Math 234, Midterm Exam, October 23, 2001

Answers

I. (30 points.) (1) Find the gradient $\nabla U = \operatorname{grad} U$ of the function

$$U = (x^2 + y^2 + z^2)^{-1/2}.$$

Answer: $U_x = -(x^2+y^2+z^2)^{-3/2}x$, $U_y = -(x^2+y^2+z^2)^{-3/2}y$, and $U_z = -(x^2+y^2+z^2)^{-3/2}z$, so

$$\nabla U = U_x\mathbf{i} + U_y\mathbf{j} + U_z\mathbf{k} = -\frac{x\mathbf{i}}{\rho^3} - \frac{y\mathbf{j}}{\rho^3} - \frac{z\mathbf{k}}{\rho^3}$$

where $\rho = (x^2+y^2+z^2)^{1/2}$.

(2) Find the total differential dU of this function.

Answer: $dU = U_x\,dx + U_y\,dy + U_z\,dz = -\dfrac{x\,dx}{\rho^3} - \dfrac{y\,dy}{\rho^3} - \dfrac{z\,dz}{\rho^3}$

II. (30 points.) A planet moves in space according to Newton's law $m\,d^2\mathbf{r}/dt^2 = -mGM|\mathbf{r}|^{-3}\mathbf{r}$ where $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ is the position vector of the planet. Let $\mathbf{v}$ be the velocity vector, $K = m|\mathbf{v}|^2/2$, and $W = -mGM/|\mathbf{r}|$. Show that the quantity $E = K + W$ is constant.

Answer: By the previous problem

$$\nabla W = -\nabla\left(\frac{mGM}{\rho}\right) = \frac{mGM}{\rho^3}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}) = \frac{mGM}{|\mathbf{r}|^3}\mathbf{r} = -\mathbf{F} = -m\mathbf{a}.$$

Now by the product rule for differentiation

$$\frac{dK}{dt} = \frac{m}{2}\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = \frac{m}{2}\left(\frac{d\mathbf{v}}{dt}\cdot\mathbf{v} + \mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right) = m\mathbf{a}\cdot\mathbf{v}$$

and by the chain rule

$$\frac{dW}{dt} = (\nabla W)\cdot\mathbf{v} = -m\mathbf{a}\cdot\mathbf{v}$$

so

$$\frac{dE}{dt} = \frac{dK}{dt} + \frac{dW}{dt} = m\mathbf{a}\cdot\mathbf{v} - m\mathbf{a}\cdot\mathbf{v} = 0.$$

III. (70 points.) The position vector $\mathbf{R}$ of a moving point is given by

$$\mathbf{R} = (2t+3)\mathbf{i} + (t^2-1)\mathbf{j}.$$

(This is example 3 on page 554 and problem 6 on page 564 of the text.)

(1) Find the velocity vector $\mathbf{v}$.

Answer: $\mathbf{v} = \dfrac{d\mathbf{R}}{dt} = 2\mathbf{i} + 2t\mathbf{j}.$

(2) Find the acceleration vector $\mathbf{a}$.

Answer: $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = 2\mathbf{j}.$

(3) Find the speed.

Answer: $\dfrac{ds}{dt} = |\mathbf{v}| = 2\sqrt{1+t^2}.$

(4) Find the unit tangent vector $\mathbf{T}$.

Answer: $\mathbf{T} = \mathbf{v}/|\mathbf{v}| = \dfrac{\mathbf{i}}{\sqrt{1+t^2}} + \dfrac{t\mathbf{j}}{\sqrt{1+t^2}}.$

(5) Find the unit normal vector $\mathbf{N}$.

Answer: The unit normal vector is perpendicular to the unit tangent vector so

$$\mathbf{N} = \pm\left(\frac{-t\mathbf{i}}{\sqrt{1+t^2}} + \frac{\mathbf{j}}{\sqrt{1+t^2}}\right)$$

This vector points to the concave side of the curve so that $\dfrac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$ where $\kappa > 0$. The curve is a parabola which opens up so the plus sign is correct.

(6) Find the normal and tangential components of the acceleration.

Answer: The tangential component is $\mathbf{a}\cdot\mathbf{T} = 2\mathbf{j}\cdot\mathbf{T} = \dfrac{2t}{\sqrt{1+t^2}}$. The normal component is $\mathbf{a}\cdot\mathbf{N} = 2\mathbf{j}\cdot\mathbf{N} = \dfrac{2}{\sqrt{1+t^2}}$. Hence $\mathbf{a} = \dfrac{2t}{\sqrt{1+t^2}}\mathbf{T} + \dfrac{2}{\sqrt{1+t^2}}\mathbf{N}.$

(7) Find the curvature.

Answer: From the formula $\mathbf{a} = \dfrac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\dfrac{ds}{dt}\right)^2\mathbf{N}$ we get that

$$\kappa = \frac{\mathbf{a}\cdot\mathbf{N}}{(ds/dt)^2} = \frac{2}{\sqrt{1+t^2}}\cdot\frac{1}{4(1+t^2)} = \frac{1}{2(1+t^2)^{3/2}}.$$

IV. (30 points.) Find the local maxima and local minima of the function

$$f(x, y) = x^2 - xy + y^2 + 2x + 2y - 4.$$

(This is the example on page 611 of the text.)

Answer: $f_x = 2x - y + 2$, $f_y = -y + 2y + 2$, so $f_x = f_y = 0$ if and only of $x = -2$, $y = -2$. For (x, y) near the point $(a, b) = (-2, -2)$ we have the quadratic approximation

$$Q(x, y) = -8 + (x + 2)^2 - (x + 2)(y + 2) + (y + 2)^2.$$

(In this example $f(x, y) = Q(x, y)$.) The discriminant $b^2 - 4ac = -3$ is negative so we have with a local maximum or a local minimum. When $y = -2$ we have $Q(x, y) = -8 + (x + 2)^2 \geq -8 = Q(-2, -2)$ so $(-2, -2)$ is a local minimum. (In fact it is a global minimum.) As $(-2, -2)$ is the only critical point there is no other local extremum.

V. (30 points.) The temperature T at any point (x, y, z) in space is

$$T = 400xyz^2.$$

Find the highest temperature on the unit sphere

$$x^2 + y^2 + z^2 = 1.$$

(This is problem 57 on page 643 of the text.)

Answer: At the maximum we have $\nabla T = \lambda \nabla (x^2 + y^2 + z^2 - 1)$ or

$$\begin{aligned} 400yz^2 &= \lambda 2x \\ 400xz^2 &= \lambda 2y \\ 4002xyz &= \lambda 2z \\ x^2 + y^2 + z^2 &= 1. \end{aligned}$$

If $x = 0$ or $y = 0$ or $z = 0$, then $T = 0$. But the function T takes positive values so we may assume that $x \neq 0$ and $y \neq 0$ and $z \neq 0$. Dividing the first two equations by the third gives

$$\frac{z}{2x} = \frac{x}{z}, \qquad \frac{z}{2y} = \frac{y}{z}$$

so $2x^2 = z^2$ and $2y^2 = z^2$. From the last equation we get $1 = x^2 + y^2 + z^2 = 2z^2$ so $z = \pm 1/\sqrt{2}$. The sign of x and y must be the same (otherwise $T < 0$) so the maximum occurs at one of the four points

$$(x, y, z) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{\sqrt{2}}\right).$$

The value of T at each of these four points is $400/8 = 50$.

VI. (30 points.) **(a)** Find an equation of the plane tangent to the surface

$$x^3 + xy^2 + y^3 + z^3 + 1 = 0$$

at the point $P_0(-2, 1, 2)$. (This is problem 10 on page 641 of the text.)

Answer: The point $P(x, y, z)$ lies on this plane if and only if the vector from P_0 to P is perpendicular to the gradient at P_0 of the defining function $F(x, y, z) = x^3 + xy^2 + y^3 + z^3 + 1$. But

$$\nabla F = (3x^2 + y^2)\mathbf{i} + (2xy + 3y^2)\mathbf{j} + 3z^2\mathbf{k}$$

so

$$\nabla F|_{(-2,1,2)} = 13\mathbf{i} - \mathbf{j} + 12\mathbf{k}$$

so the tangent plane is $13(x + 2) - (y - 1) + 12(z - 2) = 0$.

(b) Find equations of the straight line through $P_0(-2, 1, 2)$ perpendicular to the tangent plane found in part (a).

Answer: A point P lies on the line if and only if the vector from P_0 to P is a multiple of $t\nabla F$ of the gradient vector ∇F at P_0; hence the line has parametric equations $x = -2 + 13t$, $y = 1 - t$, $z = 2 + 12t$. Equations for the line are

$$\frac{x + 2}{13} = 1 - y = \frac{z - 2}{12}.$$

VII. (30 points.) **(a)** Find the linear function which best approximates

$$f(x, y) = \frac{1}{1 + x - y}$$

near the point $(x, y) = (2, 1)$. (This is example 3 on page 589 of the text.)

Answer: $L(x, y) = f(2, 1) + f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1) = \frac{1}{2} - \frac{x - 2}{4} + \frac{y - 1}{4}$.

(b) Find the tangent plane at the point $(x, y) = (2, 1)$ to the surface

$$z = \frac{1}{1 + x - y}$$

Answer: $z = L(x, y)$, i.e. $z = \frac{1}{2} - \frac{x - 2}{4} + \frac{y - 1}{4}$.

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There are  261 scores
grade      range       count      percent
   A     225...250     59         22.6%
  AB     215...224     16          6.1%
   B     200...214     30         11.5%
  BC     160...199     77         29.5%
   C     125...159     48         18.4%
   D     100...124     11          4.2%
   F       0... 99     20          7.7%
Mean score = 181.5. Mean grade = 2.61.
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