Exams from Calculus 223

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Warning: Some of this material may not be included in Math 234 for 2001.

Calculus 223 First Exam Thursday September 29, 1994

I Find an equation for the tangent plane to the surface

$$x^2 - y^2 + z^2 = 4$$

at the point $(2, -3, 3)$.

II (a) Find the unique critical point of the function

$$f(x, y) = x^2 + 3xy + 2y^2 - 8x - 11y + 30.$$

(b) Is this critical point a minimum, maximum, or saddle?

(c) Does the function $f(x, y)$ take negative values? (I.e. is there a point (x, y) where $f(x, y) < 0$?). *Justify your answer.*

III Find $\left(\frac{\partial w}{\partial y}\right)_x$ at $(w, x, y, z) = (0, 1, 2, 3)$ if

$$4x + 5y + 6z = 32 + w \qquad \text{and} \qquad 7x^2 + 8y^2 + 9z^2 = w + 120e^w.$$

IV Find the polynomial of degree 2 which best approximates

$$f(x, y) = \sin(xy)$$

near $(x, y) = (1, \pi)$.

V Find the absolute maximum and the absolute minimum of

$$f(x, y) = (x - 1)(y - 2)$$

in the closed triangle $0 \le x$, $0 \le y$, $x + y \le 7$ bounded by the x-axis, the y-axis, and the line $x + y = 7$.

VI Find the point on the ellipse $2x^2 + 3y^2 = 11$ where the function $f(x, y) = 8x - 6y$ achieves its maximum.

VI[1] Let $T = f(x, y)$ be the temperature at the point (x, y) on the circle

$$x = \cos\theta, \qquad y = \sin\theta,$$

and suppose that

$$\frac{\partial T}{\partial x} = 2x - y, \qquad \frac{\partial T}{\partial y} = 2y - x.$$

Find where the maximum temperature on the circle occurs. *Be sure not to specify where the maximum occurs and not the minimum. Be sure to justify your answer.*

Calculus 223 Second Exam Tuesday November 1, 1994

I Evaluate $\displaystyle\int_{x=0}^{1}\int_{y=2x}^{1}\int_{z=x^3+y}^{x^2+2y} y\,dz\,dy\,dx.$

II The force at the point (x, y) is

$$\mathbf{F}(x, y) = x^2 y\mathbf{i} + 2xy^2\mathbf{j}.$$

Find the work

$$W = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$$

done in moving a particle from $(0, 0)$ to $(2, 4)$ along the curve $y = x^2$.

III Evaluate the integral $\displaystyle\int\int_R x\,dx\,dy$ where R is the triangle with vertices $(1, 2)$, $(3, 3)$, $(4, 5)$.

IV The transformation $(x, y) = T(u, v)$ is given by

$$x = 1 + 2v + u^2, \qquad y = \frac{u}{3}.$$

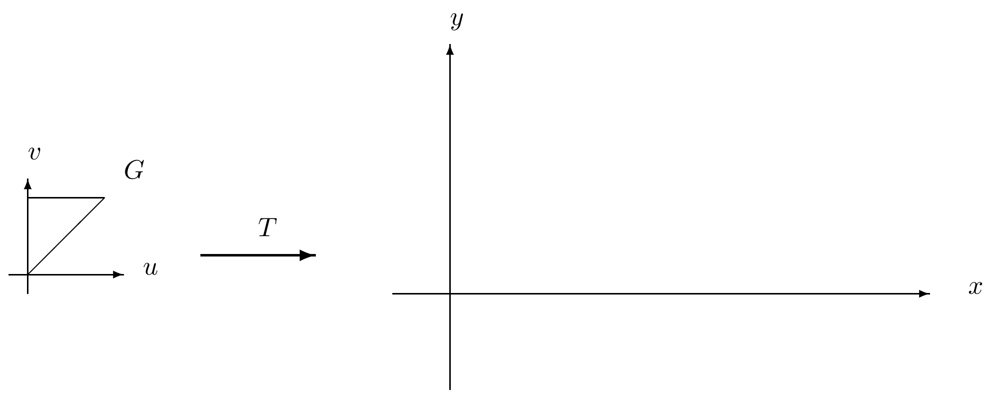
It transforms the triangle

$$G: \qquad 0 \le u \le v \le 1$$

in (u, v) space to a region R in (x, y) space. The region R is bounded by three curves.

[1] This problem was not included on the exam.

(a) Sketch the region R on the axes provided. Indicate clearly which points of the boundary of R correspond to the vertices $(0,0)$, $(0,1)$, $(1,1)$ of the triangle G. Find equations in (x,y) for each of the boundary curves. (Indicate clearly which boundary curve of R corresponds to which equation.)



(b) Evaluate $\displaystyle\int\int_R (x+y^2)\,dx\,dy$.

V Let C be the cardioid with polar equation

$$r = 1 - \cos\theta, \qquad 0 \le \theta \le 2\pi.$$

(a) Evaluate the integral

$$\oint_C x\,dy$$

using the definition of line integral. You may leave your answer in the form of a single definite integral.

(b) Evaluate the integral using Green's theorem. A numerical answer is required here.

VI The gravitational potential at the point $(0,0,z_0)$ due to a uniform mass distribution in the spherical shell

$$R: \qquad a \le \sqrt{x^2+y^2+z^2} \le b$$

is given by the triple integral

$$\int\int\int_R \frac{dx\,dy\,dz}{f(x,y,z)}$$

where $f(x,y,z)$ is the distance from the point (x,y,z) to the point $(0,0,z_0)$.

(a) Show that $f(x,y,z) = \sqrt{\rho^2 + z_0^2 - 2\rho z_0 \cos\phi}$.

(b) Evaluate the potential as a function of z_0. Hint: $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$. *Leave your answer as a definite integral in ρ. You will be asked to evaluate this integral on the next page.*

(c) Evaluate the integral in (b) when $z_0 > b$

(d) Evaluate the integral in (b) when $0 < z_0 < a$.

Some questions from the Final 223 Exam in 1994

I Find an equation for the tangent plane to the surface $z = 1 + x^2 + y^3$ at the point $(x, y, z) = (2, 1, 6)$.

II Find a function $w = f(x, y)$ whose first partials are

$$\frac{\partial w}{\partial x} = 1 + e^x \cos y, \qquad \frac{\partial w}{\partial y} = 2y - e^x \sin y$$

or prove that there is no such function.

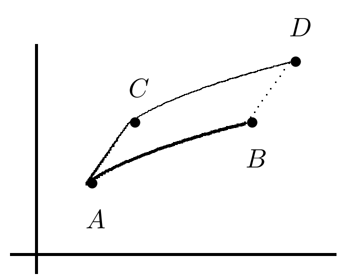
III Let C be the triangle with vertices $(1, 2)$, $(3, 2)$, $(2, 5)$. Evaluate the line integrals $\oint_C x\,dx - y\,dy$ and $\oint_C y\,dx - x\,dy$. Both integrals are to be traversed in the counterclockwise direction.

IV The transformation $(x, y) = T(u, v)$ is defined by

$$x = e^{2u} + e^v, \qquad y = e^u + e^v$$

carries the unit square $0 \le u \le 1$, $0 \le v \le 1$ in the (u, v) plane to a region R in the (x, y) plane shown in the diagram.

(1) Complete the table to give the coordinates of the vertices of R.



P	(x, y)
A	
B	
C	
D	

(2) Which of the four sides of R are straight line segments? (Circle one.)

none   all AB and CD AC and BD

(3) Find the area of R.

A question on the Divergence Theorem

Let $W = \rho^{-1}$ where $\rho = \sqrt{x^2 + y^2 + z^2}$.

(1) Calculate the gradient ∇W and the divergence of the gradient $\nabla \cdot \nabla W$,

(2) Calculate the outward flux

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

over the sphere $\rho = h$ of radius h. Here $d\sigma$ denotes the area element on the sphere and $\mathbf{n}$ denotes the outward unit normal to the sphere.

(3) Calculate the outward flux

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

over the ellipsoid S defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Here $d\sigma$ denotes the area element on the ellipsoid and $\mathbf{n}$ denotes the outward unit normal to the ellipsoid.