Math 221 Review Sheet

BN

Fall 2000

Disclaimer: *This may not be comprehensive! If a topic is not on this review sheet, it might still be on the exam!*

Advice: *Learn how to do the problems you missed on the earlier exams and quizzes. Study the examples in the text. Many have appeared on earlier exams.*

1. Write the formula which defines the derivative $f'(a)$ of the function $f(x)$ at the point $x = a$.

2. (i) Define the terms *differentiable function* and *continuous function*. (ii) Give an example of a continuous function which is not differentiable. (iii) Prove that a differentiable function is continuous. (Use the definitions you gave in part (i). You may use without proof the theorem that the limit of a product is the product of the limits.)

3. State the Mean Value Theorem.

4. State the Intermediate Value Theorem.

5. (i) Complete the definition: The *Taylor polynomial* of the function f of degree n centered at a is

$$P(x) = \sum_{k=0}^{n} \ldots$$

(ii) In what sense is the Taylor polynomial close to the function f?

6. Let $f(x)$ be a function defined on the interval $a \le x \le b$. A *Riemann sum* for f on the interval $[a, b]$ is an expression of form

$$S = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$$

where $a = x_0 < x_1 < \cdots < x_n = b$ and $x_{k-1} \le c_k \le x_k$ for $k = 1, 2, \ldots, n$. Complete the sentence: When the quantity ________________ is small, the Riemann sum is close to ________________.

7. (i) Find the Taylor Polynomial of degree 5 for $f(x) = e^x$ at $a = 0$. (ii) Estimate the error if you use part (i) to approximate e^{-1}.

8. Find the polynomial of degree 4 which best approximates the function $f(x) = \tan x$ near the point $x = \frac{\pi}{4}$.

9. Graph, indicate limits at $x = \pm\infty$ and the one sided limits where the function is undefined, and label all maxima, minima, and points of inflection:

$$\text{(i) } y = xe^{-x^2}. \qquad \text{(ii) } y = \frac{1}{(x-1)(x-2)}.$$

10. Find the limits:

$$\text{(i) } \lim_{x\to\infty} \frac{e^x}{x^5} \qquad \text{(ii) } \lim_{x\to\infty} \frac{x^2}{2^x} \qquad \text{(iii) } \lim_{x\to 0} x \ln x$$

$$\text{(iv) } \lim_{x\to a} \frac{2^x - 2^a}{x - a} \qquad \text{(v) } \lim_{x\to 0} \frac{e^x - 1}{\sin x} \qquad \text{(vi) } \lim_{x\to 1} \frac{x^a - 1}{x^b - 1}$$

$$\text{(vii) } \lim_{x\to 0} \frac{1}{x} - \csc x \qquad \text{(viii) } \lim_{x\to\infty} \frac{(\ln x)^3}{x^2} \qquad \text{(ix) } \lim_{x\to 1} \frac{a^x - 1}{b^x - 1}$$

11. The altitude of a triangle is increasing at a rate of 1 cm/min while the area is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm, and the area is 100 cm ^{2}.

12. Find the maximum value and the minimum value of $f(x) = \dfrac{\ln x}{x}$ on $[1, 3]$.

13. Find the point of the graph of $x + y^2 = 0$ that is closest to the point $(-3, 0)$.

14. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 5.

15. Find the area under the curve $y = 2x + 1$, from $a = 0$ to $b = 5$, using Riemann sums. Hint: $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

16. Evaluate $\displaystyle\int_1^1 x^2 \cos x \, dx$.

17. Given that $\displaystyle\int_4^9 \sqrt{x}\, dx = \frac{38}{3}$, what is $\displaystyle\int_9^4 \sqrt{t}\, dt$?

18. Use a Riemann sum with four terms to find a number slightly larger than $\displaystyle\int_0^1 \sqrt{1 + x^3}\, dx$. Illustrate this with a picture.

19. Find $\dfrac{dy}{dx}$ if $y = \displaystyle\int_{x^2}^{\pi} \frac{\sin t}{t}\, dt$.

20. Find the interval on which the curve $y = \int_0^x \frac{1}{1+t+t^2}\,dt$ is concave upward.

21. Verify that $\int x^2 \sin x\,dx = -x^2 \cos x + 2\int x \cos x\,dx$

22. Evaluate:

(i) $\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}\,dx$ (ii) $\int \sin(3A) - \sin(3x)\,dx$

(iii) $\int \frac{\tan^{-1} x}{1+x^2}\,dx$ (iv) $\int \sec^2(3\theta)\,d\theta$

23. Find the values of c such that the area of the region bounded by $y = x^2 - c^2$ and $y = c^2 - x^2$ is 576.

24. Set up integrals for the volume of the solid obtained by rotating the region bounded by the given curves about the given line:

(i) $y = \ln x, y = 1, x = 1$, about the x-axis.

(ii) $2x + 3y = 6, (y-1)^2 = 4 - x$, about $x = -5$.

(iii) $y = \cos x, y = 0, x = 0, x = \frac{\pi}{2}$, about $y = 1$.

(iv) $y = \frac{1}{1+x^2}, y = 0, x = 0, x = 3$, about y-axis.

25. The temperature of a metal rod, 5 m long, is $4x^o$ C at a distance x meters from one end of the rod. What is the average temperature of the rod?

26. If $\int_0^{x^2} f(t)\,dt = x\sin(\pi x)$ find $f(4)$.

27. The displacement in meters of a particle moving in a straight line is given by $s = t^2 - 8t + 18$, where t is measured in seconds.

28. Simplify:

(i) $\tan^{-1}(\sqrt{3})$ (ii) $\sin[\sin^{-1}(\frac{1}{3}) + \sin^{-1}(\frac{2}{3})]$ (iii) $\sin(\tan^{-1} x)$

29. Find the equation of the tangent line to the curve $y = \tan^{-1}(3x-2)$ at $x = 1$

30. Find the derivative:

(i) $f(x) = \cos^{-1}(\sin x)$

(ii) $g(x) = x \tan^{-1} x$

(iii) $h(x) = (\sin^{-1} x)(\ln x)$

31. Find the length of the curve $y = \ln(\cos x)$ from $0 \le x \le \frac{\pi}{4}$.

32. Find the area and centroid of the region bounded by the given curves:

(i) $y = x^2$, $y = 0$, $x = 2$. (ii) $y = x$, $y = 0$, $y = \dfrac{1}{x}$, $x = 2$.

33. Given that $\dfrac{d}{dy}(\ln y) = \dfrac{1}{y}$, prove that $\dfrac{d}{dx}(e^x) = e^x$.

34. Find $\dfrac{dy}{dx}$ if $\tan^{-1} x + \sin^{-1} y = \ln(xy)$

35. If $x[f(x)]^3 + xf(x) = 6$ and $f(3) = 1$, find $f'(3)$.

36. Find z if

$$\frac{dz}{dx} = e^{-x^2/2}x \qquad \text{and } z = 1 \text{ when } x = 0.$$

37. A population Y is growing exponentially in time. (i) If $Y = 100$ at time $t = 0$ and $Y = 300$ at time $t = 2$ what is Y at time $t = 5$? (ii) Express dY/dt as a function of Y.