Taylor's Formula

(The Extended Mean Value Theorem)

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§1 When f is a function and $k \geq 0$ is an integer the notation $f^{(k)}$ denotes kth derivative of f. Thus

$$f^{(0)}(x) = f(x), \qquad f^{(1)}(x) = f'(x), \qquad f^{(2)}(x) = f''(x),$$

and so on. Given a number a in the domain of f and an integer $n \geq 0$, the polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!} \qquad (\#)$$

is called the **degree** n **Taylor polynomial of** f **centered at** a. The Taylor polynomial $P_n(x)$ is the unique polynomial of degree n which has the same derivatives as f at a up to order n:

$$P_n^{(k)}(a) = f^{(k)}(a) \qquad \text{for } k = 0, 1, 2, \ldots, n.$$

§2 The letter $\sum$ is the Greek S (for *sum*) and is pronounced *sigma* so the notation used in (#) is called **sigma notation**. It is a handy notation but if you don't like it you can indicate the summation with dots:

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_{n-1} + a_n.$$

Hence the first few Taylor polynomials are

$$P_0(x) = f(a),$$

$$P_1(x) = f(a) + f'(a)(x-a),$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2},$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2} + \frac{f'''(a)(x-a)^3}{6}.$$

§3 The Taylor polynomial $P_n(x)$ for $f(x)$ centered at a is the polynomial of degree n which best approximates $f(x)$ for x near a. The precise statement is

Taylor's Theorem. *Suppose that f is $n+1$ times differentiable and that $f^{(n+1)}$ is continuous. Let a be a point in the domain of f. Then*

$$\lim_{x\to a} \frac{f(x) - P_n(x)}{(x-a)^n} = 0. \qquad (\heartsuit)$$

§4 In order to use Taylor's formula approximate a function f we pick a point a where the value of f and of its derivatives is known exactly. Then the Taylor polynomial $P_n(x)$ can be evaluated exactly for any x. We then need to "estimate the error" $f(x) - P_n(x)$, i.e. to find an inequality

$$|f(x) - P_n(x))| \le M|x-a|^{n+1}$$

which tells us how small the error $f(x) - P_n(x)$ is, i.e. how close $P_n(x)$ is to $f(x)$. A value for M is usually found via

§5 The Extended Mean Value Theorem. *Suppose that f is $n+1$ times differentiable and that $f^{(n+1)}$ is continuous on an interval, let a and b be two numbers in that interval, and let $P(x)$ be the Taylor polynomial of f centered at a. Let b be a point in the domain of f. Then for each b there is a number c_{n+1} between a and b such that*

$$f(b) - P_n(b) = \frac{f^{(n+1)}(c_{n+1})(b-a)^{n+1}}{(n+1)!}.$$

§6 Note that the formula for the error $f(b) - P_n(b)$ is the same as the next term in the series (#) except that the $n+1$st derivative $f^{(n+1)}$ is evaluated at the unknown point c_{n+1} instead of a. The Extended Mean Value Theorem is proved in problem 74 on page 174 of the text. Equation ($\heartsuit$) is an immediate consequence.

§7 Exercise. Evaluate $\sum_{k=3}^{5} \frac{1}{k}$.

§8 Exercise. Let $f(x) = \sqrt{x}$. Find the polynomial $P(x)$ of degree three such that $P^{(k)}(4) = f^{(k)}(4)$ for $k = 0, 1, 2, 3$.

§9 Exercise. Let $f(x) = x^{1/3}$. Find the polynomial $P(x)$ of degree two which best approximates $f(x)$ near $x = 8$.

§10 Exercise. Let $f(x)$ and $P(x)$ be as in §9. Evaluate $P(10)$ and use the Extended Mean Value Theorem to prove that

$$|10^{1/3} - P(10)| \le \frac{10}{6 \cdot 27 \cdot 32}.$$

Hint: The function $g(x) = x^{-8/3}$ is decreasing so $g(10) < g(8)$.

§11 Exercise. Find a polynomial $P(x)$ of degree three such that

$$\lim_{x\to 0} \frac{\sin(x) - P(x)}{x^3} = 0.$$

Use the Extended Mean Value Theorem to show that

$$|\sin(x) - P(x)| \leq \frac{|x|^4}{24}.$$