Caculus 221

Possible questions for Exam II

March 19, 2002

These notes cover the recent material in a style more like the lecture than the book. The proofs in the book are in section 1-11. At the end there is a list of questions. At least one of these will appear on the next exam.

§1 The Max-Min Existence Theorem. *Let the function $f(x)$ be continuous on the closed finite interval $a \leq x \leq b$. Then there is at least one number c such that $a \leq c \leq b$ and $f(x) \leq f(c)$ for all x in the interval $a \leq x \leq b$ and there is at least one number d such that $a \leq d \leq b$ and $f(d) \leq f(x)$ for all x in the interval $a \leq x \leq b$.*

§2 We say that f **attains** its maximum at c and its minimum at d and that $f(c)$ is the **maximum value** of f and $f(d)$ is the **minimum value** of f on the interval. Often the maximum is attained at an endpoint, i.e. $c = a$ or $c = b$ and similarly for the minimum. For example, on the interval $2 \leq x \leq 3$ the function $f(x) = x^2$ attains its minimum value $4 = f(2)$ at the left endpoint and its maximum value $9 = f(3)$ at the right endpoint. When $a < c < b$ we say that c is an **interior minimum**; when $a < d < b$ we say that d is an **interior maximum**. For example, on the interval $-2 \leq x \leq 1$ the function $f(x) = x^2$ attains its maximum $f(-2) = 4$ at the left endpoint and its minimum value $f(0) = 0$ at the interior point 0. On an interval which is not closed a continuous function need not assume its maximum or minimum. For example, the function $g(x) = 1/x$ is continuous on the interval $0 < x \leq 1$ but there is no number c satisfying $0 < c \leq 1$ and $g(x) \leq g(c)$ for all x. This is because no matter what c we pick in the interval $0 < x \leq 1$ we have that $f(c') > f(c)$ when c' is nearer 0 then c, e.g. when $c' = c/2$.

The Max-Min Existence Theorem is normally proved in more advanced courses like Math 521.

Calculus can help us find the maximum and the minimum when the continuous function is differentiable. The key tool is the following

§3 First Derivative Test. *Suppose that a function $f(x)$ defined on an interval $a \leq x \leq b$ attains its minimum (or its maximum) at a point c in the interval. Then either*

(1) *c is an endpoint, i.e. $c = a$ or $c = b$; or*

(2) *f is not differentiable at c; or*

(3) *c is a* **critical point** *of f, i.e. $f'(c) = 0$.*

Proof: Assume that (1) and (2) fail, i.e. that $a < c < b$ and that $f'(c)$ exists. We will prove (3). Since $f'(c)$ exists we have that

$$\frac{f(x) - f(c)}{x - c} \approx f'(c) \tag{$*$}$$

for $x \approx c$. If the ratio on the left is positive, then the numerator $f(x) - f(c)$ and the denominator $x - c$ have the same sign; if the ratio on the left is negative, then $f(x) - f(c)$ and $x - c$ have opposite signs. Since c is an interior point, there are numbers in the interval to the right of c and close to c and other numbers in the interval to the left of c and close to c. Thus

- If $f'(c)$ is positive, then $f(x) - f(c) > 0$ for x near and to the right of c, so c is not a minimum.
- If $f'(c)$ is positive, then $f(x) - f(c) < 0$ for x near and to the left of c, so c is not a maximum.
- If $f'(c)$ is negative, then $f(x) - f(c) < 0$ x near and to the right of c, so c is not a minimum.
- If $f'(c)$ is negative, then $f(x) - f(c) > 0$ for x near and to the left of c, so c is not a maximum.

Thus the only possibility is $f'(c) = 0$.

§4 Examples. (1) The function $L(x) = 2x + 3$ satisfies $L'(x) = 2$ and so is differentiable and has no critical point. On any interval it attains its minimum at the left endpoint and its maximum at the right endpoint.

(2) The absolute value function $g(x) = |x|$ satisfies

$$g'(x) = \frac{x}{|x|}$$

for $x \neq 0$ but $g'(0)$ does not exist. On any interval containing 0 it attains its minimum at 0 and its maximum at one of the two endpoints.

(3) The derivative of the function $f(x) = x^3 - 3x$ is $f'(x) = 3(x^2 - 1)$ and $f'(x) = 0$ for $x = \pm 1$. The point $x = -1$ does not lie in the interval $0 \leq x \leq 2$ and $f(0) = 0$, $f(1) = -2$, and $f(2) = -1$ so on the interval $0 \leq x \leq 2$ the function attains its minimum value at 1 and its maximum value at 0.

§5 Mean Value Theorem. *Assume $f(x)$ is continuous on the closed interval $a \leq x \leq b$ and differentiable on $a < x < b$. Then there is a point c with $a < c < b$ and*

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

i.e. the tangent line to the graph at $(c, f(c))$ is parallel to the "secant line" joining $(a, f(a))$ and $(b, f(b))$.

Proof: Consider the linear function

$$W(x) = f(a) + \left(\frac{f(b) - f(a)}{b - a}\right) \cdot (x - a).$$

The graph $y = W(x)$ is the line joining $(a, f(a))$ and $(b, f(b))$, i.e.

$$W(a) = f(a), \qquad W(b) = f(b).$$

The function

$$g(x) = f(x) - W(x)$$

satisfies

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}, \qquad g(a) = g(b) = 0.$$

By the Max-Min Existence Theorem the function g attains its maximum and its minimum. Since $g(a) = g(b) = 0$ either the maximum or the minimum (or both) must occur at an interior point c. By the First Derivative Test we have

$$g'(c) = 0$$

as required. (The special case of the Mean Value Theorem where the values at the endpoints are the same is called **Rolle's Theorem**.)

§6 Definition. A function $y = f(x)$ is said to be **increasing** on an interval iff

$$x_1 < x_2 \implies f(x_1) < f(x_2)$$

for any two points x_1, x_2 of the interval. (The symbol $\implies$ means *implies*.) Similarly f is said to be **decreasing** on an interval iff

$$x_1 < x_2 \implies f(x_1) > f(x_2).$$

A function is **monotonic** on an interval iff either it is increasing on that interval or else it is decreasing on that interval.

§7 Monotonicity Theorem. *If* $\left\{ \begin{array}{c} f'(x) > 0 \\ f'(x) < 0 \\ f'(x) = 0 \end{array} \right\}$ *for all* x *in an interval* I*, then* f *is* $\left\{ \begin{array}{l} \textit{increasing} \\ \textit{decreasing} \\ \textit{constant} \end{array} \right\}$ *on that interval.*

Proof. Choose x_1 and x_2 in the interval I with $x_1 < x_2$. By the Mean Value Theorem there is a c with $x_1 < c < x_2$ and

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c). \tag{\#}$$

Since c is between x_1 and x_2 it lie in the interval I and hence $f'(c)$ has the sign (positive, negative, or zero) of the hypothesis. Hence the ratio on the left in equation (#) has this same sign. Since $x_1 < x_2$ the denominator $x_2 - x_1$ is positive and hence the numerator $f(x_2) - f(x_1)$ has this same sign. If the sign is positive, then $f'(c) > 0$ so $f(x_2) - f(x_1) > 0$ so $f(x_1) < f(x_2)$. If the sign is negative, then $f'(c) < 0$ so $f(x_2) - f(x_1) < 0$ so $f(x_1) > f(x_2)$. If the derivative is identically zero, then $f'(c) = 0$ so $f(x_2) - f(x_1) = 0$ so $f(x_1) = f(x_2)$.

§8 Definition. A function $y = f(x)$ is **concave up** on an interval iff the second derivative $f''(x)$ is positive at every point x in the interval. The derivative of f' is f'' so, by the Monotonicity Theorem, the derivative f' of a concave up function is increasing. Similarly, the function is **concave down** iff its second derivative is negative. For example, the function $f(x) = x^2$ is concave up on any interval and the function $g(x) = -x^2$ is concave down.

§9 Secant Concavity Theorem. *Suppose that* $f(x)$ *is concave up on the interval* $a \le x \le b$ *and define*

$$W(x) = f(a) + \left(\frac{f(b) - f(a)}{b - a} \right) \cdot (x - a)$$

so that the graph $y = W(x)$ is the secant line joining the points $(a, f(a))$ and $(b, f(b))$. Then the graph $y = f(x)$ lies below the graph of the secant line, i.e.

$$f(x) < W(x).$$

for $a < x < b$. Similarly, if $f(x)$ is concave down, the graph of the function lies above the graph of the secant line.

Proof: Define $g(x) = f(x) - W(x)$ as in the proof of the Mean Value Theorem. Since $W(x)$ is a linear function, its second derivative is zero so $g''(x) = f''(x)$ and g is also concave up. By the Mean Value Theorem there is a point c with $a < c < b$ and $g'(c) = 0$. Since g' is increasing this means that $g'(x) < g'(c) = 0$ for $a < x < c$ and $g'(x) > 0$ for $c < x < b$. Hence g is decreasing on the interval $a < x < c$ and increasing on the interval $c < x < b$. Hence $g(a) > g(x)$ for $a < x < c$ and $g(x) < g(b)$ for $c < x < b$, i.e. $g(x) < 0$ for $a < x < b$. As $g = f - W$ we get $f(x) < W(x)$ for $a < x < b$ as required. If f is concave down, then $-f$ is concave up, so $-f < -W$, and hence $W < f$.

§10 Tangent Concavity Theorem. *Suppose that $f(x)$ is concave up on the interval $a \leq x \leq b$ and let c be an interior point of that interval, i.e. $a < c < b$. Define*

$$L(x) = f(c) + f'(c)(x - c)$$

so that the graph $y = L(x)$ is the tangent line to the graph $y = f(x)$ at the point $(c, f(c))$. Then the graph $y = f(x)$ lies above the graph of the tangent line, i.e.

$$L(x) \leq f(x)$$

for $a \leq x \leq b$. Similarly if $f(x)$ is concave down, the graph of the function lies below the graph of the tangent line.

Proof: Consider the function $g(x) = f(x) - L(x)$. As the function $L(x)$ is linear, its second derivative is zero, so $g''(x) = f''(x)$ and g is also concave up. Moreover $f'(c) = L'(c) =$ the slope of the tangent line at c, so $g'(c) = 0$. Since g' is increasing this means that $g'(x) < g'(c) = 0$ for $a < x < c$ and $g'(x) > 0$ for $c < x < b$. Hence g is decreasing on the interval $a < x < c$ and increasing on the interval $c < x < b$. Hence $g(x) > g(c) = 0$ for $a < x < c$ and $0 = g(c) < g(x)$ for $c < x < b$, i.e. $g(x) > 0$ for $a < x < b$. As $g = f - L$ we get $L(x) < f(x)$ for $a < x < b$ as required. If f is concave down, then $-f$ is concave up, so $-L < -f$, and hence $f < L$.

§11 Definition. Let $f(x)$ be a differentiable function and a a point in its domain. The **linear approximation** to $f(x)$ at $x = a$ is the linear function $L(x)$ whose graph is the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$, i.e.

$$L(x) = f(a) + f'(a)(x - a)$$

Note that $L(a) = f(a)$ and $L'(a) = f'(a)$.

§12 Linear Approximation Theorem. *The linear approximation $L(x)$ is the linear function which best approximates $f(x)$ near $x = a$ in the sense that*

$$\lim_{x\to a} \frac{f(x) - L(x)}{x - a} = 0.$$

Proof: $\displaystyle\lim_{x\to a} \frac{f(x) - L(x)}{x - a} = \lim_{x\to a} \left(\frac{f(x) - f(a)}{x - a} - f'(a) \right) = 0.$

§13 Chain Rule. *Suppose that the function $y = f(x)$ is differentiable at $x = a$, that $b = f(a)$, and that the function $z = g(x)$ is differentiable at $y = b$. Then the composition $g \circ f$ is differentiable at $x = a$ and*

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

Denote the linear approximations to f near a and to g near b by

$$L(x) = f(a) + f'(a)(x - a), \qquad M(y) = g(b) + g'(b)(y - b).$$

By high school algebra

$$M \circ L(x) = g(f(a)) + g'(f(a))f'(a)(x - a).$$

Thus the Chain Rule says that $M \circ L$ is the linear approximation to $g \circ f$, i.e. the linear approximation to the composition is the composition of the linear approximations.

§14 When f is a function and $k \geq 0$ is an integer the notation $f^{(k)}$ denotes kth derivative of f. Thus

$$f^{(0)}(x) = f(x), \qquad f^{(1)}(x) = f'(x), \qquad f^{(2)}(x) = f''(x),$$

and so on. Given a number a in the domain of f and an integer $n \geq 0$, the polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!} \tag{\#}$$

is called the **degree** n **Taylor polynomial of** f **centered at** a. The Taylor polynomial $P_n(x)$ is the unique polynomial of degree n which has the same derivatives as f at a up to order n:

$$P_n^{(k)}(a) = f^{(k)}(a) \qquad\qquad \text{for } k = 0, 1, 2, \ldots, n.$$

§15 The letter $\sum$ is the Greek S (for *sum*) and is pronounced *sigma* so the notation used in (#) is called **sigma notation**. It is a handy notation but if you don't like it you can indicate the summation with dots:

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_{n-1} + a_n.$$

Hence the first few Taylor polynomials are

$$P_0(x) = f(a),$$

$$P_1(x) = f(a) + f'(a)(x-a),$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2},$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2} + \frac{f'''(a)(x-a)^3}{6}.$$

§16 The Taylor polynomial $P_n(x)$ for $f(x)$ centered at a is the polynomial of degree n which best approximates $f(x)$ for x near a. The precise statement is

Taylor's Theorem. *Suppose that f is $n+1$ times differentiable and that $f^{(n+1)}$ is continuous. Let a be a point in the domain of f. Then*

$$\lim_{x \to a} \frac{f(x) - P_n(x)}{(x-a)^n} = 0. \tag{$\heartsuit$}$$

Possible Questions

§17 State the First Derivative Test and prove it. **Answer:** Write what is in §3

§18 State and prove the Mean Value Theorem. In your proof you may use (without proof) the Max-Min Existence Theorem and the First Derivative Test. **Answer:** Write what is in §5.

§19 State and prove the Monotonicity Theorem. In your proof you may use (without proof) the Mean Value Theorem. **Answer:** Write what is in §7.

§20 Prove that if the derivative of a function is positive on an interval that function is increasing on that interval. In your proof you may use (without proof) the Mean Value Theorem. **Answer:** Write an appropriately simplified version of what in §7.

§21 State and prove the Secant Concavity Theorem. In your proof you may use (without proof) the Mean Value Theorem and the Monotonicity Theorem. **Answer:** Write what is in §9.

§22 State the Tangent Concavity Theorem. **Answer:** Write what is in §10 before the word "Proof".

§23 Let f be a function and a a number in its domain. Write a formula for the linear function which best approximates $f(x)$ near a. Explain what "best approximates" means and prove it. **Answer:** Write what is in §11 and §12.

§24 Let f be a function, a a number in its domain, and n a nonnegative integer. Write a formula for the polynomial of degree $\leq n$ which best approximates $f(x)$ near a. **Answer:**

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!}.$$