sigma$ for a surface of form $z = f(x, y)$.

5. Evaluate $\iint\limits_S (xy + z)\, d\sigma$ where S is the portion of the plane

$$2x - y + z = 3$$

above the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$ in the (x, y)-plane. (See [VPR] Example 1 on page 755.)

6. The cylinder $x^2 + y^2 = 2x$ cuts a portion of a surface G from the upper nappe of the cone $x^2 + y^2 = z^2$. Compute the value of the surface integral

$$\iint\limits_G (x^4 - y^4 + y^2 z^2 - z^2 x^2 + 1)\, d\sigma$$

where $d\sigma$ is the infinitesimal area element.

7. A square hole of side $2\sqrt{2}$ is cut symmetrically through a sphere of radius 2. Show that the area of the surface removed is $16\pi(\sqrt{2}-1)$. (See problem 55 page 678 of [TF].)

8. Find the area above the xy-plane cut from the cone $x^2 + y^2 = z^2$ by the cylinder $x^2 + y^2 = 2ax$. (See problem 50 page 678 of [TF].)

9. Find the area of the surface $y^2 + z^2 = 2x$ cut off by the plane $x = 1$. (See problem 48 page 678 of [TF].)

10. Find the surface area of the sphere $r^2 + z^2 = a^2$ that is inside the cylinder $r = a \sin\theta$. (See problem 51 page 678 of [TF].)

11. A torus surface is generated by moving a sphere of unit radius whose center travels around a closed plane circle of radius 2. Find the area of this surface. (See problem 56 page 678 of [TF].)

12. Calculate the area of the surface $(x^2+y^2+z^2)^2 = x^2 - y^2$. (See problem 57 page 678 of [TF].)

25 Green's Theorem

1. State Green's Theorem for a plane region.

Answer: Let R be a plane region whose boundary is a simple closed curve C and $M(x,y)$ and $N(x,y)$ be continuously differentiable functions defined on R. Then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where the boundary curve is traversed in the counter clockwise direction. □

2. Let R be a plane region whose boundary is a simple closed curve C. Show that the area of R is

$$\text{Area}\,(R) = \tfrac{1}{2}\oint_C (x\,dy - y\,dx).$$

(See [TF] page 712 or Example 5 page 752 of [VPR].)

3. **(a)** Evaluate the line integral $\oint_C (y^2\,dx + x^2\,dy)$ where C is the triangle bounded by the three lines $x = 0$, $y = 0$, $x + y = 1$, by writing it as the sum of three line integrals. (Use the counter clockwise orientation.)

(b) Evaluate this integral using Green's Theorem.

4. **(a)** Evaluate the line integral $\oint_C (3y\,dx + 2x\,dy)$ where C is the boundary of the region $0 \le x \le \pi$ and $0 \le y \le \sin x$ by writing it as the sum of two line integrals. (Use the counter clockwise orientation.)

(b) Evaluate this integral using Green's Theorem.

5. **(a)** Evaluate the line integral $\oint_C y\,dx$ where C is the boundary of the region $a \le x \le b$ and $0 \le y \le f(x)$ by writing it as the sum of four line integrals. (Use the counter clockwise orientation; assume that $f(x) > 0$ for all x.)

(b) Evaluate this integral using Green's Theorem.

6. Let R be a plane region whose boundary is a simple closed curve C and $\mathbf{F}$ be a continuously differentiable force field defined on R. A particle moves once around the curve C in the counter clockwise direction. Prove that the work done by $\mathbf{F}$ is

$$W = \iint_R (\nabla \times \mathbf{F}) \cdot d\mathbf{A}$$

where $d\mathbf{A} = dA\,\mathbf{k}$.

Answer: Suppose the force field is $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$, and that $\mathbf{R}$ is the position vector of a point on the curve. Then

$$dW = \mathbf{F} \cdot d\mathbf{R} = \big(M\,\mathbf{i} + N\,\mathbf{j}\big) \cdot \big(dx\,\mathbf{i} + dy\,\mathbf{j}\big) = M\,dx + N\,dy$$

is the work done in moving the particle from (x, y) to $(x + dx, y + dy)$. By Green's Theorem the total work is

$$\oint_C dW = \oint M\,dx + N\,dy \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA.$$

But since $M_z = N_z = 0$ we have that

$$\nabla \times \mathbf{F} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}$$

and hence that

$$\oint_C dW = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\,dA = \iint_R (\nabla \times \mathbf{F}) \cdot d\mathbf{A}$$

as claimed. □

7. Let R be a plane region whose boundary is a simple closed curve C and $\mathbf{F}$ be a continuously differentiable flow field defined on R. The flow field represents the flow of a fluid in the plane. The **flux** across the boundary is the rate at which fluid flows across the boundary. Define the flux across C as an integral over C and show that it is the integral of the divergence of $\mathbf{F}$ over R.

Answer: The amount of fluid which flows across an infinitesimal piece $d\mathbf{R}$ of the boundary in an infinitesimal time dt the signed area $\pm|\mathbf{F}\,dt \times d\mathbf{R}|$ of the infinitesimal parallelogram with edge vectors $\mathbf{F}\,dt$ and $d\mathbf{R}$; the flux (rate of flow) across this piece is thus $\pm|\mathbf{F} \times d\mathbf{R}|$. The sign in the signed area is positive if the fluid is flowing out and negative if the fluid is flowing in, and the outward unit normal vector $\mathbf{N}$ is obtained by rotating the unit tangent $\mathbf{T}$ clockwise through ninety degrees. Hence in either case the infinitesimal flux is

$$\pm|\mathbf{F} \times d\mathbf{R}| = \pm|\mathbf{F} \times d\mathbf{T}|\,ds = \mathbf{F} \cdot \mathbf{N}\,ds.$$

The total flux across the boundary is thus defined to be

$$\text{flux}_C(\mathbf{F}) = \int_C \mathbf{F} \cdot \mathbf{N}\,ds.$$

Now suppose that $\mathbf{F} = P(x,y)\,\mathbf{i} + Q(x,y)\mathbf{j}$. Since

$$\mathbf{T}\,ds = d\mathbf{R} = dx\,\mathbf{i} + dy\,\mathbf{j}$$

we have

$$\mathbf{N}\,ds = dy\,\mathbf{i} - dx\,\mathbf{j}$$

so

$$\mathbf{F} \cdot \mathbf{N}\,ds = -Q\,dx + P\,dy$$

and by Green's Theorem (with $M = -Q$ and $N = P$) we have

$$\text{flux}_C(\mathbf{F}) = \int_C -Q\,dx + P\,dy = \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\,dy = \iint_R (\nabla \cdot \mathbf{F})\,dA$$

as claimed. □

Remark 8. The formula in problem 6 can be written

$$\oint_C \mathbf{F} \cdot \mathbf{T}\,ds = \iint_R (\nabla \times \mathbf{F}) \cdot d\mathbf{A}.$$

The formula in problem 7 is

$$\oint_C \mathbf{F} \cdot \mathbf{N}\,ds = \iint_R (\nabla \cdot \mathbf{F})\,dA.$$

9. Let C be the triangle with vertices $(1,2)$, $(3,2)$, $(2,5)$. Evaluate the line integrals $\oint_C x\,dx - y\,dy$ and $\oint_C y\,dx - x\,dy$. Both integrals are to be traversed in the counterclockwise direction.

10. Use Green's Theorem to evaluate the integral

$$\oint_{\partial D} 2xy^2\,dx + 3x^2y^3\,dy$$

(taken in the counter clockwise direction) where D is the plane region given by

$$x^3 \leq y \leq x.$$

Answer:

$$\begin{aligned}
\oint_{\partial D} 2xy^2\,dx + 3x^2y^3\,dy &= \iint_D (6xy^3 - 4xy)\,dx\,dy \\
&= \int_{x=0}^{1}\int_{y=x^3}^{x} (6xy^3 - 4xy)\,dy\,dx \\
&= \int_{x=0}^{1} \left(\frac{2xy^4}{3} - 2xy^2\right)\Bigg|_{y=x^3}^{x} dx \\
&= \int_{x=0}^{1} \left(\frac{2x^5}{3} - 2x^3 - \frac{2x^{13}}{3} + 2x^7\right) dx
\end{aligned}$$

□

11. (a) Give parametric equations for the circle C with equation $(x-3)^2 + (y-4)^2 = 25$.

(b) Express the line integral

$$\oint_C (6y + x)\,dx + (y + 2x)\,dy$$

in terms of the parameterization given in part (a) and evaluate it. (The integral is taken in the counter clockwise direction.)

12. Let C be the semicircle comprised of the graph $y = \sqrt{1-x^2}$ and the line segment $y = 0$, $-1 < x < 1$. Let $\mathbf{F}$ be the vector field given by

$$\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j}.$$

Evaluate the integral $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds$. In the integral, the semicircle C is traversed in the counterclockwise direction, ds denotes arclength, and $\mathbf{T}$ denotes the unit tangent vector.

13. Let $\mathbf{F}$ be the force field given by

$$\mathbf{F}(x,y) = 3y\,\mathbf{i} + 2x\,\mathbf{j}.$$

Calculate the work done when moving a particle clockwise once around the boundary of the region $0 \le x \le \pi$, $0 \le y \le \sin x$. Find the flux of the field $\mathbf{F} = 2x\,\mathbf{i} - 3y\,\mathbf{j}$ outward across the ellipse with parametric equations

$$x = \cos t, \qquad y = 4\sin t, \qquad 0 \le t \le 2\pi.$$

14. (See Example 3 page 708 of [TF].)

15. Suppose that C is the boundary of a region R in the plane and that $\mathbf{T}$ and $\mathbf{n}$ are respectively the unit tangent and normal vectors to C so oriented that $\mathbf{n}$ points out and $(\mathbf{n}, \mathbf{T})$ is a positively oriented frame. Suppose that

$$\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}, \qquad \mathbf{G} = -N\,\mathbf{i} + M\,\mathbf{j}$$

are two vector fields.

(a) Show that $\displaystyle\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C \mathbf{G} \cdot \mathbf{T}\, ds$.

(b) Which integral represents work? Which represents flux?

(c) Using Green's theorem, express the integral in part (a) as an integral over R.

16. Use Green's Theorem to find the area enclosed by the ellipse given parametrically by

$$x = a\cos t, \qquad y = b\sin t, \qquad 0 \le t \le 2\pi.$$

(See Example 2 page 711 in [TF].)

17. Let C be the cardioid with polar equation

$$r = 1 - \cos\theta, \qquad 0 \le \theta \le 2\pi.$$

(a) Evaluate the integral

$$\oint_C x\, dy$$

using the definition of line integral. You may leave your answer in the form of a single definite integral.

(b) Evaluate the integral using Green's Theorem. A numerical answer is required here.

18.

26 The Gauss Divergence Theorem

1. State the the two dimensional Gauss Divergence Theorem.

Answer: See problem 7 of section 25. □

2. State the three dimensional Gauss Divergence Theorem.

Answer: Let B be a bounded region in three dimensional space whose boundary ∂B is a smooth surface. Let $\mathbf{n}$ be the unit outward normal vector field on ∂B and $\mathbf{F}$ be a continuously differentiable vector field defined ob B. Then

$$\iint_{\partial B} (\mathbf{F} \cdot \mathbf{n})\, d\sigma = \iiint_{B} (\nabla \cdot \mathbf{F})\, dV$$

where $d\sigma$ denotes the area element of the boundary ∂B and dV denotes the volume element of the region B. □

3. (a) Find the outward unit normal vector $\mathbf{n}$ to the ellipse C with equation $9x^2 + y^2 = 1$.

(b) Find the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$$

(taken with the counter clockwise orientation) where the vector field $\mathbf{F}$ is given by

$$\mathbf{F} = \mathbf{i} + 2xy\mathbf{j}.$$

(As usual, the ds is the integral indicates arclength.)

Answer: (a) Let $f(x, y) = 9x^2 + y^2$ so that the ellipse C is a level curve $f(x, y) = 1$ of f. The gradient

$$\nabla f = 18x\mathbf{i} + 2y\mathbf{j}$$

is normal to the ellipse at the point (x, y) and points out so we find $\mathbf{n}$ by normalizing:

$$\mathbf{n} = \frac{9x}{\sqrt{81x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{81x^2 + y^2}}.$$

(b) By the Divergence Theorem

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy$$

when $\partial R = C$ and $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$. Since $M = 1$ and $N = 2xy$ we get

$$\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0 + 2x = 2x$$

and the inside of the ellipse is given by $9x^2 + 2y^2 \leq 1$ or

$$-1 \leq y \leq 1, \qquad -a(y) \leq x \leq a(y)$$

where $a(y) = \sqrt{1 - y^2}/3$ so the integral over R, the inside of the ellipse, is

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy = \int_{y=-1}^{1} \int_{x=-a(y)}^{a(y)} 2x\, dx\, dy = \int_{y=-1}^{1} x^2 \Big|_{x=-a(y)}^{a(y)} dy = 0.$$

□

Remark 4. One can prove directly that the answer is zero by symmetry as follows:

$$\mathbf{F} \cdot \mathbf{n} = \frac{9x + 2xy^2}{\sqrt{9x^2 + y^2}}$$

This shows that

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \oint_C g(x, y)\, ds$$

where $g(-x, y) = -g(x, y)$. Now the ellipse is symmetric about the y-axis, so each element $g(x, y)\, ds$ in the integral is cancelled by an equal and opposite element $g(-x, y)\, ds$. Be careful when making an argument like this; for example, the symmetry $g(x, -y) = g(x, y)$ and symmetry of the ellipse about x-axis are irrelevant.

5. The semicircular region R in the xy-plane lies inside the circle $x^2 + y^2 = 1$ and above the line $y = x$. Find the outward flux across the boundary of R produced by the vector field

$$\mathbf{F} = xy\mathbf{i} + y\mathbf{j}.$$

27 Stokes Theorem

1. State the special case of Stokes' Theorem for plane regions.

Answer: See problem 6 of section 25. □

2. State Stokes' Theorem.

Answer: Let S be a smooth surface with boundary ∂S. Let $\mathbf{n}$ be a unit normal vector field on S and $\mathbf{F}$ be a continuously differentiable vector field defined on S. Then

$$\oint_{\partial S} (\mathbf{F} \cdot \mathbf{T})\, ds = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$$

where $\mathbf{T}$ is the unit tangent vector to the boundary ∂S, ds is the arclength element on the boundary, $d\sigma$ is the area element on the surface, and the direction in the line integral on the left is such that at any point on the boundary, the inward pointing vector, the unit tangent $\mathbf{T}$, and the unit normal $\mathbf{n}$ (in that order) form a right hand frame. □

3. Let S be the portion of the paraboloid $z = 4 - x^2 - y^2$ that lies above the plane $z = 0$ and

$$\mathbf{F} = (z - y)\,\mathbf{i} + (z + x)\,\mathbf{j} - (x + y)\,\mathbf{k}.$$

Calculate

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{R} \qquad \text{and} \qquad \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma.$$

(See Example 2 page 731 of [TF].)

4. Let $W = \rho^{-1}$ where $\rho = \sqrt{x^2 + y^2 + z^2}$ and let $\mathbf{F} = \nabla W$ be the gradient of W.

(a) Calculate the gradient $\mathbf{F}$ and the divergence of the gradient $\nabla \cdot \mathbf{F}$.

(b) Calculate the outward flux

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

over the sphere $\rho = h$ of radius h. Here $d\sigma$ denotes the area element on the sphere and $\mathbf{n}$ denotes the outward unit normal to the sphere.

(c) Calculate the outward flux

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

over the ellipsoid S defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Here $d\sigma$ denotes the area element on the ellipsoid and $\mathbf{n}$ denotes the outward unit normal to the ellipsoid.

5. Prove that if $\nabla \times \mathbf{F} = \mathbf{0}$ throughout a simply connected region D, then

$$\int_P^Q \mathbf{F} \cdot \mathbf{v} \, dt, \qquad \mathbf{v} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k}$$

has the same value for every curve $(x(t), y(t), z(t)$ in D with endpoints P and Q. Hint: The term *simply connected* means that given two curves from P to Q there is a surface in D whose boundary is the union of these curves.

6. True or false? Suppose that S is a portion of a level surface of the function $f(x, y, z)$ and that $\mathbf{n}$ is a unit normal to S. Then

$$\iint\limits_S \nabla f \cdot \mathbf{n} \, d\sigma = \pm \iint\limits_S |\nabla f| \, d\sigma.$$

7. True or false? Let $\mathbf{F} = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$ be a vector field defined for $(x, y) \neq (0, 0)$ and such that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. Then $\mathbf{F}$ is the gradient of a scalar function.

8. True or false? For any vector field $\mathbf{F} = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$ defined in the whole plane and any curve C the integral

$$\int_C \mathbf{F} \cdot \mathbf{v}\,dt, \qquad \mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$$

is independent of the choice of the parameterization $(x(t), y(t))$ of the curve C.

9. Calculate $\int_C \mathbf{F} \cdot d\mathbf{R}$ where and C is the unit circle

$$x = \cos t, \qquad y = \sin t, \qquad 0 \le t \le 2\pi$$

and

(a) $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$ **(b)** $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j}$ **(c)** $\mathbf{F} = (x - y)\,\mathbf{i} + (x + y)\,\mathbf{j}$.