Kepler's Laws*

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- *Kepler's first law:* A planet moves in a plane in an ellipse with the sun at one focus.
- *Kepler's second law:* The position vector from the sun to a planet sweeps out area at a constant rate.
- *Kepler's third law:* The square of the period of a planet is proportional to the cube of its mean distance from the sun. (The mean distance is the average of the closest distance and the furthest distance. The period is the time required to go once around the sun.)

1. Let (x, y, z) be the position of a planet in space where x, y, and z are all function of time t. Assume the sun is at the origin (0,0,0). We define the position vector $\mathbf{r}$, the velocity vector $\mathbf{v}$, and the acceleration vector $\mathbf{a}$ by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \qquad \mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}, \qquad \mathbf{a} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}.$$

Newton's law of motion is

$$\mathbf{F} = m\mathbf{a}$$

where $\mathbf{F}$ is the force on the planet and m is the mass of the planet. Newton's inverse square law of gravity is

$$\mathbf{F} = -\frac{GMm}{|\mathbf{r}|^3}\mathbf{r}$$

*These notes are adapted from: Milnor, J: On the geometry of the Kepler problem. *Amer. Math. Monthly* **90** (1983) 353-365.

where G is a universal gravitational constant and M is the mass of the sun. (The inverse square law is so called because the magnitude $|\mathbf{F}| = GMm|\mathbf{r}|^{-2}$ of the force $\mathbf{F}$ is proportional to the reciprocal of the square of the distance $|\mathbf{r}|$ from the planet to the sun.) Newton's laws imply

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = -\frac{GM\mathbf{r}}{|\mathbf{r}|^3}. \tag{1}$$

Note that m cancelled. This means that the mass of the planet does not affect its motion. (We are assuming that the sun is motionless. In more advanced treatments, this assumption is not made.)

2. First we show that the planet moves in a plane. By the product rule for differentiation

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a} \tag{2}$$

By (1) and the fact that the cross product of parallel vectors is $\mathbf{0}$ the right hand side of (2) is $\mathbf{0}$. It follows that the vector

$$\mathbf{h} = \mathbf{r} \times \mathbf{v}$$

is constant. We conclude that both the position and velocity vector lie in a plane normal to $\mathbf{h}$. Choose coordinates so that this plane is the xy-plane. Then

$$\mathbf{h} = h\mathbf{k}$$

for some constant scaler h. (We have assumed $\mathbf{h} \neq \mathbf{0}$; in case $\mathbf{h} = \mathbf{0}$ it can be shown that the planet moves on a straight line.)

3. Since the planet moves in the xy plane we have

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} = r\cos(\theta)\mathbf{i} + r\sin(\theta)\mathbf{j} \tag{3}$$

where the polar coordinates r and θ are functions of t. The derivative of $\mathbf{r}$ is

$$\mathbf{v} = \left[\frac{dr}{dt}\cos(\theta) - r\sin(\theta)\frac{d\theta}{dt}\right]\mathbf{i} + \left[\frac{dr}{dt}\sin(\theta) + r\cos(\theta)\frac{d\theta}{dt}\right]\mathbf{j} \tag{4}$$

From $\mathbf{r} \times \mathbf{v} = h\mathbf{k}$ we get

$$r\cos(\theta)\left[\frac{dr}{dt}\sin(\theta) + r\cos(\theta)\frac{d\theta}{dt}\right] - r\sin(\theta)\left[\frac{dr}{dt}\cos(\theta) - r\sin(\theta)\frac{d\theta}{dt}\right] = h.$$

After multiplying out and simplifying this reduces to

$$r^2 \frac{d\theta}{dt} = h \tag{5}$$

The area A swept out from time t_0 to time t_1 by a curve in polar coordinates is

$$A = \frac{1}{2} \int_{t_0}^{t_1} r^2 \frac{d\theta}{dt}\, dt$$

so $A = h(t_1 - t_0)/2$ by (5). This is Kepler's second law.

Theorem 4 (Hamilton). *The velocity vector* $\mathbf{v}$ *moves on a circle.*

Proof. Since $r = |\mathbf{r}|$ equation (1) can be written

$$\frac{d\mathbf{v}}{dt} = -\frac{GM}{r^2}\Big[\cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}\Big]. \tag{6}$$

Divide (6) by (5) and use the chain rule:

$$\frac{d\mathbf{v}}{d\theta} = -\frac{GM}{h}\Big[\cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}\Big].$$

Now integrate to obtain

$$\mathbf{v} = \frac{GM}{h}\Big[-\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j}\Big] + \mathbf{c} \tag{7}$$

where $\mathbf{c}$ is the constant of integration. Hence $|\mathbf{v} - \mathbf{c}| = GM/h$ so $\mathbf{v}$ moves on a circle centered at $\mathbf{c}$ with radius GM/h. □

5. Now we can prove Kepler's first law. Choose coordinates (in the xy plane) so that $\mathbf{c}$ is parallel to $\mathbf{j}$ (and in the same direction) and let $e = (\mathbf{c}\cdot\mathbf{j})h/GM$ so that (7) takes the form

$$\mathbf{v} = \frac{GM}{h}\Big[-\sin(\theta)\mathbf{i} + (\cos(\theta) + e)\mathbf{j}\Big]. \tag{8}$$

The cross product of (3) and (8) is

$$h\mathbf{k} = \mathbf{h} = \mathbf{r} \times \mathbf{v} = \frac{GMr}{h}(1 + e\cos(\theta))\,\mathbf{k}$$

so

$$r = \frac{k}{1 + e\cos\theta} \tag{9}$$

where $k = h^2/GM$. Equation (9) is the polar equation of a conic section with eccentricity e.

6. Assume that the conic section (9) is an ellipse, i.e. that $e < 1$. The ellipse (9) has one focus at the origin and the other on the negative x-axis so the closest and farthest the planet comes to the sun are given by

$$r_{\min} = r\Big|_{\theta=0} = \frac{k}{1+e}, \qquad r_{\max} = r\Big|_{\theta=\pi} = \frac{k}{1-e}.$$

These are the values of $x = r\cos(\theta)$ where the $\mathbf{i}$ component dx/dt of $\mathbf{v}$ vanishes, i.e. by equation (7) where $\sin(\theta) = 0$. The quantity

$$a = \frac{r_{\min} + r_{\max}}{2} = \frac{k}{1-e^2} \tag{10}$$

is the major semi-axis of the ellipse. The minor semi-axis b can be found by maximizing y on the orbit. This maximum value of $y = r\sin(\theta)$ occurs when $dy/dt = 0$, i.e. when the $\mathbf{j}$ component of $\mathbf{v}$ vanishes, i.e. by equation (7) when $\cos(\theta) + e = 0$. Thus

$$b = y\Big|_{\theta=\cos^{-1}(-e)} = \frac{k\sin(\cos^{-1}(-e))}{1 + e\cos(\cos^{-1}(-e))} = \frac{k\sqrt{1-e^2}}{1-e^2} = \frac{k}{\sqrt{1-e^2}}. \tag{11}$$

The equation for the ellipse on rectangular coordinates is

$$\frac{(x+ea)^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{12}$$

7. We now prove Kepler's third law in the form

$$T^2 = \frac{4\pi^2}{GM}\left(\frac{r_{\min} + r_{\max}}{2}\right)^3 \tag{13}$$

where T is the period of the planet, i.e. the time it takes the planet to go around the sun one time. The area of the ellipse is πab so by the second law we get

$$\pi ab = \frac{1}{2}\int_0^{2\pi} r^2\,d\theta = \frac{1}{2}\int_0^T r^2\frac{d\theta}{dt}\,dt = \frac{1}{2}\int_0^T h\,dt = \frac{Th}{2} \tag{14}$$

Thus

$$T^2 = \left(\frac{2\pi ab}{h}\right)^2 = \frac{4\pi^2 k^4}{(1-e^2)^3 h^2} = \frac{4\pi^2 a^3 k}{h^2} = \frac{4\pi^2 a^3}{GM}. \tag{15}$$

The first equality in (15) comes from (14), the second from (10) and (11), the third from (10), and the fourth from the definition $k = h^2/GM$ of k given after equation (9). Equation (13) results by substituting (10) in (15).

8. Historical Remark. Kepler published his laws in 1609. Newton's *Principia* was published in 1687. The proof of Hamilton was published in 1846. Exercise 15 below asks you to prove Newton's inverse square law assuming (a) Kepler's first law (9) and (b) that the force on the planet is directed towards the sun. Perhaps this is how Newton discovered the inverse square law of gravitation.

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9. In the special case where the orbit of the planet is a circle Kepler's third law is much easier to prove. Show that $T^2 = (4\pi^2/GM)a^3$ under the assumption that

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{GM\mathbf{r}}{a^3}, \qquad \mathbf{r} = a\cos(\omega t)\mathbf{i} + a\sin(\omega t)\mathbf{j}$$

where a and ω are constants.

10. The proof of Kepler's second law did not use the full force of (1). Prove Kepler's second law under the hypothesis that

$$m\frac{d^2\mathbf{r}}{dt^2} = g\mathbf{r} \tag{16}$$

where $g = g(x, y, z)$ is any function. The proof in the text is the special case $g = -GMm|\mathbf{r}|^{-3}$.

11. The planet earth is 93 million miles from the sun and orbits the sun in one year. The planet Pluto takes 248 years to orbit the sun. How far is Pluto from the sun?

12. Halley's comet goes once around the sun every 77 years. Its closest approach is 53 million miles. What is its furthest distance from the sun? What is the maximum speed of the comet and what is the minimum speed?

13. Calling the quantity a defined in equation (10) the *mean distance* is a misnomer. Show that

$$\frac{1}{T}\int_0^T r\,dt = a(1 + e^2/2).$$

Hint: By equation (12) the ellipse has parametric equations

$$x = -ea + a\cos\phi, \qquad y = b\sin\phi.$$

Express the integral first in terms of θ and then in terms of ϕ.

14. Show that

$$\frac{1}{T}\int_0^T r^{-1}\,dt = a^{-1}.$$

15. Assume that the motion of a particle satisfies Kepler's first law and that the force is directed toward the origin; i.e. assume Equations (9) and (16). Show that $f = cr^{-2}$ (along the orbit) where c is a constant. (See Exercise 7 on page 566 of Thomas & Finney fifth edition.)

16. The quantity

$$W = -\frac{GMm}{|\mathbf{r}|}$$

is called the *potential energy* of the planet, the quantity

$$K = \frac{m|\mathbf{v}|^2}{2}$$

is called the *kinetic energy*, and the quantity

$$E = K + W = \frac{m|\mathbf{v}|^2}{2} - \frac{mGM}{|\mathbf{r}|}$$

is called the *energy*. Show that

$$E = \frac{m}{2}\left(\frac{GM}{h}\right)^2 (e^2 - 1).$$

Conclude that E is constant along solutions of (1) and that the orbit (9) is an ellipse, a parabola, or a hyperbola according as the energy E is negative, zero, or positive. Hint: Use (7). (9), and the definition of k. Also show that E is negative, zero, or positive according as the origin lies inside, on, or outside the velocity circle of Theorem 4.

17. Show that the force

$$\mathbf{F} = -\frac{GMm}{|\mathbf{r}|^3}\mathbf{r}$$

in the Kepler problem is the negative gradient

$$\mathbf{F} = -\operatorname{grad} W$$

of the potential energy. Conclude that

$$\frac{dE}{dt} = 0$$

along any solution of (1). This confirms the result of Exercise 16. (Do this exercise after you have learned partial differentiation.)

Answers to the Exercises

18. *Answer to Exercise 9.* The formula $\mathbf{r} = a\cos(\omega t)\mathbf{i} + a\sin(\omega t)\mathbf{j}$ says that the planet moves on a circle of radius a with angular velocity ω. Differentiating twice gives that the acceleration vector is $\mathbf{a} = -\omega^2\mathbf{r}$. From Newton's law the acceleration vector is $\mathbf{a} = -GM|\mathbf{r}|^{-3}\mathbf{r} = -GMa^{-3}\mathbf{r}$ so $\omega^2 = GMa^{-3}$. The period is the time required to go around once so $\omega T = 2\pi$ and hence $\omega^2 = 4\pi^2/T^2$. Equate the two formulas for ω^2 to get $T^2 = GMa^3/4\pi^2$ which is Kepler's third law.

19. *Answer to Exercise 10. Part I. Suppose that $m\mathbf{a} = g\mathbf{r}$; we show that $\mathbf{r}$ lies in a plane.* By the product rule

$$\frac{d}{dt}(\mathbf{r}\times\mathbf{v}) = \frac{d\mathbf{r}}{dt}\times\mathbf{v} + \mathbf{r}\times\frac{d\mathbf{v}}{dt} = \mathbf{v}\times\mathbf{v} + \mathbf{r}\times\mathbf{a} = \mathbf{v}\times\mathbf{v} + \frac{g}{m}\mathbf{r}\times\mathbf{r} = \mathbf{0}$$

as the cross product of a vector with itself is $\mathbf{0}$. Hence the vector $(\mathbf{r}\times\mathbf{v})$ is constant; we choose coordinates so that it is a multiple of $\mathbf{k}$ say

$$\mathbf{r}\times\mathbf{v} = h\mathbf{k}.$$

Since $\mathbf{r}\perp\mathbf{r}\times\mathbf{v}$ we conclude that (if $h\neq 0$) $\mathbf{r}\perp\mathbf{k}$, i.e. that $\mathbf{r}$ lies in the (x,y)-plane.

20. *Answer to Exercise 10. Part II. Suppose that $\mathbf{r}\times\mathbf{v} = h\mathbf{k}$ where h is constant so that $\mathbf{r}$ lies in the (x,y)-plane. We show that the radius vector $\mathbf{r}$ sweeps out area at a constant rate.* The radius vector $\mathbf{r}$ is given in polar coordinates by the formula

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} = r(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

where x, y, r, θ depend on t. Thus by the product rule for differentiation

$$\mathbf{v} = \frac{dr}{dt}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) + r\frac{d\theta}{dt}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})$$

The first term on the left is proportional to $\mathbf{r}$ and the second is perpendicular to $\mathbf{r}$ so

$$h\mathbf{k} = \mathbf{r}\times\mathbf{v} = r(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})\times r\frac{d\theta}{dt}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = r^2\frac{d\theta}{dt}\mathbf{k}.$$

so

$$h = r^2\frac{d\theta}{dt}.$$

The rate at which the planet sweeps out area is

$$\frac{dA}{dt} = \frac{r^2}{2}\frac{d\theta}{dt} = \frac{h}{2},$$

i.e. dA/dt is constant.

21. *Answer to Exercise 11.* The period of the earth is T_1 = one year and its mean distance to the sun is $a_1 = 93 \cdot 10^6$ miles. The period of Pluto is $T_2 = 248$ years; we are to find its mean distance a_2 to the sun. Using Kepler's third law $T_1^2 = 4\pi^2 a_1^3/(GM)$ and the data for the earth we can find the constant GM:

$$GM = \frac{4\pi^2 a_1^3}{T_1^2} = 4\pi^2 \cdot 93^3 \cdot 10^{18} \tag{17}$$

so using Kepler's law with the data for Pluto determines its mean distance

$$a_2 = \left(\frac{T_2^2 GM}{4\pi^2}\right)^{1/2} = 248^{2/3} \cdot 93 \cdot 10^6 \text{ miles.}$$

22. *Answer to Exercise 12.* As in the previous exercise the period of Halley's comet is $T = 77$ years; let a be its mean distance to the sun. Then

$$77^2 = T^2 = \frac{4\pi^2 a^3}{GM} = \frac{a^3}{93^3 \cdot 10^{18}}$$

so $a^3 = 77^2 \cdot 93^3 \cdot 10^{18}$ so $a = 77^{2/3} \cdot 93 \cdot 10^6$ miles. Now $a = (r_{\min} + r_{\max})/2$ so $r_{\max} = 2a - r_{\min} = (2 \cdot 77^{2/3} \cdot 93 - 53) \cdot 10^6$. By Equation (7) the speed is

$$|\mathbf{v}| = (GM/h)\sqrt{\sin^2\theta + (\cos(\theta) + e)^2} = (GM/h)\sqrt{1 + 2e\cos\theta + e^2} \tag{18}$$

and is largest when $\theta = 0$ and smallest when $\theta = \pi$. From the values $r_{\min}$ and $r_{\max}$ we can determine the constants e and k in Equation (9); we determined GM in Equation (17), and we can determine h from the definition $k = h^2/GM$ of k which appears after Equation (9). Specifically, $a = r_{\min} + ea$ so $e = 1 - (r_{\min}/a)$ and $k = a(1 - e^2)$ so the largest and smallest values of $|\mathbf{v}|$ are

$$\frac{GM\sqrt{1 \pm 2e + e^2}}{h} = \frac{GM(1 \pm e)}{h}$$

where $GM = 4\pi^2 \cdot 93^3 \cdot 10^{18}$, $e = 1 - (53/(77^{2/3} \cdot 93))$, and $k = 77^{2/3} \cdot 93 \cdot 10^6(1 - e^2)$.

23. *Answer to Exercises 13 and 14.* Exercise 13 asks us to evaluate the integral

$$I_n = \frac{1}{T}\int_0^T r^n \, dt$$

for $n = 1$; Exercise 14 asks us to evaluate this integral for $n = -1$. The planet lies on the ellipse

$$r = \frac{k}{1 + e\cos\theta}$$

where $k = h^2/GM$ and h is the constant in Kepler's second law $r^2 d\theta/dt = h$. Thus $dt/d\theta = r^2/h$ so

$$I_n = \frac{1}{hT}\int_0^{2\pi} r^{n+2}\, d\theta. \tag{19}$$

The semimajor axis $a = (r_{\min} + r_{\max})/2$ of the ellipse, the semiminor axis b, and the distance $2c$ between the foci are given by

$$a = \frac{k}{1-e^2}, \qquad b = \sqrt{a^2 - c^2}, \qquad c = ea.$$

In rectangular coordinates $x = r\cos\theta$ and $y = r\sin\theta$ the equation of the ellipse is

$$\frac{(x+c)^2}{a^2} + \frac{y^2}{b^2} = 1$$

so the ellipse has parametric equations

$$x = -c + a\cos\phi, \qquad y = b\sin\phi, \qquad 0 \le \phi \le 2\pi.$$

We calculate $d\theta/d\phi$. Differentiate the equation

$$-c + a\cos\phi = x = r\cos\theta = \frac{k\cos\theta}{1 + e\cos\theta}$$

to get

$$-a\sin\phi\frac{d\phi}{d\theta} = -\frac{k\sin\theta}{(1+e\cos\theta)^2} = -\frac{yr}{k}.$$

substitute $y = b\sin\phi$ and divide by $-a\sin\phi$ to get

$$\frac{d\phi}{d\theta} = \frac{br}{ak}$$

and hence from Equation (19) we get

$$I_n = \frac{1}{hT}\int_0^{2\pi} r^{n+2}\frac{d\theta}{d\phi}\, d\phi = \frac{ak}{hTb}\int_0^{2\pi} r^{n+1}\, d\phi. \tag{20}$$

We express the constant ak/hTb in terms of a and e using the formulas

$$a = \frac{k}{1-e^2}, \qquad T = \frac{2\pi a^{3/2}}{\sqrt{GM}}, \qquad k = \frac{h^2}{GM}.$$

Then $k = a(1-e^2)$ and $h = \sqrt{kGM} = \sqrt{a(1-e^2)GM}$ so

$$\frac{ak}{hTb} = a \cdot a(1-e^2) \cdot \frac{1}{\sqrt{a(1-e^2)GM}} \cdot \frac{\sqrt{GM}}{2\pi a^{3/2}} \cdot \frac{1}{a\sqrt{1-e^2}} = \frac{1}{2\pi a}$$

so Equation (20) simplifies to

$$I_n = \frac{1}{2\pi a}\int_0^{2\pi} r^{n+1}\,d\phi. \tag{21}$$

Taking $n = -1$ in Equation (21) gives

$$I_{-1} = \frac{1}{2\pi a}\int_0^{2\pi} d\phi = \frac{1}{a}$$

which completes Exercise 14. For $n = 1$ we use $r^2 = x^2 + y^2$, $a^2 = b^2 + c^2$, and $c = ea$ so from the parametric equations for the ellipse and the half angle formula we get

$$\begin{aligned} r^2 &= c^2 - 2ac\cos\phi + a^2\cos^2\phi + b^2\sin^2\phi \\ &= a^2(1 - 2e\cos\phi + e^2\cos^2\phi) \\ &= a^2\left(1 - 2e\cos\phi + \frac{e^2(1-\cos 2\phi)}{2}\right). \end{aligned}$$

Now $\int_0^{2\pi}\cos\phi\,d\phi = \int_0^{2\pi}\cos 2\phi\,d\phi = 0$ so by Equation (21) we have

$$I_1 = a\left(1 + \frac{e^2}{2}\right).$$

This completes Exercise 13.

24. *Answer to Exercise 15.* We assume that the force is directed toward the origin

$$m\mathbf{a} = m\frac{d^2\mathbf{r}}{dt} = g\mathbf{r} \tag{16}$$

and that the planet moves in an ellipse

$$r = \frac{k}{1 + e\cos\theta} \tag{9}$$

We are to prove that $|m\mathbf{a}|$ is proportional to $1/r^2$. We introduce the unit vector

$$\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$$

in the direction $\mathbf{r}$. Then $\mathbf{r} = r\mathbf{u}$ and the unit vector

$$\frac{d\mathbf{u}}{d\theta} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$$

is perpendicular to $\mathbf{u}$, and a calculation (see Thomas Finney) shows that

$$\mathbf{a} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{u} + \left(\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\frac{d\mathbf{u}}{d\theta}$$

From Equation (16) we get

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = gr \tag{22}$$

$$\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = 0. \tag{23}$$

As is shown in Thomas Finney, Equation (23) leads to another proof of Kepler's second law $r^2 d\theta/dt = h$. Differentiate Equation (9) and use (9) and Kepler's second law to get

$$\frac{dr}{dt} = \frac{ke\sin\theta}{(1+e\cos\theta)^2}\cdot\frac{d\theta}{dt} = \frac{k^2}{(1+e\cos\theta)^2}\frac{d\theta}{dt}\frac{e\sin\theta}{k} = r^2\frac{d\theta}{dt}\frac{e\sin\theta}{k} = \frac{he\sin\theta}{k}$$

and hence

$$\frac{d^2r}{dt^2} = \frac{he\cos\theta}{k}\cdot\frac{d\theta}{dt}. \tag{24}$$

By Kepler's second law again

$$r\left(\frac{d\theta}{dt}\right)^2 = \frac{r^2}{r}\left(\frac{d\theta}{dt}\right)^2 = \frac{h}{r}\cdot\frac{d\theta}{dt} = \frac{h(1+e\cos\theta)}{k}\cdot\frac{d\theta}{dt} \tag{25}$$

so subtracting (25) from (24) gives

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{h}{k}\cdot\frac{d\theta}{dt} = \frac{h^2}{kr^2}. \tag{26}$$

By (22) and (23) the left hand side of (26) is $|\mathbf{a}|$ so $|\mathbf{a}|$ is inversely proportional to r^2 as claimed.

25. *Answer to Exercise 16.* By Equation (8)

$$K = \frac{m|\mathbf{v}|^2}{2} = \frac{m}{2}\left(\frac{GM}{h}\right)^2(\sin^2\theta + (\cos\theta + e)^2) = \frac{m}{2}\left(\frac{GM}{h}\right)^2(1 + 2e\cos\theta + e^2).$$

Recall that $k = h^2/GM$ [see the definition after Equation (9)] so by the definition of W and k and Equation (9) we have

$$W = -\frac{mGM}{r} = -\frac{mGM(1+e\cos\theta)}{k} = -m\left(\frac{GM}{h}\right)^2(1+e\cos\theta)$$

so the total energy

$$E = K + W = \frac{m}{2}\left(\frac{GM}{h}\right)^2 (e^2 - 1)$$

is constant along the orbit. When the orbit is an ellipse (i.e. $e < 1$) the energy is negative and taking $\theta = 0, \pi/2, \pi, 3\pi/2$ in Equation (8) shows that $\mathbf{v}$ passes through all four quadrants so the origin lies inside the velocity circle.

26. *Answer to Exercise 17.* Where $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$ we have

$$\nabla r = \frac{\partial r}{\partial x}\,\mathbf{i} + \frac{\partial r}{\partial y}\,\mathbf{j} + \frac{\partial r}{\partial z}\,\mathbf{k} = \frac{\mathbf{r}}{r}$$

so by the chain rule

$$\nabla W = -\nabla\left(\frac{mGM}{r}\right) = \frac{mGM}{r^2}\nabla r = \frac{mGM}{r^3}\mathbf{r} = -\mathbf{F} = -m\mathbf{a}$$

so

$$\frac{dE}{dt} = \frac{dK}{dt} + \frac{dW}{dt} = m\mathbf{v}\cdot\mathbf{a} + \nabla W\cdot\mathbf{v} = m\mathbf{v}\cdot\mathbf{a} - m\mathbf{a}\cdot\mathbf{v} = 0.$$

Possible Examination Questions

27. A particle moves in space according to a central force law $m\,d^2\mathbf{r}/dt^2 = g\mathbf{r}$ where $g = g(x, y, z)$ is any function and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the position vector of the particle. Prove that the particle stays in a plane. **Answer:** Write what is in section 19 above.

28. Suppose that the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ of a particle in the (x, y)-plane satisfies $\mathbf{r}\times\mathbf{v} = h\mathbf{k}$ where h is constant and $\mathbf{v}$ is the velocity vector. Show that the radius vector $\mathbf{r}$ sweeps out area at a constant rate **Answer:** Write what is in section 20 above.

29. A planet moves in space according to Newton's law $m\,d^2\mathbf{r}/dt^2 = -mGM|\mathbf{r}|^{-3}\mathbf{r}$ where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the position vector of the planet. Let $\mathbf{v}$ be the velocity vector, $K = m|\mathbf{v}|^2/2$, and $W = -mGM/|\mathbf{r}|$. Show that the quantity $E = K + W$ is constant. **Answer:** Write what is in section 26 above.