Math 234

What you should know on day one

August 28, 2001

1. You should be able to use general principles like

$$\text{Length} = \int ds, \qquad \text{Area} = \int dA, \qquad \text{Volume} = \int dV.$$

For example the length of the semi circle

$$x = \cos t, \qquad y = \sin t, \qquad 0 \le t \le \pi$$

is

$$L = \int \sqrt{(dx)^2 + (dy)^2} = \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^\pi 1\, dt = \pi$$

2. You should be able to derive a formula like

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

from more general formulas. For example, the last formula can be derived by taking $t = \theta$ in the formula

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt, \qquad x = r\cos\theta, \quad y = r\sin\theta.$$

3. You should be comfortable with the idea of approximating a function by a (Taylor) polynomial. The polynomial of degree n which best approximates the function $f(x)$ near $x = a$ is the Taylor polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

The linear approximation is the case $n = 1$ and is given by

$$L(x) = f(a) + f'(a)(x-a).$$

Thus $y = L(x)$ is the equation for the tangent line to $y = f(x)$ at the point $(x, y) = (a, f(a))$. The quadratic approximation is the case $n = 2$ and is given by

$$Q(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2.$$

If $f'(a) = 0$ and $f''(a) \neq 0$ then $f(x)$ has a (local) minimum (or maximum) at $x = a$ if and only if Q does.

4. You should be comfortable with differentials and simple differential equations.

5. You should be comfortable with the distinction between vectors and points. The notation $P(x, y, z)$ means that P is the point with coordinates x, y, z. The vector from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}.$$

A translation which moves P_1 to P_1' and P_2 toP_2' leaves the vector from P_1 to P_2 unchanged; i.e. $\overrightarrow{P_1P_2} = \overrightarrow{P_1'P_2'}$ if and only if P_1P_2 and $P_1'P_2'$ are opposite edges of a parallelogram. (In particular, if $\overrightarrow{P_1P_2} = \overrightarrow{P_1'P_2'}$, then the four points P_1, P_2, P_1', P_2' are coplanar.)

6. You should know the fundamental vector operations. Suppose c is a scalar (number) and

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \qquad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},$$

are vectors. There are five vector operations:

Vector Addition.	$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}$.
Scalar Product.	$c\mathbf{a} = ca_1\mathbf{i} + ca_2\mathbf{j} + ca_3\mathbf{k}$.
Dot Product.	$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.
Cross Product.	$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_2)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$.
Length.	$\lvert\mathbf{a}\rvert = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

The formula for the cross product can best be remembered with the determinant formula

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{vmatrix}.$$

7. You should be familiar with the geometric interpretations of the vector operations.

- To add $\mathbf{a}$ and $\mathbf{b}$ translate $\mathbf{b}$ so its tail is the tip of $\mathbf{a}$ and draw the vector for the tail of $\mathbf{a}$ to the tip of $\mathbf{b}$. In other words, $\overrightarrow{P_1P_2} + \overrightarrow{P_2P_3} = \overrightarrow{P_1P_3}$.

- The vector $c\mathbf{a}$ points in the same direction as $\mathbf{a}$ if $c > 0$ and in the opposite direction if $c < 0$. Its length is $|c|$ times the length of $\mathbf{a}$, i.e. $|c\mathbf{a}| = |c|\,|\mathbf{a}|$.

- If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\cos\theta$$

  and

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta.$$

- $\mathbf{a} \times \mathbf{b} \perp \mathbf{a}$, $\mathbf{a} \times \mathbf{b} \perp \mathbf{b}$, and $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

- $|\mathbf{a}|$ is the distance from the tail of $\mathbf{a}$ to the tip of $\mathbf{a}$; i.e. $d(P_1, P_2) = |\overrightarrow{P_1P_2}|$ is the distance from the point P_1 to the point P_2.

8. You should know how to use the geometric interpretation ro guide calculations. For example,

- The point P lies on the line parallel to the vector $\mathbf{L}$ through the point P_0 if and only if $\overrightarrow{P_0P} = t\mathbf{L}$ for some number t. In particular the vector equation $\overrightarrow{P_0P} = t\overrightarrow{P_0P_1}$ gives parametric equations for the line through the points P_0 and P_1.

- Thus the line through the points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ has the parametric equations

$$x = x_0 + t(x_1 - x_0), \qquad y = y_0 + t(y_1 - y_0), \qquad z = z_0 + t(z_1 - z_0),$$

  i.e. a point $P(x, y, z)$ lies on this line if and only if its coordinates satisfy these parametric equations for some value of t. Eliminating t gives the "symmetric equations" for the line:

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}.$$

- The point P lies on the plane perpendicular to the vector $\mathbf{N}$ through the point P_0 if and only if $\overrightarrow{P_0P} \perp \mathbf{N}$, i.e. $\mathbf{N} \cdot \overrightarrow{P_0P} = 0$. Thus the point $P(x, y, x)$ lies on plane perpendicular to $\mathbf{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and through $P_0(x_0, y_0, z_0)$ if and only if its coordinates satisfy the equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

- If the angle between $\mathbf{a}$ and $\mathbf{b}$ is θ, the length of the projection of $\mathbf{b}$ along $\mathbf{a}$ is $|\mathbf{b}|\cos\theta$ and the area of the parallelogram with edges $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a}|\,|\mathbf{b}|\,\sin\theta$, The area of a triangle is one half the area of the corresponding parallelogram, and the volume of a parallelopiped is the area of the base time the altitude. (The altitude is the length of the projection of the third edge along the perpendicular to the first two.)

9. You should be familiar with parametric representation of a curve. When the point P is a function of time (or some other parameter) t the equations

$$x = x(t), \qquad y = y(t), \qquad z = z(t),$$

are parametric equations for the curve. The position vector for the point $P(x, y, z)$ is

$$\mathbf{r} = \overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

where $O(0, 0, 0)$ is the origin. The *velocity vector* of the parametric curve is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}.$$

The *acceleration vector* is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}.$$

10. You should have seen conic sections. (We will meet conic sections in 234: a planet moves in an ellipse with the sun at a focus.) If F_1 and F_2 are given points (in a plane) and $2a > d(F_1, F_2)$, then the set of all points P such that

$$d(P, F_1) + d(P, F_2) = 2a$$

is the ellipse with foci F_1 and F_2 and major axis $2a$. If F is a point, L is a line, and $0 < e < 1$, then the set of all points P such that

$$d(P, F) = e\, d(P, L)$$

is the ellipse with eccentricity e, directrix L, and corresponding focus at F.

1 Practice Questions

11. The following question involve the three points

$$A(-1, 0, 2), \qquad B(2, 1, -1), \qquad C(1, -2, 2).$$

(i) Find the angle ABC.

Answer: The angle θ is given by the formula

$$\cos\theta = \left(\frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\left|\overrightarrow{BA}\right| \; \left|\overrightarrow{BC}\right|} \right)$$

Now

$$\begin{aligned}
\overrightarrow{BA} &= (-1-2)\mathbf{i} + (0-1)\mathbf{j} + (2-2)\mathbf{k} &&= -3\mathbf{i} - \mathbf{j} \\
\overrightarrow{BC} &= (1-2)\mathbf{i} + (-2-1)\mathbf{j} + (2-(-1))\mathbf{k} &&= -\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}
\end{aligned}$$

so

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = (-3)(-1) + (-1)(-3) + (0)(3) = 6,$$

$$\left|\overrightarrow{BA}\right| = \sqrt{9+1+0} = \sqrt{10}, \qquad \left|\overrightarrow{BC}\right| = \sqrt{1+9+9} = \sqrt{19}$$

and $\theta = \cos^{-1}(6/\sqrt{190})$.

(ii) Find an equation for the plane containing the three points A, B, and C.

Answer: The vector

$$\mathbf{N} = \overrightarrow{BA} \times \overrightarrow{BC},$$

is perpendicular to the plane and B is a point on the plane so the point $P(x, y, z)$ lies on the plane if and only if the vector $\overrightarrow{BP}$ is perpendicular to the vector $\mathbf{N}$ so the equation of the plane is

$$\mathbf{N} \cdot \overrightarrow{BP} = 0.$$

Now

$$\mathbf{N} = \begin{vmatrix} -3 & -1 & 0 \\ -1 & -3 & 3 \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{vmatrix} = -3\mathbf{i} + 9\mathbf{j} + 8\mathbf{k}$$

and $\overrightarrow{BP} = (x-2)\mathbf{i} + (y-1)\mathbf{j} + (z+1)\mathbf{k}$ so the equation is

$$-3(x-2) + 9(y-1) + 8(z+1) = 0.$$

(iii) Find the area of the triangle ABC.

Answer: The length of $\mathbf{N}$ is the area of the parallelogram with edges $\overline{BA}$ and $\overline{BC}$ so the area of the parallelogram is half this. The area is $\frac{1}{2}\sqrt{9+81+64}$.

12. Find the distance from the origin to (the nearest point on) the line

$$\frac{x-2}{3} = \frac{y-1}{4} = \frac{2-z}{5}.$$

Answer: There are several possible approaches to the problem of finding the distance from the point $P_1(x_1, y_1, z_1)$ to the line L whose equation is

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}. \qquad (L)$$

(i) The line has parametric equations

$$x = at + x_0, \qquad y = bt + y_0, \qquad z = ct + z_0,$$

and the distance from P_1 to $P(x, y, z)$ is

$$r(t) = \sqrt{(at + x_0 - x_1)^2 + (bt + y_0 - y_1)^2 + (ct + z_0 - z_1)^2}.$$

The minimum value of the function r is the distance from P_1 to L. To find this solve the equation $r'(t) = 0$ for t and plug the answer into $r(t)$.

(ii) The vector
$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$
is parallel to the line. Let $P(x, y, z)$ be the closest point of the line L to the point P_1. Then the vectors $\overrightarrow{P_1P}$ and $\mathbf{v}$ are orthogonal so
$$0 = \overrightarrow{P_1P} \cdot \mathbf{v} = a(x - x_1) + b(y - y_1) + c(z - z_1). \qquad (*)$$
The two equations (L) and the equation $(*)$ give three equations in three unknowns x, y, z. Solve to find P. The distance from P_1 to L is $|\overrightarrow{P_1P}|$.

(iii) Pick a point P_0 on the line; for example, the point $P_0(x_0, y_0, z_0)$ and let P be the closest point of the line L to the point P_1 as in (ii). Then P_1PP_0 is a right triangle so
$$|\overrightarrow{P_1P_0}|^2 = |\overrightarrow{P_1P}|^2 + |\overrightarrow{PP_0}|^2.$$
Then $|\overrightarrow{P_1P_0}|^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2$,
$$|\overrightarrow{PP_0}| = \frac{\overrightarrow{PP_0} \cdot \mathbf{v}}{|\mathbf{v}|},$$
and the distance from P_1 to L is $|\overrightarrow{P_1P}|$.

(iv) The vector $\mathbf{M} = \mathbf{v} \times \overrightarrow{P_1P_0}$ is orthogonal to the plane containing the line and the point P_1 so the vector $\mathbf{N} = \mathbf{v} \times \mathbf{M}$ lies in this plane and is orthogonal to $\mathbf{v}$ and hence the line. The distance from P_1 to the line is the length of the projection of $\overrightarrow{P_1P_0}$ on $\mathbf{N}$; i.e. the distance from P_1 to L is
$$|\overrightarrow{P_1P}| = \frac{\mathbf{N} \cdot \overrightarrow{P_1P_0}}{|\mathbf{N}|}.$$

(v) Let P_0 and Q be two points on the line, e.g. $P_0(x_0, y_0, z_0)$ and $Q(x_0 + a, y_0, +b, z_0 + c)$. The area of the parallelogram with vertices P_0, P_1, and Q is $|\overrightarrow{P_0P_1} \times \overrightarrow{P_0Q}|$. But this area is also the length of the base $|\overrightarrow{P_0Q}|$ times the height and the height is the distance from P_1 to the line. Hence the distance from P_1 to L is
$$\frac{|\overrightarrow{P_0P_1} \times \overrightarrow{P_0Q}|}{|\overrightarrow{P_0Q}|}.$$

13. The following questions concern the four points
$$P_1(1, 2, 3), \quad P_2(4, 1, 1), \qquad Q_1(2, 3, 4), \quad Q_2(1, 5, 1).$$

(i) Find parametric equations for the line P_1P_2 through P_1 and P_2, and for the line Q_1Q_2 through Q_1 and Q_2.

(ii) Find a unit vector $\mathbf{u}$ perpendicular to both the lines P_1P_2 and Q_1Q_2.

(iii) Find the distance $d(P_1P_2, Q_1Q_2)$ between the lines P_1P_2 and Q_1Q_2.

(iv) Find an equation for the plane A which contains the line P_1P_2 and is perpendicular to $\mathbf{u}$. Find an equation for the plane B which contains the line Q_1Q_2 and is perpendicular to $\mathbf{u}$.

(v) The planes A and B are parallel. Find the distance between them.

(vi) Find an equation for the plane M which contains the line P_1P_2 and is parallel to $\mathbf{u}$. Find an equation for the plane N which contains the line Q_1Q_2 and is parallel to $\mathbf{u}$. (The planes M and N intersect.)

(vii) Find the point P on the line P_1P_2 which is closest to the line Q_1Q_2. Find the point Q on the line Q_1Q_2 which is closest to the line P_1P_2.

(viii) Find the distance $d(P, Q)$ from the point P to the point Q.

(ix) How to the parts of this question relate to one another?

14. The motion of a particle in the plane is governed by the equations

$$\frac{dx}{dt} = x, \qquad \frac{dy}{dt} = -x^2.$$

(i) Find parametric equations for its coordinates (x, y) at time t if $(x, y) = (1, -4)$ when $t = 0$. (Hint: Find x first.)

Answer: We solve the equation $dx/dt = x$ by separation of variables:

$$\frac{dx}{x} = dt \implies \ln(x) = t + C \implies x = e^{t+C}.$$

When $t = 0$ we have $x = 1$ so $1 = e^{0+C}$ so $C = 0$ so

$$x = e^t.$$

Thus $\dfrac{dy}{dt} = -x^2 = -e^{2t}$ so $y = -e^{2t}/2 + C_1$. When $t = 0$ we have $y = -4$ so $-4 = -e^0/2 + C_1$ so $C_1 = -7/2$

$$y = -\frac{e^{2t} + 7}{2}.$$

(ii) Find the velocity vector $\mathbf{v}$ and the acceleration vector $\mathbf{a}$.

Answer:

$$\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} = x\mathbf{i} - x^2\mathbf{j} = e^t\mathbf{i} - e^{2t}\mathbf{j}$$

and

$$\mathbf{a} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} = e^t\mathbf{i} - 2e^{2t}\mathbf{j}.$$

15. Find the coefficients a_k in the Maclaurin series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ for the function $f(x) = (1+x)^{2/3}$.

Answer: The general formula is Taylor's series

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)(x-a)^k}{k!}.$$

Maclaurin series arises by taking $a = 0$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)x^k}{k!}.$$

In case $f(x) = (1+x)^p$ this becomes the binomial formula

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k.$$

where

$$\binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}$$

so

$$a_k = \binom{\frac{2}{3}}{k} = \frac{\frac{2}{3}(\frac{2}{3}-1)(\frac{2}{3}-2)\cdots(\frac{2}{3}-k+1)}{k!}.$$