Math 234, Final Exam, Tuesday December 18, 2001
Answers

(The page references below are to Thomas and Finney fifth edition.)

I. (40 points.) Find an equation for the plane that is tangent to the surface $z = x^2 - xy - y^2$ at the point $P_0(1, 1, -1)$.

Answer: The equation for the surface has form $z = f(x, y)$ so the answer is given by the linearization

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

i.e.

$$z = -1 + (x - 1) - 3(y - 1).$$

Alternatively the surface has equation $F(x, y, z) = 0$ where $F(x, y, z) = z - x^2 + xy + y^2$ so the equation for the tangent plane is

$$0 = \nabla F(x_0, y_0, z_0) \cdot \overrightarrow{P_0P} = -(x - 1) + 3(y - 1) + (z + 1).$$

(This is problem 3 on page 583.)

II. (50 points.) **(1)** Find the linear function $L(x, y)$ which best approximates the function

$$f(x, y) = \frac{1}{1 + x - y}$$

near the point $(2, 1)$.

Answer: This is the linearization

$$L(x, y) = f(2, 1) + f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1),$$

i.e.

$$L(x, y) = \frac{1}{2} - \frac{x - 2}{4} + \frac{y - 1}{4}.$$

(This is Example 3 on page 589.)

(2) Find the quadratic function $Q(x, y)$ which best approximates the function $f(x, y)$ of part (1) near the point $(2, 1)$.

Answer: This is the quadratic approximation

$$Q(x, y) = L(x, y) + \frac{f_{xx}(2, 1)(x - 2)^2 + 2f_{xy}(2, 1)(x - 2)(y - 1) + f_{yy}(2, 1)(y - 1)^2}{2},$$

i.e.

$$Q(x,y) = \frac{1}{2} - \frac{x-2}{4} + \frac{y-1}{4} + \frac{(x-2)^2}{8} - \frac{(x-2)(y-1)}{4} + \frac{(y-1)^2}{8}.$$

(This was explained in the lecture and in section 16-11.)

III. (50 points.) **(1)** Find the total differential dw of the function

$$w = e^{2x+3y}\cos 4z.$$

Answer: $dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz$ so

$$dw = 2e^{2x+3y}\cos 4z\,dx + 3e^{2x+3y}\cos 4z\,dy - 4e^{2x+3y}\sin 4z\,dz.$$

(2) Find the derivative $\frac{dw}{dt}$ of the function w of part (1) along the curve given by the parametric equations

$$x = \ln t, \qquad y = \ln(t^2+1), \qquad z = t.$$

Answer: $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$ so

$$\frac{dw}{dt} = 2e^{2\ln t+3\ln(t^2+1)}\cos 4t\,\frac{1}{t} + 3e^{2\ln t+3\ln(t^2+1)}\cos 4t\,\frac{2t}{t^2+1} - 4e^{2\ln t+3\ln(t^2+1)}\sin 4t.$$

Alternatively, along the curve we have

$$w = e^{2\ln t+3\ln(t^2+1)}\cos 4t = t^2(t^2+1)^3\cos 4t$$

so, by the product rule,

$$\frac{dw}{dt} = 2t(t^2+1)^3\cos 4t + t^2(t^2+1)^2 6t\cos 4t - 4t^2(t^2+1)^3\sin 4t.$$

(This is exercise 3 on page 604.)

IV. (60 points.) The unit tangent vector and curvature vector of a curve whose position vector is $\mathbf{R}$ are defined by

$$\mathbf{T} = \frac{d\mathbf{R}}{ds}, \qquad \kappa\mathbf{N} = \frac{d\mathbf{T}}{ds} = \frac{d^2\mathbf{R}}{ds^2}$$

where s is the arclength. The osculating circle to a curve at a point P on the curve is that circle through P which has the same unit tangent vector and curvature vector at P as the curve.

(1) Find $\mathbf{T}$ and $\kappa\mathbf{N}$ for the curve $y = e^x$. Express your answers as functions of the x coordinate. If either (or both) of your answers depend on the direction of parameterization so indicate by placing a $\pm$ in front of your answer.

Answer: $\mathbf{R} = x\mathbf{i} + e^x\mathbf{j}$ so

$$\mathbf{T} = \frac{d\mathbf{R}}{dx}\frac{dx}{ds} = \pm(\mathbf{i} + e^x\mathbf{j})(1 + e^{2x})^{-1/2}$$

and

$$\kappa\mathbf{N} = \frac{d\mathbf{T}}{dx}\frac{dx}{ds} = \left(e^x\mathbf{j}(1 + e^{2x})^{-1/2} - (\mathbf{i} + e^x\mathbf{j})(1 + e^{2x})^{-3/2}e^{2x}\right)(1 + e^{2x})^{-1/2}.$$

(2) Find an equation for the osculating circle to the curve of part (1) at the point $P(0, 1)$.

Answer: At the point $P(0, 1)$ we have

$$\mathbf{T} = \pm\frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}}, \qquad \kappa\mathbf{N} = \frac{\mathbf{j}}{2} - \frac{\mathbf{i} + \mathbf{j}}{4} = -\frac{\mathbf{i}}{4} + \frac{\mathbf{j}}{4}.$$

The osculating circle has curvature $\kappa = \frac{\sqrt{2}}{4}$ and hence radius $\frac{4}{\sqrt{2}}$ and thus has an equation of form $(x - a)^2 + (y - b)^2 = 8$. The center $Q(a, b)$ of this circle lies on the concave side of the curve $y = e^x$ and the line $\overline{PQ}$ from the center Q to the point P on the curve is perpendicular to the tangent line $y = x + 1$ to the curve at P. Hence the line $\overline{PQ}$ has slope $\dfrac{b - 1}{a - 0} = -1$ so $b = 1 - a$. Since the distance from Q to P is $\frac{4}{\sqrt{2}} = \sqrt{(a - 0)^2 + (b - 1)^2}$ we have $8 = a^2 + (b - 1)^2 = 2a^2$ so $a = -2$ and $b = 3$. Thus an equation for the circle is

$$(x + 2)^2 + (y - 3)^2 = 8.$$

(This is problem 13 on page 557.)

V. (50 points.) **(1)** If the vector field $\mathbf{v} = M\mathbf{i} + N\mathbf{j}$ represents the velocity vector field of a fluid in the plane then the flux across a curve C measures the rate at which the fluid flows across the curve. Find the flux of the field

$$\mathbf{v} = 2x\mathbf{i} - 3y\mathbf{j}$$

outward across the ellipse

$$16x^2 + y^2 = 16.$$

Answer: Let R denote the interior of the ellipse, $\mathbf{n}$ denote the unit outward normal to the ellipse, and ds denote the arclength element of the ellipse. Orient the ellipse counter clockwise. The unit tangent and the outward normal are

$$\mathbf{T} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}, \qquad \mathbf{n} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}.$$

By the Divergence Theorem the flux of $\mathbf{v} = M\mathbf{i} + N\mathbf{j}$ across the ellipse is

$$\oint_{\partial R} \mathbf{v} \cdot \mathbf{n}\, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA = \int_{x=-1}^{1} \int_{y=-\sqrt{16-16x^2}}^{\sqrt{16-16x^2}} (2-3)\, dy\, dx.$$

The value of the integral is -4π since the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab. (This is example 3 on page 708.)

(2) Find the work done by the force field

$$\mathbf{F} = 3y\mathbf{i} + 2x\mathbf{j},$$

in moving a particle once[1] counter clockwise around the ellipse in part (1). Hint: The vector field $\mathbf{v}$ arises from $\mathbf{F}$ by rotation through ninety degrees.

Answer: The force field is $\mathbf{F} = -N\mathbf{i} + M\mathbf{j}$ where $\mathbf{v} = M\mathbf{i} + N\mathbf{j}$ is the field in part (1). The work is

$$\oint_{\partial R} \mathbf{F} \cdot \mathbf{T}\, ds = \oint_{\partial R} \mathbf{v} \cdot \mathbf{n}\, ds$$

so the answer is the same as for part (1). (This is example 4 on page 709.)

[1] The wording of the question in Thomas Finney fifth would be "... when the point of application moves once ...".

VI. (50 points.) Find the area cut off the surface $y^2 + z^2 = 2x$ by the plane $x = 1$.

Answer: This is the same as the area of the surface $x^2 + y^2 = 2z$ cut off by the plane $z = 1$ and the area is

$$\iint_{x^2+y^2\le 2} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy = \iint_{x^2+y^2\le 2} \sqrt{1 + x^2 + y^2}\, dx\, dy.$$

In cylindrical coordinates the surface is $r^2 = 2z$ and $z \le 1$ when $r \le \sqrt{2}$ the area is

$$\iint_{x^2+y^2\le 2} \sqrt{1 + x^2 + y^2}\, dx\, dy = \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{1 + r^2}\, r\, dr\, d\theta.$$

The integral can be evaluated with the substitutions $u = 1 + r^2$, $du = 2r\, dr$ so $u = 1$ when $r = 0$ and $u = 3$ when $r = \sqrt{2}$ and

$$\int_0^{\sqrt{2}} \sqrt{1 + r^2}\, r\, dr = \frac{1}{2} \int_1^3 \sqrt{u}\, du = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big|_1^3 = \frac{\sqrt{27} - 1}{3}.$$

(This is problem 48 on page 678.)

VII. (50 points.) Let A, B, C be twice continuously differentiable functions of (x, y, z) and let $\mathbf{F}$ be the vector field defined by

$$\mathbf{F} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}.$$

The curl of $\mathbf{F}$ is denoted by $\nabla \times \mathbf{F}$ and the divergence of $\mathbf{F}$ is denoted by $\nabla \cdot \mathbf{F}$.

(1) Write formulas for the divergence and curl of the vector field $\mathbf{F}$ and show that the divergence of the curl is zero.

Answer:

$$\nabla \cdot \mathbf{F} = \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z},$$

$$\nabla \times \mathbf{F} = \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right)\mathbf{j} + \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)\mathbf{k},$$

so

$$\nabla \cdot (\nabla \times \mathbf{F}) = (C_y - B_z)_x + (A_z - C_x)_y + (B_x - A_y)_z = 0.$$

(This is problem 24 on page 702.)

(2) State Stokes' Theorem and the Divergence Theorem for the vector field $\mathbf{F}$. Be sure to define any notation you use.

Answer: For any three dimensional region R with boundary ∂R the Divergence Theorem (see page 722) says that

$$\iiint_R \nabla \cdot \mathbf{F}\, dV = \iint_{\partial R} (\mathbf{F} \cdot \mathbf{n})\, d\sigma$$

where dV is the volume element, $\mathbf{n}$ is the outward unit normal, and $d\sigma$ is the area element. For any surface S with boundary ∂S Stokes' Theorem (see page 729) says that

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_{\partial S} \mathbf{F} \cdot d\mathbf{R}$$

where $d\mathbf{R} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k} = \mathbf{T}\, ds$; here $\mathbf{T}$ is the unit tangent vector to the boundary $\mathbf{R}$. The relation between the direction of the unit normal $\mathbf{n}$ and the direction of the unit tangent $\mathbf{T}$ is such that if the normal is the thumb of your right hand, the forefinger of your right hand indicates the direction around the boundary.

VIII. (50 points.) Let $\mathbf{F}$ denote the force field in the Kepler problem, i.e.

$$\mathbf{F} = -\frac{x\mathbf{i}}{\rho^3} - \frac{y\mathbf{j}}{\rho^3} - \frac{z\mathbf{k}}{\rho^3}, \qquad \rho = \sqrt{x^2 + y^2 + z^2}.$$

(1) Is there a function W such $\nabla W = \mathbf{F}$? If so, find it; if not, say why not.

Answer: As we learned in our study of the Kepler problem, $\mathbf{F} = \nabla W$ where $W = \rho^{-1}$.

(2) Write an iterated integral (you need not evaluate it) for the outward flux

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

over the ellipsoid S defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Here $d\sigma$ denotes the area element on the ellipsoid and $\mathbf{n}$ denotes the outward unit normal to the ellipsoid. If you don't see how to do this, do the special case of the sphere ($a = b = c$) for partial credit.[2]

[2] In grading this question, it was sometimes difficult to decide if the student was doing the special case or the general case.

Answer: The problem is easy when $a = b = c$; then the outward unit normal is $\mathbf{n} = a^{-1}\mathbf{R}$ and $\mathbf{F} = -a^{-3}\mathbf{R} = -a^{-2}\mathbf{n}$ so $\mathbf{F}\cdot\mathbf{n} = -a^{-2}$. As this is constant, the integral is this constant times the area of the sphere, i.e.

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = -a^{-2}(4\pi a^2) = -4\pi.$$

For the general case, parameterize the ellipsoid by the equations

$$x = a\cos\theta\sin\phi, \qquad y = b\sin\theta\sin\phi, \qquad z = c\cos\phi.$$

Let $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector. Then

$$\frac{\partial \mathbf{R}}{\partial \phi} = a\cos\theta\cos\phi\mathbf{i} + b\sin\theta\cos\phi\mathbf{j} - c\sin\phi\mathbf{k}$$

and

$$\frac{\partial \mathbf{R}}{\partial \theta} = -a\sin\theta\sin\phi\mathbf{i} + b\cos\theta\sin\phi\mathbf{j}$$

so

$$\frac{\partial \mathbf{R}}{\partial \phi}\times\frac{\partial \mathbf{R}}{\partial \theta} = bc\sin\theta\sin^2\phi\mathbf{i} + ac\cos\theta\sin^2\phi\mathbf{j} + ab\cos\phi\sin\phi\mathbf{k}.$$

Now $\mathbf{n}\,d\sigma = \dfrac{\partial \mathbf{R}}{\partial \phi}\times\dfrac{\partial \mathbf{R}}{\partial \theta}d\phi\,d\theta$ and

$$\mathbf{F} = -\frac{a\cos\theta\sin\phi\mathbf{i}}{\rho^3} - \frac{b\sin\theta\sin\phi\mathbf{j}}{\rho^3} - \frac{c\cos\phi\mathbf{k}}{\rho^3},$$

where

$$\rho = \sqrt{x^2+y^2+z^2} = (a^2\cos^2\theta\sin^2\phi + b^2\sin^2\theta\sin^2\phi + c^2\cos^2\phi)^{1/2}$$

so

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = -abc\int_0^{2\pi}\int_0^{\pi}\frac{2\cos\theta\sin\theta\sin^3\phi + \cos^2\phi}{(a^2\cos^2\theta\sin^2\phi + b^2\sin^2\theta\sin^2\phi + c^2\cos^2\phi)^{3/2}}\,d\phi\,d\theta$$

Remark. Some of you may have learned in a course in electrostatics that the integral is 4π times the total charge inside. This is actually a consequence of the Divergence Theorem as follows. A direct calculation (see problem 11 on page 632) shows that for $W = \rho^{-1}$ we have

$$\nabla\cdot\mathbf{F} = \nabla\cdot\nabla W = \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} = 0.$$

Let R be the region outside the tiny sphere $\rho = h$ and inside the ellipsoid S. Then

$$0 = \iiint_R \nabla \cdot \mathbf{F}\, dV = \iint_{\partial R} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma - \iint_{\rho=h} \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

The integral over the sphere $\rho = h$ is -4π. If you used this method to answer the question you will get full credit provided you indicated that the divergence of $\mathbf{F}$ vanishes and that you are using the Divergence Theorem.

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There are  253 scores
grade      range       count      percent
   A     320...400     31         12.3%
  AB     300...319     15          5.9%
   B     230...299     65         25.7%
  BC     180...229     60         23.7%
   C     150...179     30         11.9%
   D     100...149     32         12.6%
   F       0... 99     20          7.9%
Mean score = 217.1. Mean grade = 2.42.
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The cutoffs for the toal score (out of 1000) were
A 865, AB 785, B 705, BC 603, C 500, D 385.