

# Calculus 221 - First Exam (50 Minutes)

Friday October 4 1996

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I Find the limit or show that it does not exist. Justify your answer.

(1)  $\lim_{t \rightarrow -\infty} \frac{t^3 - 1}{t^2 - 1}$

Answer:

$$\lim_{t \rightarrow -\infty} \frac{t^3 - 1}{t^2 - 1} = \lim_{t \rightarrow -\infty} \frac{t - \frac{1}{t^2}}{1 - \frac{1}{t^2}} = -\infty$$

(2)  $\lim_{t \rightarrow 3} \frac{t^3 - 1}{t^2 - 1}$

Answer:

$$\lim_{t \rightarrow 3} \frac{t^3 - 1}{t^2 - 1} = \frac{3^3 - 1}{3^2 - 1} = \frac{26}{8}$$

(3)  $\lim_{t \rightarrow 1} \frac{t^3 - 1}{t^2 - 1}$

Answer:

$$\lim_{t \rightarrow 1} \frac{t^3 - 1}{t^2 - 1} = \lim_{t \rightarrow 1} \frac{(t - 1)(t^2 + t + 1)}{(t - 1)(t + 1)} = \lim_{t \rightarrow 1} \frac{(t^2 + t + 1)}{(t + 1)} = \frac{3}{2}$$

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II Find the limit or show that it does not exist. Justify your answer.

(1)  $\lim_{x \rightarrow \infty} \frac{\sin(3x)}{x}$

Answer: Since  $-1 \leq \sin(3x) \leq 1$  for all  $x$  we have

$$-\frac{1}{x} \leq \frac{\sin(3x)}{x} \leq \frac{1}{x}$$

for  $x > 0$ . Hence  $\lim_{x \rightarrow \infty} \frac{\sin(3x)}{x} = 0$  by the Squeeze theorem.

(2)  $\lim_{x \rightarrow 0} \frac{\sin(x^2)}{x}$

Answer:

$$\lim_{x \rightarrow 0} \frac{\sin(x^2)}{x} = \lim_{x \rightarrow 0} \frac{\sin(x^2)}{x^2} \cdot x = \lim_{x \rightarrow 0} \frac{\sin(x^2)}{x^2} \cdot \lim_{x \rightarrow 0} x = \lim_{x \rightarrow 0} \frac{\sin(u)}{u} \cdot \lim_{x \rightarrow 0} x = 1 \cdot 0 = 0.$$

(3)  $\lim_{x \rightarrow \pi/3} \frac{\sin(x) - \sin(\pi/3)}{x - \pi/3}$

Answer: Let  $f(x) = \sin(x)$  and  $a = \pi/3$ . Then  $f'(x) = \cos(x)$  and

$$\lim_{x \rightarrow \pi/3} \frac{\sin(x) - \sin(\pi/3)}{x - \pi/3} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = f'(a) = \cos(\pi/3) = \frac{1}{2}.$$

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**III** (1) Find  $f'(x)$  and  $f''(x)$  if  $f(x) = \sin(x^3 - 2)$ .

Answer:

$$f'(x) = (\cos(x^3 - 2)) 3x^2, \quad f''(x) = -(\sin(x^3 - 2)) (3x^2)^2 + (\cos(x^3 - 2)) 6x.$$

(2) Find  $g'(3)$  if  $h'(9) = 17$  and  $g(x) = h(x^2)$ .

Answer:

$$g'(x) = h'(x^2)2x, \quad g'(3) = h'(9)6 = 102.$$

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**IV** Find the constant  $c$  which makes  $g$  continuous on  $(-\infty, \infty)$ ,

$$g(x) = \begin{cases} x^2 & \text{if } x < 4, \\ cx + 20 & \text{if } x \geq 4. \end{cases}$$

Answer: By the limit laws,  $g(x)$  is continuous at any  $x \neq 4$ .

$$\lim_{x \rightarrow 4^-} g(x) = \lim_{x \rightarrow 4^-} x^2 = 16, \quad \lim_{x \rightarrow 4^+} g(x) = \lim_{x \rightarrow 4^+} cx + 20 = 4c + 20.$$

The function  $g(x)$  is continuous when these are equal, i.e. when  $c = -1$ .

Answer:

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**V** Consider the curve

$$y^2 + xy - x^2 = 11.$$

(1) Find the equation of the tangent line to the curve at the point  $P(2, 3)$ .

Answer: Differentiate:

$$2y \frac{dy}{dx} + y + x \frac{dy}{dx} - 2x = 0.$$

Evaluate at the point  $P(2, 3)$ :

$$6 \left. \frac{dy}{dx} \right|_{(x,y)=(2,3)} + 3 + 2 \left. \frac{dy}{dx} \right|_{(x,y)=(2,3)} - 4 = 0.$$

Solve:

$$\left. \frac{dy}{dx} \right|_{(x,y)=(2,3)} = \frac{1}{8}.$$

This is the slope of the tangent line. The point  $P(2, 3)$  lies on the tangent line so the equation of the tangent line is

$$(y - 3) = \frac{(x - 2)}{8}.$$

(2) Find  $\frac{d^2y}{dx^2}$  at the point  $P(2, 3)$ .

Answer: Differentiate again:

$$2 \left[ \left( \frac{dy}{dx} \right)^2 + y \frac{d^2y}{dx^2} \right] + \frac{dy}{dx} + \left[ \frac{dy}{dx} + x \frac{d^2y}{dx^2} \right] - 2 = 0.$$

Evaluate at  $(x, y) = (2, 3)$ :

$$2 \left[ \left( \frac{1}{8} \right)^2 + 3 \frac{d^2y}{dx^2} \Big|_{(x,y)=(2,3)} \right] + \frac{1}{8} + \left[ \frac{1}{8} + 2 \frac{d^2y}{dx^2} \Big|_{(x,y)=(2,3)} \right] - 2 = 0.$$

Solve:

$$\frac{d^2y}{dx^2} \Big|_{(x,y)=(2,3)} = \frac{-2 \left( \frac{1}{8} \right)^2 - \frac{1}{8} - \frac{1}{8} + 2}{6 + 2}.$$

Answer:

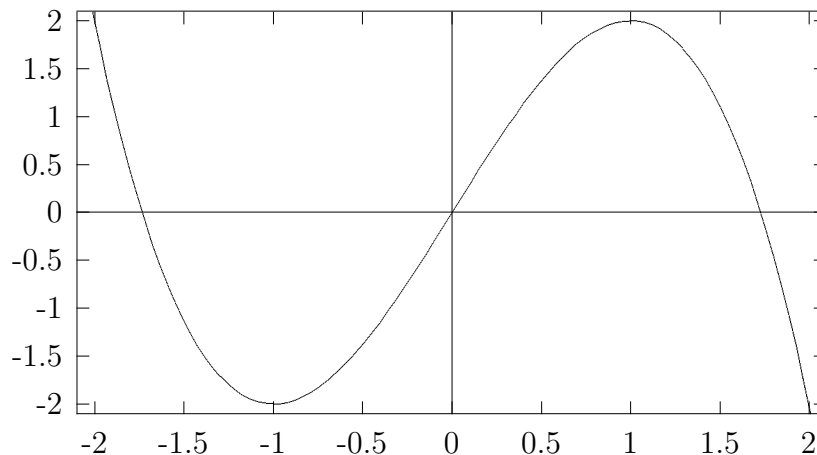
**VI** Consider the function  $y = f(x)$  whose graph is shown below. Match the expression in the left column with the correct corresponding value in the right column.

$$f'(0) \qquad -6$$

$$f'(0.9) \qquad 0$$

$$f'(1) \qquad 3$$

$$f'(1.732) \qquad 0.6$$



Answer: For positive  $x$ , the slope of the tangent line is decreasing as  $x$  increases, so  $f'(0) > f'(0.9) > f'(1) > f'(1.732)$ . The only possibility is  $f'(0) = 3$ ,  $f'(0.9) = 0.6$ ,  $f'(1) = 0$ ,  $f'(1.732) = -6$ .

Answer:

**VII** Prove the product rule  $(fg)' = f'g + fg'$  using the definition of the derivative, high school algebra, and the appropriate limit laws. Justify each step.

Answer: Let  $w(x) = f(x)g(x)$  and assume that  $f$  and  $g$  are differentiable.

$$w'(x) = \lim_{h \rightarrow 0} \frac{w(x+h) - w(x)}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} g(x+h) + f(x) \frac{g(x+h) - g(x)}{h} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \lim_{h \rightarrow 0} g(x+h) + f(x) \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \quad (4)$$

$$= f'(x)g(x) + f(x)g'(x). \quad (5)$$

Reasons:

- (1) Definition of  $w'(x)$ .
- (2) Definition of  $w(x)$ .
- (3) High school algebra.

(4) Limit laws.

(5) Definition of  $f'(x)$ , continuity of  $g(x)$ , and definition of  $g'(x)$ .

In step (5) we used the fact that a differentiable function is continuous. Here is the proof:

$$\begin{aligned}\left(\lim_{h \rightarrow 0} g(x+h)\right) - g(x) &= \lim_{h \rightarrow 0} (g(x+h) - g(x)) \\ &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \cdot h \\ &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \cdot \lim_{h \rightarrow 0} h \\ &= g'(x) \cdot 0 = 0.\end{aligned}$$

so  $\lim_{h \rightarrow 0} g(x+h) = g(x)$ , i.e.  $g$  is continuous.

## Calculus 221 - Second Exam (Two Hours)

Thursday October 24 1996

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I (a) Find  $f'(x)$  if  $f(x) = \frac{e^{5x}}{1+e^{2x}}$ .

$$\text{Answer: } f'(x) = \frac{5e^{5x}(1+e^{2x}) - e^{5x} \cdot 2e^{2x}}{(1+e^{2x})^2}.$$

(b) Find  $\frac{d\theta}{dt}$  when  $\theta = \sin^{-1}(t^2)$ .

Answer: Let  $y = t^2$  so  $\theta = \sin^{-1}(y)$ . Then

$$\frac{d\theta}{dt} = \frac{d\theta}{dy} \frac{dy}{dt} = \frac{1}{\sqrt{1-y^2}} \cdot 2t = \frac{2t}{\sqrt{1-t^4}}.$$

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II After 3 days a sample of radon-222 decayed to 58% of its original amount.

(a) What is the half life of radon-222?

Answer: Let  $Y$  be the amount of radon-222 at time  $t$  and  $Y_0$  be the amount when  $t = 0$ . Thus  $Y = Y_0 e^{ct}$  for all time  $t$ . When  $t = 3$  we have  $Y = 0.58Y_0$ . Hence  $0.58Y_0 = Y_0 e^{c3}$ . Apply  $\ln$  to find  $c$ :  $c = \frac{\ln(0.58)}{3}$ . The half life is the time  $\tau$  when

$$Y = 0.5Y_0 \text{ i.e. } 0.5Y_0 = Y_0 e^{c\tau}. \text{ Apply } \ln \text{ to find } \tau: \tau = \frac{\ln(0.5)}{c} = \frac{3 \ln(0.5)}{\ln(0.58)}.$$

(b) How long would it take for the sample to decrease to 10% of its original amount?

Answer: The desired time  $t$  satisfies  $0.10Y_0 = Y_0e^{ct}$ . To find  $t$  cancel  $Y_0$  and apply  $\ln$ :  $t = \frac{\ln(0.10)}{c} = \frac{3 \ln(0.10)}{\ln(0.58)}$ .

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**III** (a) Find the inverse function  $g(y)$  to the function  $f(x) = \sqrt{9+x}$ .

Answer: For  $y \geq 0$  we have  $y = \sqrt{9+x} \iff y^2 = 9+x \iff x = y^2 - 9$ . Thus  $g(y) = f^{-1}(y) = y^2 - 9$ .

(b) What is the domain of  $f(x)$ ?

Answer: When a function is given by a formula, the domain is the set of  $x$  for which the formula is meaningful, *unless the contrary is asserted*. The domain of  $f$  is  $x \geq -9$ .

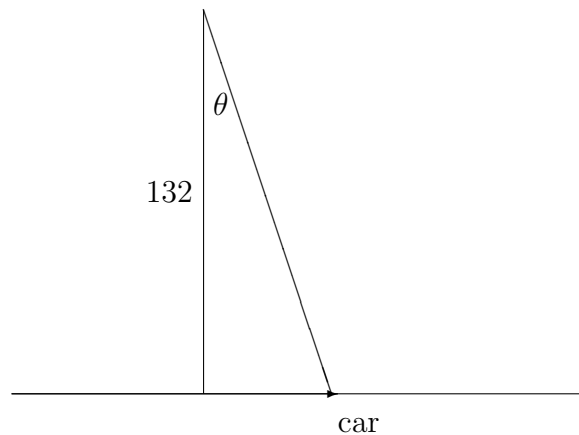
(c) What is the domain of  $g(y)$ ?

Answer: The formula defining  $g(y)$  is meaningful for all  $y$ , but  $g$  is defined to be the inverse function to  $f$ . Hence  $domain(g) = range(f)$ . Thus  $domain(g)$  is the set of all  $y \geq 0$ .

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**IV** You are videotaping a race from a stand 132 feet from the track, following a car that is moving at a constant velocity along a straight track. When the car is directly in front of you, the camera angle  $\theta$  is changing at a rate of

$$\left. \frac{d\theta}{dt} \right|_{\theta=0} = \frac{6}{43} \text{ radians per second.}$$



(a) How fast is the car going?

Answer: Choose coordinates so that the car moves along the  $x$ -axis and the viewing stand is on the  $y$ -axis at the point  $Q(0, 132)$ . Let  $P(x, 0)$  be the position of the car. Let  $t$  be the time in seconds and suppose  $t = 0$  is the time when the car is directly in front of you. Let  $\theta$  be the angle between the  $y$ -axis and  $QP$ . We are given that

$$\frac{dx}{dt} = v \text{ is constant, and } \left. \frac{d\theta}{dt} \right|_{t=0} = \frac{6}{43}.$$

Since

$$x = 132 \tan \theta$$

we have

$$v = \frac{dx}{dt} = (132 \sec^2 \theta) \cdot \frac{d\theta}{dt}$$

so evaluating at  $t = 0$  gives:

$$v = 132 \cdot \left. \frac{d\theta}{dt} \right|_{t=0} = 132 \cdot \frac{6}{43}.$$

(b) How fast will the camera angle  $\theta$  be changing a half second later?

Answer: The position of the car at time  $t$  is

$$x = vt$$

since  $v$  is constant. Thus

$$\theta = \tan^{-1} \left( \frac{vt}{132} \right).$$

By the chain rule:

$$\frac{d\theta}{dt} = \frac{1}{1 + \left(\frac{vt}{132}\right)^2} \cdot \left(\frac{v}{132}\right).$$

Evaluate at  $t = 1/2$  and plug in the value of the constant  $v$  which we found above:

$$\left. \frac{d\theta}{dt} \right|_{t=1/2} = \frac{1}{1 + \left(\frac{v}{2 \cdot 132}\right)^2} \cdot \left(\frac{v}{132}\right) = \frac{1}{1 + \left(\frac{3}{43}\right)^2} \cdot \left(\frac{6}{43}\right).$$

**V** Find  $\sin^{-1}(\sin(3))$  exactly. Justify your reasoning.

Answer: For  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$  we have

$$\theta = \sin^{-1}(\sin(3)) \iff \sin(\theta) = \sin(3).$$

Also  $\sin(\pi - 3) = \sin(3)$  and  $-\frac{\pi}{2} \leq \pi - 3 \leq \frac{\pi}{2}$ . Therefore  $\sin^{-1}(\sin(3)) = \pi - 3$ . (The horizontal line  $y = \sin(3)$  intersects the curve  $y = \sin(\theta)$  infinitely often.)

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**VI** (a) Find the equation for the tangent line to  $y = \ln(x)$  at the point  $x = 1$ .

Answer: The equation for the tangent line is  $y = L(x)$  where  $L(x)$  is the linear approximation of  $y = \ln(x)$  at  $x = 1$ . Using the formula

$$L(x) = f(a) + f'(a)(x - a)$$

with  $f(x) = \ln x$  and  $a = 1$  and  $f'(x) = 1/x$  we get  $f(1) = \ln(1) = 0$ ,  $f'(1) = 1$  and so

$$L(x) = 0 + 1 \cdot (x - 1) = (x - 1).$$

(b) Find the quadratic approximation  $Q(x)$  to the function  $f(x) = \ln x$  at 1.

Answer: Using the formula

$$Q(x) = f(a) + f'(a)(x - a) + \frac{f''(a)(x - a)^2}{2}$$

with  $f(x) = \ln(x)$ . we get  $f'(x) = 1/x$ ,  $f''(x) = -1/x^2$  so  $f''(1) = -1$  and hence

$$Q(x) = (x - 1) - \frac{(x - 1)^2}{2}.$$

(c) Estimate  $\ln(1.1)$  without a calculator.

Answer:  $\ln(1.1) \approx Q(1.1) = 0.1 - (0.1)^2/2 = 0.095$ .

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**VII** Find  $\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n$ . Justify your steps.

Answer: Using the formula

$$e = \lim_{m \rightarrow \infty} \left(1 + \frac{1}{m}\right)^m \quad (\heartsuit)$$

implies

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^{n/2} = \left(\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n\right)^{1/2}.$$

so squaring gives:

$$\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n = e^2$$

The wording of the question leaves some doubt as to whether the professor wants a proof of  $(\heartsuit)$ . To be on the safe side we'll provide it:

$$\ln \left( \lim_{m \rightarrow \infty} \left(1 + \frac{1}{m}\right)^m \right) = \lim_{m \rightarrow \infty} \ln \left(1 + \frac{1}{m}\right)^m = \lim_{m \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{m}\right) - \ln(1)}{1/m} = \lim_{h \rightarrow 0} \frac{\ln(1+h) - \ln(1)}{h}.$$



The limit on the right is the derivative of the  $\ln(x)$  evaluated at  $x = 1$ . This is  $\ln'(1) = 1/1 = 1$  so

$$\ln\left(\lim_{m \rightarrow \infty} 1 + \frac{1}{m}\right)^m = 1 \quad \text{so} \quad \lim_{m \rightarrow \infty} 1 + \frac{1}{m} = e.$$

You can also use l'Hospital's rule.

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**VIII** Suppose  $g$  is the inverse function to  $f(x) = x^5 + x + 1$ . Find  $g'(1)$  and  $g'(3)$

Answer:  $x = g(y) \iff y = f(x)$ . Since  $f(0) = 1$  we have  $g(1) = 0$ . Since  $f(1) = 3$  we have  $g(3) = 1$ . Now  $g(f(x)) = x$  so by the chain rule

$$g'(f(x)) = \frac{1}{f'(x)} = \frac{1}{5x^4 + 1}.$$

When  $x = 0$  this gives

$$g'(1) = g'(f(0)) = \frac{1}{5 \cdot 0^4 + 1} = 1$$

When  $x = 1$  this gives

$$g'(3) = g'(f(1)) = \frac{1}{5 \cdot 1^4 + 1} = \frac{1}{6}.$$

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**IX** (a) If  $f(x) = x^{\ln x}$  what is  $f'(x)$ ?

Answer:  $x = e^{\ln x}$  so  $x^{\ln x} = e^{(\ln x)^2}$ . Hence by the Chain Rule

$$f'(x) = \frac{d}{dx} e^{(\ln x)^2} = e^{(\ln x)^2} \frac{d}{dx} (\ln x)^2 = e^{(\ln x)^2} 2(\ln x) \frac{d}{dx} \ln x = \frac{e^{(\ln x)^2} 2(\ln x)}{x}.$$

(b) Express  $\frac{du}{dt}$  in terms of  $t$  and  $u$  if  $u = t \ln(t + u)$ .

Answer: Use implicit differentiation:

$$\frac{du}{dt} = \ln(t + u) + \frac{t}{t + u} \cdot \left(1 + \frac{du}{dt}\right).$$

Solve for  $du/dt$ :

$$\left(1 - \frac{t}{t + u}\right) \frac{du}{dt} = \ln(t + u) + \frac{t}{t + u};$$

so

$$\frac{du}{dt} = \left(1 - \frac{t}{t + u}\right)^{-1} \left(\ln(t + u) + \frac{t}{t + u}\right).$$

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**X** (a) Find  $\lim_{t \rightarrow 0} \frac{\sin t}{t^3 - t}$ .

Answer: By l'Hospital's Rule:

$$\lim_{t \rightarrow 0} \frac{\sin t}{t^3 - t} = \lim_{t \rightarrow 0} \frac{\cos t}{3t^2 - 1} = -1.$$

(b) Find  $\lim_{x \rightarrow \infty} \ln(2x + 1) - \ln(x)$ .

Answer:  $\ln(2x + 1) - \ln(x) = \ln\left(\frac{2x + 1}{x}\right) = \ln\left(2 + \frac{1}{x}\right)$ . Hence

$$\lim_{x \rightarrow \infty} \ln(2x + 1) - \ln(x) = \ln\left(\lim_{x \rightarrow \infty} 2 + \frac{1}{x}\right) = \ln(2).$$

## Calculus 221 - Third Exam (50 Minutes)

Friday December 6 1996

**I** The second derivative of a function  $f(x)$  satisfies

$$f''(x) = 12x^2 - 4.$$

Moreover,  $f'(0) = 0$  and  $f(1) = 0$ .

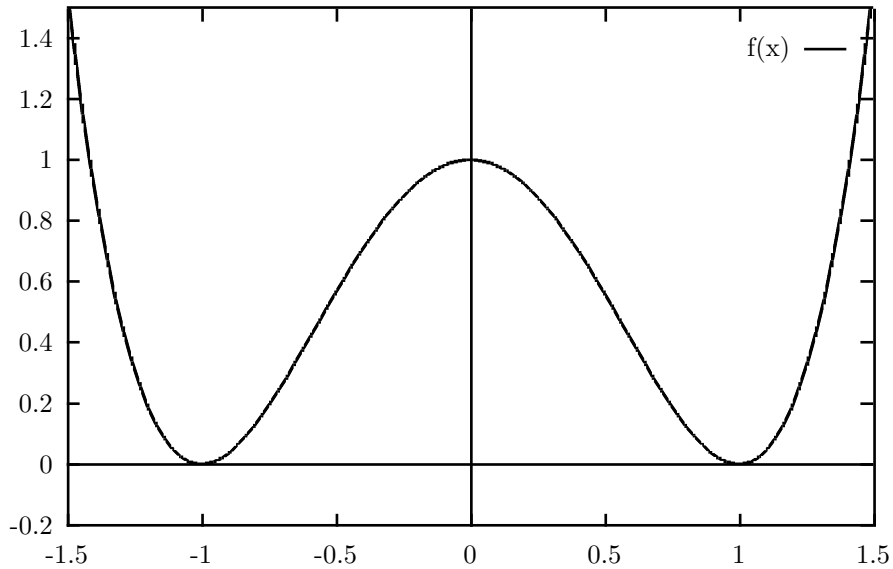
(a) Find the function  $f(x)$ .

Answer:  $f'(x) = 4x^3 - 4x + C_1$  so  $0 = f'(0) = C_1$  so  $f'(x) = 4x^3 - 4x$ .  $f(x) = x^4 - 2x^2 + C_2$  so  $0 = f(1) = 1^4 - 2 \cdot 1^2 + C_2$  so  $f(x) = x^4 - 2x^2 + 1$ .

(b) Draw a graph of  $f(x)$ . Indicate all asymptotes (if any), local maxima and minima, inflection points, intervals where  $f$  is increasing, intervals where  $f$  is concave up like a cup.

Answer:

$x$	$f(x)$	$f'(x)$	$f''(x)$	
$-\infty$	$\infty$			no hor asymp
$x < -1$		-	+	↓, CU
$-1$	$0$	$0$	+	local min
$-1 < x < -1/\sqrt{3}$		+	+	↑, CU
$-1/\sqrt{3}$		+	$0$	inflection
$-1/\sqrt{3} < x < 0$		+	-	↑, CD
$0$	$1$	$0$	-	local max
$0 < x < 1/\sqrt{3}$		-	-	↓, CD
$1/\sqrt{3}$		-	$0$	inflection
$1/\sqrt{3} < x < 1$		-	+	↓, CU
$1$	$0$	$0$	+	local min
$1 < x$		+	+	↑, CU
$\infty$	$\infty$			no hor asymp



**II** (a) Evaluate  $\int_3^4 \sqrt{x-1} dx$ .

Answer:

$$\int_3^4 \sqrt{x-1} dx = \frac{2}{3}(x-1)^{3/2} \Big|_3^4 = \frac{2}{3} \cdot 3^{3/2} - \frac{2}{3} \cdot 2^{3/2}$$

(b) Evaluate  $\int_2^3 \frac{\cos(1/t)}{t^2} dt$ .

Answer: Let  $u = 1/t$  so  $du = -dt/t^2$  and  $t = 2 \implies u = 1/2$ ,  $t = 3 \implies u = 1/3$ . Thus

$$\int_2^3 \frac{\cos(1/t)}{t^2} dt = \int_{1/2}^{1/3} -\cos u du = -\sin(1/3) + \sin(1/2).$$

**III** (a) Evaluate  $\frac{d}{dx} \int_{x^2}^x \sin(t^4) dt$ .

Answer: If  $F'(t) = \sin(t^4)$  then

$$\frac{d}{dx} \int_{x^2}^x \sin(t^4) dt = \frac{d}{dx} (F(x) - F(x^2)) = F'(x) - F'(x^2)2x = \sin(x^4) - \sin(x^8)2x.$$

(b) Evaluate  $\int_{x^2}^x \frac{d}{dt} \sin(t^4) dt$

Answer: If  $F(t) = \sin(t^4)$  then  $\frac{d}{dt} \sin(t^4) = F'(t)$  so

$$\int_{x^2}^x \frac{d}{dt} \sin(t^4) dt = F(x) - F(x^2) = \sin(x^4) - \sin(x^8).$$

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**IV** Evaluate  $\lim_{h \rightarrow 0} \frac{1}{h} \int_3^{3+h} x^3 \sqrt{x^2 - 1} dx$ .

Answer: The key step in the proof of the Fundamental Theorem is

$$\lim_{h \rightarrow 0} \int_a^{a+h} f(x) dx = f(a).$$

Hence

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_3^{3+h} x^3 \sqrt{x^2 - 1} dx = 3^3 \sqrt{3^2 - 1}.$$

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**V** (a) The numbers  $a = x_0 < x_1 < x_2 < \cdots < x_n = b$  divide the closed interval  $[a, b]$  into  $n$  tiny intervals of equal length. Give a formula for  $x_i$ , the right endpoint of the  $i$ -th interval.

Answer:  $x_i = a + \frac{i(b-a)}{n}$ . Note that  $\Delta x_i = x_i - x_{i-1} = \frac{b-a}{n}$

(b) Evaluate  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(5 + \frac{3i}{n}\right)^7 \frac{3}{n}$ . Hint: This is a Riemann sum

where  $x_i^* = x_i$ .

Answer: If  $a = 5$  and  $b = 8$  then  $x_i = 5 + \frac{3i}{n}$  and  $\Delta x_i = 3/n$ . If  $f(x) = x^7$  then

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(5 + \frac{3i}{n}\right)^7 \frac{3}{n} = \int_a^b f(x) dx = \int_5^8 x^7 dx = \frac{x^8}{8} \Big|_5^8 = \frac{8^8}{8} - \frac{5^8}{8}.$$

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**VI** An athlete is at point  $A$  on a bank of a straight river, 3 km wide and wants to reach  $B$ , 8 km downstream on the opposite bank, as quickly as possible. He will row to a point  $X$ ,  $x$  km downstream and on the side opposite  $A$  and then run along the shore to  $B$ . If he rows at 6 km/hour and runs at 8 km/hour, what should  $x$  be?

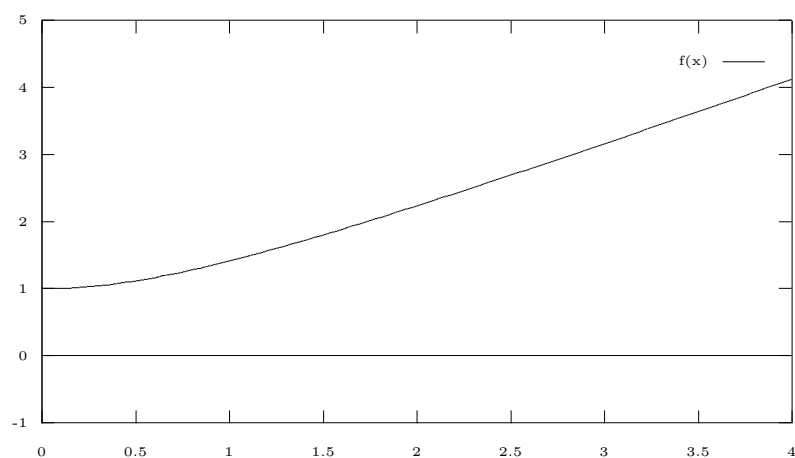
Answer: See Text Page 297 Example 4.

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**VII** Find positive numbers  $A$  and  $B$  such that

$$0 < A \leq \int_1^3 \sqrt{1+x^2} dx \leq B$$

The graph of  $f(x) = \sqrt{1+x^2}$  is shown below. There is no ‘best’ answer here. What counts is the reasoning. The answers  $A = 0.000001$  and  $B = 1000000$  are correct provided that you can prove the above inequality.



Answer: A crucial property of integration is that  $\int_a^b f(x) dx \leq \int_a^b g(x) dx$  if  $f(x) \leq g(x)$  for  $a \leq x \leq b$ . Since  $\sqrt{2} \leq \sqrt{1+x^2} \leq \sqrt{10}$  for  $1 \leq x \leq 3$ . We have

$$2\sqrt{2} = \int_1^3 \sqrt{2} dx \leq \int_1^3 \sqrt{1+x^2} dx \leq \int_1^3 \sqrt{10} dx = 2\sqrt{10}.$$

## Calculus 221 - Final Exam (Two Hours)

Monday December 16 1996

---

**I** In each of the following, find  $dy/dx$ .

(a)  $y = \sin x$ .

Answer:  $\frac{dy}{dx} = \cos x$ .

(b)  $y = (\sin x)^{-1}$ .  
Answer:  $\frac{dy}{dx} = \frac{-\cos x}{(\sin x)^2}$ .

(c)  $y = \sin(x^{-1})$ .  
Answer:  $\frac{dy}{dx} = (\cos(x^{-1}))(-x^{-2})$

(d)  $y = \sin^{-1}(x)$ .  
Answer:  $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ .

---

**II** Evaluate the integral.

(a)  $\int \frac{\cos x \, dx}{\sin x}$ .

(b)  $\int_1^2 x \sin(x^2) \, dx$ .

(c)  $\int_{-1}^2 |x^3| \, dx$

---

**III** Evaluate the limit.

(a)  $\lim_{x \rightarrow 1} \frac{x^3 - 1}{x^2 - 1}$ .

(b)  $\lim_{x \rightarrow 0} \frac{3^{\sin x} - 1}{\sin x}$ .

(c)  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \cos\left(3 + \frac{4i}{n}\right) \frac{4}{n}$

---

**IV** (a) Find an equation for the tangent line to the curve  $y^2 + x \cos(y) = \pi^2 - 3$  at the point  $(x, y) = (3, \pi)$ .

Answer:  $y - \pi = \frac{x - 3}{2\pi}$ .

(b) A function  $f(x)$  satisfies the identity  $f(x)^2 + x \cos(f(x)) = \pi^2 - 3$  and moreover  $f(3) = \pi$ . Find the linearization  $L(x)$  of  $f(x)$  at 3. Hint: Note the similarity to part (a).

Answer:  $L(x) = \pi + \frac{x-3}{2\pi}$ .

---

**V** (a) Find the point on the curve  $y = \sqrt{x}$ ,  $0 \leq x \leq 4$  which is closest to  $(2, 0)$ .

(b) Find the point on the curve  $y = \sqrt{x}$ ,  $0 \leq x \leq 4$  which is farthest from  $(2, 0)$ .

---

**VI** The count in a bacteria culture was 400 after 2 hours and 25,600 after 8 hours.

(a) Give a formula for the count after  $t$  hours.

Answer: The general formula is  $N = N_0 e^{kt}$ . We are given  $t = 2 \implies N = 400$  and  $t = 8 \implies N = 25,600$ ; i.e.

$$400 = N_0 e^{2k}, \quad 25,600 = N_0 e^{8k}.$$

Dividing gives  $64 = e^{6k}$  so (as  $2^6 = 64$ )  $e^k = 2$  and  $N_0 = 400/2^2 = 100$ . Thus

$$N = 100 \cdot 2^t.$$

(b) How long does it take for the count to double?

Answer: From  $N = 100 \cdot 2^t$  it follows that the count doubles every hour.

---

**VII** Let  $f(x) = \frac{x}{16 + x^3}$

(a) Find the absolute maximum and the absolute minimum of  $f(x)$  on the interval  $1 \leq x \leq 4$ ,

Answer: The derivative

$$f'(x) = \frac{(16 + x^3) - x(3x^2)}{(16 + x^3)^2} = \frac{16 - 2x^3}{(16 + x^3)^2}$$

vanishes at  $x = 2$ . Moreover  $f(1) = 1/17$ ,  $f(2) = 1/12$ , and  $f(4) = 1/20$ . Therefore the maximum value is  $1/12$  and the minimum is  $1/20$ .

(b) Find positive numbers  $A$  and  $B$  such that

$$0 < A \leq \int_1^4 \frac{x dx}{16 + x^3} \leq B.$$

You must justify your answer.

Answer: If  $m \leq f(x) \leq M$  for  $a \leq x \leq b$  then

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a).$$

Therefore

$$\frac{1}{20}(4-1) \leq \int_1^4 \frac{x dx}{16+x^3} \leq \frac{1}{12}(4-1).$$

---

**VIII** A particle moves along a straight line so that its velocity at time  $t$  is

$$v = \frac{ds}{dt} = t^3 - 4t = (t^2 - 4)t$$

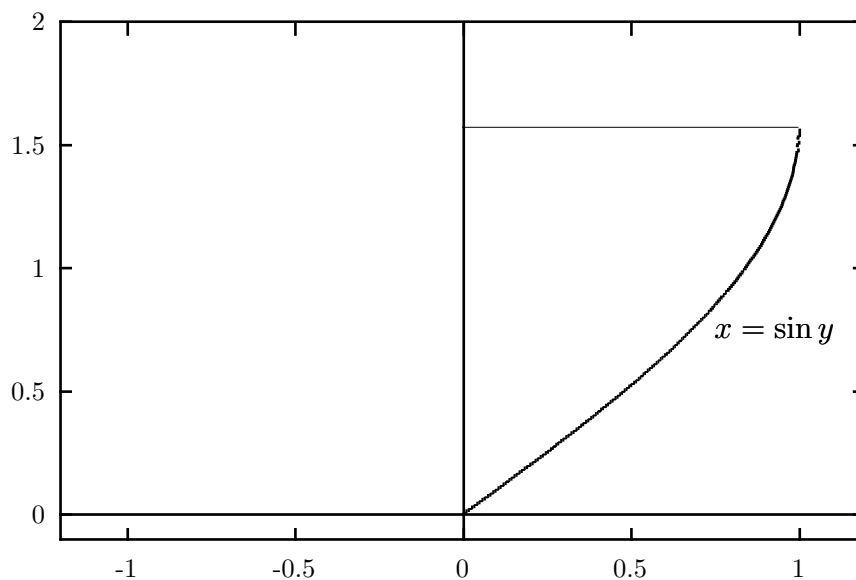
where  $t$  is the time in minutes.

(a) Write an integral which gives the displacement (the net change in position) during the first five minutes.

(b) Write an integral which gives the total distance travelled during the first five minutes.

---

**IX** A tank is constructed by rotating the area bounded by the  $y$ -axis, the line  $y = \pi/2$  and the curve  $x = \sin y$  about the  $y$ -axis. (Note: *Not* about the  $x$ -axis.) Assume that the unit of length is feet.





- (a) Write a definite integral which gives the volume of this tank. You need not evaluate the integral.
- (b) The tank is being filled with water at the rate of 7 cubic feet per minute. How fast is the water level  $h$  increasing when  $h = \pi/3$  feet? Hint: What is the volume of the water when the water level is  $h$ ?

## Calculus 221 Quiz

Monday Sep 15, 1997 (25 Minutes)

---

1  $\frac{z}{a\left(\frac{x}{y}\right)} =$

a)  $\frac{z}{axy}$    b)  $\frac{yz}{ax}$    c)  $\frac{az}{xy}$    d)  $\frac{ayz}{x}$

e) None of the above.

Answer: (b)  $\frac{z}{a\left(\frac{x}{y}\right)} = \frac{z}{a} \cdot \frac{y}{\left(\frac{x}{y}\right)} = \frac{zy}{ax}$ .

---

2  $\frac{(a+3)^8}{(a+3)^2} =$

a)  $(a+3)^4$    b)  $(a+1)^6$    c)  $a^6 + 3^6$    d)  $a^4 + 3^4$

e) None of the above.

Answer: (e)  $\frac{(a+3)^8}{(a+3)^2} = (a+3)^6$ .

---

2  $\frac{2w}{3} - \frac{6t}{w} =$

a)  $\frac{2w-6t}{3-w}$    b)  $\frac{2w-6t}{3w}$    c)  $\frac{2(w-t)}{3w}$    d)  $2w^2 - 18t$

e) None of the above.

Answer: (e)  $\frac{2w}{3} - \frac{6t}{w} = \frac{2w^2}{3w} - \frac{18t}{3w} = \frac{2w^2 - 18t}{3w}$ .

---

4  $\lim_{x \rightarrow 1} \frac{x-1}{x^2-1} =$

a) 0   b) 1   c)  $\infty$    d)  $\frac{1}{2}$

e) None of the above.

Answer: (d)  $\lim_{x \rightarrow 1} \frac{x-1}{x^2-1} = \lim_{x \rightarrow 1} \frac{x-1}{(x-1)(x+1)} = \lim_{x \rightarrow 1} \frac{1}{x+1} = \frac{1}{2}$ .

---

5  $\frac{18vw^2}{(3v^2w^2)^3} =$

a)  $\frac{6}{v^3}$    b)  $\frac{6}{v^4w^3}$    c)  $\frac{2}{3v^4w^3}$    d)  $\frac{2}{3v^5w^4}$

e) None of the above.

Answer: (d)  $\frac{18vw^2}{(3v^2w^2)^3} = \frac{18vw^2}{27v^6w^6} = \frac{2}{3v^5w^4}$ .

---

6  $\sqrt{4a^2 + 16} =$

a)  $2\sqrt{a+2}$    b)  $2\sqrt{a^2+4}$    c)  $2a+4$    d)  $a\sqrt{20}$

e) None of the above.

Answer: (b)  $\sqrt{4a^2 + 16} = \sqrt{4(a^2 + 4)} = \sqrt{4}\sqrt{a^2 + 4} = 2\sqrt{a^2 + 4}$ .

*The following two questions are related.*

---

7  $\frac{x - \frac{4}{x}}{2 - \frac{4}{x}} =$

a)  $\frac{x}{2}$    b)  $\frac{x+2}{2}$    c)  $\frac{x^2-1}{2x-1}$    d)  $\frac{(x^2-4)(2x-4)}{x^2}$

e) None of the above.

Answer: (b)  $\frac{x - \frac{4}{x}}{2 - \frac{4}{x}} = \frac{x - \frac{4}{x}}{2 - \frac{4}{x}} \cdot \frac{x}{x} = \frac{x^2 - 4}{2x - 4} = \frac{(x+2)(x-2)}{2(x-2)} = \frac{x+2}{2}$ .

8  $\lim_{x \rightarrow 2} \frac{x - \frac{4}{x}}{2 - \frac{4}{x}} =$

a) 1   b)  $\infty$    c) 2   d) Does not exist

e) None of the above.

Answer: (c)  $\lim_{x \rightarrow 2} \frac{x - \frac{4}{x}}{2 - \frac{4}{x}} = \lim_{x \rightarrow 2} \frac{x+2}{2} = \frac{2+2}{2} = 2$ .

*The following two questions are related.*

9  $(2x - a)(x^{-1} - 2a^{-1}) =$

a) 0   b) 4   c)  $\frac{2x-a}{x-2a}$    d)  $-\frac{(2x-a)^2}{ax}$

e) None of the above.

Answer: (d)  $(2x-a)(x^{-1}-2a^{-1}) = (2x-a)(x^{-1}-2a^{-1}) \cdot \frac{ax}{ax} = \frac{(2x-a)(a-2x)}{ax} = -\frac{(2x-a)^2}{ax}$ .

10  $\lim_{x \rightarrow a} (2x - a)(x^{-1} - 2a^{-1}) =$

a) 0   b) 4   c) Does not exist   d)  $\infty$

e) None of the above.

Answer: (e)  $\lim_{x \rightarrow a} (2x - a)(x^{-1} - 2a^{-1}) = \lim_{x \rightarrow a} -\frac{(2x - a)^2}{ax} = -\frac{(2a - a)^2}{a^2} = -\frac{a^2}{a^2} = -1.$

---

11  $\lim_{x \rightarrow 1} \left( \frac{1}{x - 1} - \frac{2}{x^2 - 1} \right) =$

a) 0   b)  $\frac{1}{x + 1}$    c)  $\infty$    d)  $\frac{1}{2}$

e) None of the above.

Answer: (d)  $\lim_{x \rightarrow 1} \left( \frac{1}{x - 1} - \frac{2}{x^2 - 1} \right) = \lim_{x \rightarrow 1} \left( \frac{x + 1}{x^2 - 1} - \frac{2}{x^2 - 1} \right) = \lim_{x \rightarrow 1} \frac{x - 1}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{1}{x + 1} = \frac{1}{2}.$

---

12  $\lim_{x \rightarrow \infty} \frac{2x + 3}{5x + 6} =$

a) 0   b)  $1/2$    c)  $\infty$    d) Does not exist

e) None of the above.

Answer: (e)  $\lim_{x \rightarrow \infty} \frac{2x + 3}{5x + 6} = \lim_{x \rightarrow \infty} \frac{2 + \frac{3}{x}}{5 + \frac{6}{x}} = \frac{2}{5}.$

*Read carefully. The following two questions are different.*

---

13  $\lim_{h \rightarrow 0} \frac{\sqrt{2 + h} - \sqrt{2}}{h} =$

a)  $\sqrt{2}$    b)  $\frac{1}{2\sqrt{2}}$    c)  $\infty$    d) 0

e) None of the above.

Answer: (b)  $\lim_{h \rightarrow 0} \frac{\sqrt{2 + h} - \sqrt{2}}{h} = \lim_{h \rightarrow 0} \frac{(\sqrt{2 + h} - \sqrt{2}) \cdot (\sqrt{2 + h} + \sqrt{2})}{h(\sqrt{2 + h} + \sqrt{2})} = \lim_{h \rightarrow 0} \frac{(2 + h) - 2}{h(\sqrt{2 + h} + \sqrt{2})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{2 + h} + \sqrt{2}} = \frac{1}{2\sqrt{2}}.$

---

**14**  $\lim_{h \rightarrow \infty} \frac{\sqrt{2+h} - \sqrt{2}}{h} =$

a)  $\sqrt{2}$  b)  $\frac{1}{2\sqrt{2}}$  c)  $\infty$  d)  $0$

e) None of the above.

Answer: (d)  $\lim_{h \rightarrow \infty} \frac{\sqrt{2+h} - \sqrt{2}}{h} = \lim_{h \rightarrow \infty} \sqrt{\frac{2}{h^2} + \frac{1}{h}} - \sqrt{\frac{2}{h^2}} = \sqrt{0+0} - \sqrt{0} = 0.$

---

**15**  $\lim_{x \rightarrow \infty} \cos\left(\frac{1}{x}\right) =$

a)  $0$  b)  $1$  c)  $\infty$  d) Does not exist

e) None of the above.

Answer: (b)  $\lim_{x \rightarrow \infty} \cos\left(\frac{1}{x}\right) = \cos\left(\lim_{x \rightarrow \infty} \frac{1}{x}\right) = \cos(0) = 1$

---

**16**  $\lim_{x \rightarrow \infty} \frac{\cos x}{x} =$

a)  $0$  b)  $1$  c)  $\infty$  d) Does not exist

e) None of the above.

Answer: (a)  $\lim_{x \rightarrow \infty} \frac{\cos x}{x} = 0$  by the Squeeze Theorem as

$$-\frac{1}{x} \leq \frac{\cos x}{x} \leq \frac{1}{x}$$

for  $x > 0$ .

## Calculus 221 Exam

Tuesday October 7, 1997

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**I** Find  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  if  $y = \frac{1}{1+x^2}$ .

Answer:

---

**II** What is  $\lim_{x \rightarrow 1} \frac{x^{17} - 1}{x - 1}$ ?

Answer:

---

**III** Find  $f'(t)$  and  $f''(t)$  if  $f(t) = (1 + t^p)^q$ . (Here  $p$  and  $q$  are constants.)

Answer:

---

**IV** A function  $y = f(x)$  is implicitly given by the equation

$$y^3 + y = 2x^2$$

Note that the points  $(x, y) = (-1, 1)$ ,  $(0, 0)$ ,  $(1, 1)$  lie on the graph. Complete the following table:

$x$	-1	0	1
$f(x)$			
$f'(x)$			
$f''(x)$			

Answer:

---

**V** A tank holds 600 cubic feet of water which drains from the bottom in 10 minutes. According to Torricelli's Law, the volume of water remaining in the tank after  $t$  minutes is

$$V = 600 \left(1 - \frac{t}{10}\right)^2.$$

(a) How fast is the volume decreasing at time  $t = 2$ ?

Answer:

(b) The tank is a cylinder whose base has area 100 square feet. Find the water level  $h$  and its rate of decrease at time  $t = 2$ .

Answer:

---

**VI** The functions  $f(x)$  and  $g(x)$  are differentiable and the following table gives some of the values of  $f(x)$  and  $g(x)$  and their derivatives. Use these data to answer questions (a) - (e) below. If there is not enough information given to determine some answer, write “INSUFFICIENT DATA”.

$x$	2	3	5	7
$f(x)$	4	7	2	-2
$f'(x)$	3	5	-1	-3
$g(x)$	3	3	7	2
$g'(x)$	-2	5	3	2

(a) Find  $f(g(x^2 + 1))$  when  $x = 2$ .

Answer:

(b) Find  $f'(x^2 + 1)$  when  $x = 2$ .

Answer:

(c) Find  $\left. \frac{d}{dx} f(x^2 + 1) \right|_{x=2}$ .

Answer:

(d) Find  $\left. \frac{d}{dx} f(g(x)) \right|_{x=2}$ .

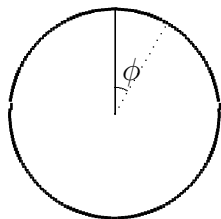
Answer:

(d) Find the equation for the tangent line to  $y = f(x)$  at the point  $P(2, f(2))$ .

Answer:

**VII** The entire length of the underside of the hour hand of a clock is coated with a thick blue dye which rubs off on the face of the clock leaving a circular blue sector. The length of the hour hand is 7 inches. The minute hand of the clock has been removed. The clock is started a noon with a clean face.

(a) How fast is the area of blue sector increasing two hours later?



Answer:

(b) How fast is the distance from the tip to the top increasing two hours later? (“tip” means the end of the hour hand, “top” is the position of the tip at noon. Both tip and top are always 7 inches from the center of the clock.)

Answer:

---

**VIII** Prove that the derivative of the sine function is the cosine function.

In your proof you may use the Limit Laws and high school algebra without further justification. You need not prove that the sine and cosine functions are continuous. You may use without proof the formulas

$$\lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1, \quad \lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} = 0.$$

Answer:

## Calculus 221 Exam

Thursday October 23, 1997

---

**I** (1) Find  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  if  $y = e^{-x^2}$ .

Answer:

(2) Find  $f'(x)$  if  $f(x) = \log_5(x^2 - x)$

Answer:

---

**II** Evaluate each limit. Simplify your answers; i.e. do not leave inverse trig functions in the final form.



(1)  $\lim_{x \rightarrow 1} \cos^{-1} \left( \frac{x}{x+1} \right)$

Answer:

(2)  $\lim_{x \rightarrow \infty} \tan^{-1}(x^4)$

Answer:

(3)  $\lim_{u \rightarrow \infty} \frac{e^u + e^{4u}}{e^{2u} + e^{3u}}$

Answer:

---

**III** Find the quadratic approximation to  $f(x) = \ln(x)$  near 1 and use it to approximate the number  $\ln(1.1)$ .

Answer:

---

**IV** If  $g$  is the inverse function to  $f(x) = x + x^2 + e^x$ , find  $g'(1)$ .

---

**V** Radioactive strontium has a half life of 25 years.

(1) Give a formula for the mass of strontium that remains from a sample of 18 grams after  $t$  years.

Answer:

(2) How long would it take for the mass to decay to 2 grams?

Answer:

(3) What is the rate of decay (in grams per year) when there are 2 grams left?

Answer:

---

**VI** Let  $C(x) = e^{3x} + e^{-3x}$  and  $S(x) = e^{3x} - e^{-3x}$ .

(1) Express  $S'(x)$  in terms of  $C(x)$ .

Answer:

(2) Draw the graph of  $y = C(x)^2 - S(x)^2$ .

Answer:

---

**VII** Let  $f(x) = \arctan(x)$ . State and derive the formula for  $f'(x)$ . You may assume without proof the formula for the derivative of the tangent function. (Note:  $\arctan(x) = \tan^{-1}(x) \neq (\tan x)^{-1}$ .)

Answer:

---

**VIII** An extraterrestrial bug swarm occupies a spherical ball in orbit about the planet Venus. The number of bugs grows exponentially at a rate of 9% per hour, i.e. the growth constant is  $k = 0.09$ . The volume of the spherical bug ball is proportional to the number of bugs in the ball, the density of bugs being  $c = 10^6$  bugs per cubic meter. There are initially  $N_0 = 10^{15}$  bugs present.

(1) Write a formula for the number  $N$  of bugs present after  $t$  hours.

Answer:

(2) Write formulas for the volume  $V$ , the radius  $R$ , and the surface area  $A$  of the bug sphere after  $t$  hours.

Answer:

(3) How fast is the surface area of the bug sphere increasing after 3 hours?

Answer:

## Calculus 221 Exam

Thursday November 20, 1997

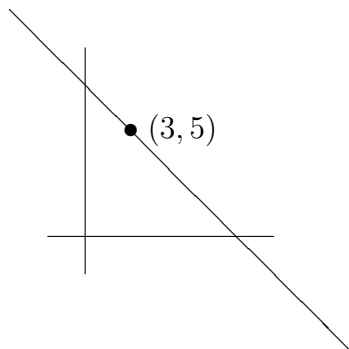
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**I** Graph  $f(x) = \frac{3}{x} - \frac{1}{x^3}$  for  $x > 0$ . Indicate all horizontal and vertical asymptotes, critical points, and points of inflection.

Answer:

---

**II** Find an equation for the line through the point  $(3, 5)$  that cuts off the least area from the first quadrant.



Answer:

---

**III** (a) Evaluate  $\int_0^6 |2x-4| dx$  by interpreting the definite integral as an area.

Answer:

(b) Evaluate  $\int_0^{1/2} \sqrt{1-x^2} dx$  by interpreting the definite integral as an area. Hint: Sketch  $y = \sqrt{1-x^2}$  and write the area in question as the area of a sector plus the area of a triangle.

Answer:

---

**IV** State the Mean Value Theorem and use it to prove that a function whose derivative is positive on an interval is increasing on that interval.

Answer:

---

**V** (a) Show that  $\int_1^3 \frac{1}{x} dx \leq \frac{1}{2} \left(1 + \frac{2}{3} + \frac{1}{2} + \frac{2}{5}\right)$ . (Give a picture and an explanation.)

Answer:

(b) Find a number  $A > 2/3$  such that  $A \leq \int_1^3 \frac{1}{x} dx$ . (Justify your answer.)

Answer:

---

**VI** (a) Find  $\int_2^9 (3f(x) - 4g(x)) dx$  given that

$$\int_2^7 f(x) dx = 3, \quad \int_7^9 f(x) dx = 4, \quad \int_2^9 g(x) dx = 2,$$

Answer:

(b) Express the limit  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{3}{n} \ln \left(5 + \frac{3k}{n}\right)$  as a definite integral.

Answer:

(c) Is the finite sum  $\sum_{k=1}^n \frac{3}{n} \ln \left( 5 + \frac{3k}{n} \right)$  bigger or smaller than the integral in part (b)? (Give reason.)

Answer:

---

**VII** Find  $f(x)$  if  $f'(x) = x^3 - 1$  and the absolute minimum value of  $f(x)$  is 0.

Answer:

## Calculus 221 Quiz

Wednesday December 10, 1997 (30 Minutes)

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**I** (a) Evaluate  $F(x) = \int_0^x \frac{t \, dt}{1 + t^2}$ .

Answer:

(b) Find  $F'(x)$ .

Answer:

---

**II** Find the volume of the solid obtained by rotating about the  $y$ -axis the region bounded on the left by the line  $x = 3$ , on the right by the line  $x = 4$ , below by the line  $y = 1$ , and above by the curve  $y = (25 - x^2)^{1/3}$ .

Answer:

---

**III** (a) Find  $\frac{d}{dx} \int_{x^2}^{x^3} \sin(t^4) \, dt$ .

Answer:

(b) Find  $\int_{t^2}^{t^3} \frac{d}{dx} \sin(x^4) \, dx$ .

Answer:

# Calculus 221 Final Exam

Monday December 15, 1997

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**I** (a) Find  $\frac{dy}{dx}$  when  $y = \ln(2 + e^x)$ .

Answer:

(b) Find  $\frac{d^2y}{dx^2}$  for  $y$  as in (a).

Answer:

---

**II** (a) Evaluate  $\int_2^3 \frac{x^2 dx}{x^3 - 1}$ .

Answer:

(b) Evaluate  $\lim_{x \rightarrow 3} \left( \frac{1}{x-3} \int_3^x \frac{\sin(t)}{t} dt \right)$ .

Answer:

---

**III** A population of bacteria triples in three hours. Assuming exponential growth, how long does it take to double?

Answer:

---

**IV** Find the points on the hyperbola  $y^2 - x^2 = 4$  which are closest to the point  $(2, 0)$ .

Answer:

---

**V** (a) Find the equation for the tangent line to the curve  $y^2 = x^3 + 3$  at the point  $(x, y) = (1, 2)$ .

Answer:

(b) Is this tangent line above the curve? Why or why not?

Answer:

---

**VI** (a) Evaluate  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \left[ 3 \left( 1 + \frac{2i}{n} \right)^5 - 6 \right] \frac{2}{n}$ .

Answer:

(b) Evaluate  $\lim_{x \rightarrow \infty} \tan^{-1}(x)$ .

Answer:

---

**VII** The velocity function for a particle moving along a line is

$$v(t) = 3t - 12, \quad 0 \leq t \leq 5.$$

(a) Find the displacement (net change in position) from  $t = 0$  to  $t = 5$ .

Answer:

(b) Find the total distance travelled from  $t = 0$  to  $t = 5$ . (Note that the velocity changes sign.)

Answer:

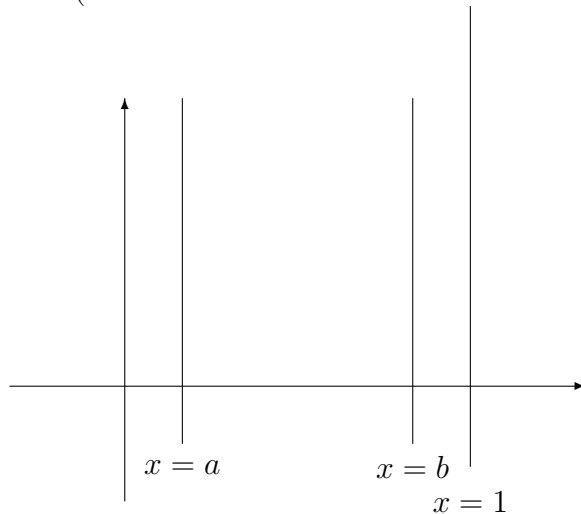
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**VIII** Find the interval on which the curve  $y = \int_0^x \frac{dt}{1+t+t^2}$  is concave up.

Answer:

---

**IX** Find the volume generated by revolving the region bounded by the lines  $x = a$ ,  $x = b$ ,  $y = 0$  and the curve  $y = \sqrt{1-x^2}$  about the  $x$ -axis. (Assume that  $a$  and  $b$  are constants and  $0 < a < b < 1$ .)



Answer:

---

**X** State and prove the formula for the derivative of the inverse sine function  $\sin^{-1}(x)$ . You may assume without proof that the derivative of the sine function is the cosine function.

Answer:

---

**EXTRA CREDIT** A high speed train accelerates at  $(1/2)$  meter/sec<sup>2</sup> until it reaches its maximum cruising speed of 30 meters per second; after it reaches this maximum cruising speed it remains at that speed. If it starts from rest, how far will it go in 10 minutes?

## Calculus 221 - First Exam

Tuesday October 6, 1998

---

**I** Find the limit. If the limit does not exist, write DNE. (Distinguish between a limit which is infinite and one which does not exist.)

a)  $\lim_{x \rightarrow \infty} \frac{5x^2 + 6}{7x^2 + 8x + 9} =$

Answer:

b)  $\lim_{x \rightarrow 3} \frac{x - \frac{9}{x}}{1 - \frac{3}{x}} =$

Answer:

c)  $\lim_{x \rightarrow \infty} \cos\left(\frac{1}{x}\right) =$

Answer:

d)  $\lim_{x \rightarrow \infty} \frac{\cos x}{x} =$

Answer:

---

**II** Suppose that  $f(x) = \sqrt{25 - x^2}$ .

a) Find the first derivative  $f'(x)$ .

Answer:

b) Find the second derivative  $f''(x)$ .

Answer:

c) Write an equation for the tangent line to the curve  $y = f(x)$  at the point  $(3, 4)$ .

Answer:

---

III Suppose that  $u(x)$  and  $v(x)$  are differentiable. Let

$$f(x) = \frac{u(x)}{v(x)}$$

Prove the quotient rule:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}$$

Answer:

---

IV Find the limit. If the limit does not exist, write DNE. (Distinguish between a limit which is infinite and one which does not exist.)

Answer:

a)  $\lim_{x \rightarrow \pi} \tan(x) =$

Answer:

b)  $\lim_{x \rightarrow \pi/2^-} \tan(x) =$

Answer:

c)  $\lim_{x \rightarrow \infty} \tan(x) =$

Answer:

d)  $\lim_{x \rightarrow -\pi/2} \tan(x) =$

Answer:

e)  $\lim_{x \rightarrow \pi/4} \left( \tan(x) + \frac{1}{\tan(x)} \right) =$

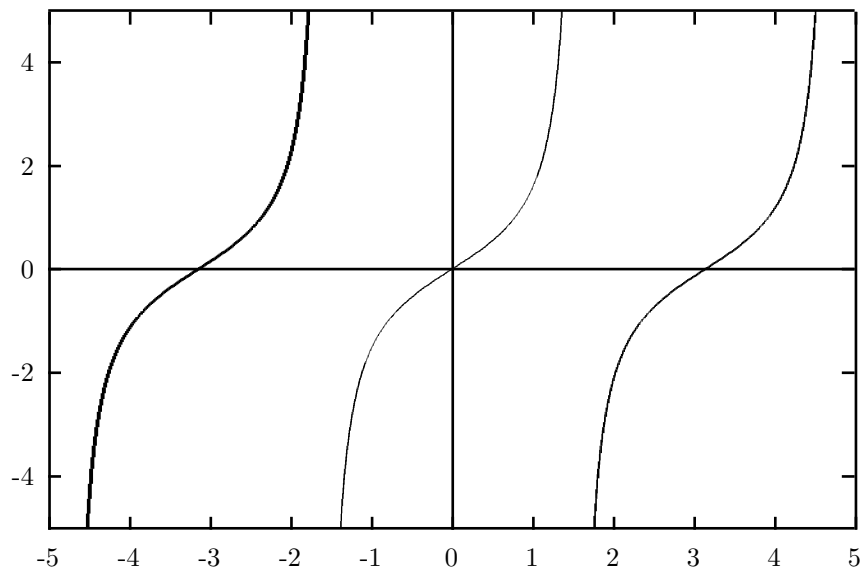
Answer:

f)  $\lim_{x \rightarrow \pi/2} \frac{1}{\tan(x)} =$

Answer:

Hint: Below is part of the graph of  $y = \tan(x)$ .






---

**V** Find an equation for the tangent line to the curve  $y^5 + y - x = 0$  at the point  $(2, 1)$ .

Answer:

---

**VI** A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at the rate of 7 feet per minute, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 feet from the wall?

Answer:

---

**VII a)** State the definition of the derivative  $f'(x)$  of a function  $f(x)$ .

Answer:

**b)**  $(125)^{4/3} = ?$  (Circle one.)

5    25    125    625    3125    none of these

Answer:

**c)** Use your knowledge of differentiation to estimate the number

$$\frac{(127)^{4/3} - (125)^{4/3}}{2}.$$

approximately without a calculator. Your answer should have the form  $p/q$  where  $p$  and  $q$  are integers.

Answer:

## Calculus 221 Exam

Thursday November 12, 1998

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**I** Graph the function  $f(x) = e^{x^3/3}$ . Be sure to indicate all asymptotes, all critical points, all points of inflection, all intervals where the function is increasing, decreasing, concave up, concave down.

Answer:

---

**II** Consider the function  $f(x) = e^{x^3/3}$  from the previous problem. Is there a function  $g(y)$  such that

$$y = f(x) \iff x = g(y)?$$

(In other words, does the function  $f$  have an inverse function?) If yes, give a formula for  $g(y)$ . If no, say why not.

Answer:

---

**III** A population of bacteria doubles every two days. How long does it take to triple? (The population grows exponentially.)

Answer:

---

**IV** The simplest version of l'Hospital's rule says that

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

whenever

- (1) the functions  $f(x)$  and  $g(x)$  are differentiable,
- (2) the derivatives  $f'(x)$  and  $g'(x)$  are continuous,
- (3)  $f(a) = g(a) = 0$ , and

(4)  $g'(a) \neq 0$ .

Prove this. Give a reason for each step.

Answer:

---

**V** Evaluate the following limits:

(a)  $\lim_{x \rightarrow 0} \frac{\sin x}{e^x}$

Answer:

(b)  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$

Answer:

(c)  $\lim_{x \rightarrow 0} (1 - 2x)^{1/x}$

Answer:

---

**VI** A cylindrical can is made to hold 50 cubic inches of soup. Find the dimensions that will minimize the area of the can. Don't forget that the top and bottom of the can have area as well as the side. (The area of the side is the perimeter of the circular top times the height of the can.)

Answer:

---

**VII** The hyperbolic functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

satisfy the identity

$$\sinh^2(x) - \cosh^2(x) = \text{constant}$$

Specify the constant and prove this identity.

Answer:

(b) Find the domain and range of the function  $f(x) = \cosh(x)$ .

Answer:

(c) Is there a function  $g$  such that

$$y = \cosh(x) \iff x = g(y)?$$

(In other words, does the function  $f(x) = \cosh(x)$  have an inverse function?) Why or why not? (The domain of  $g$  should be the range of  $f$  and the range of  $g$  should be the domain of  $f$ .)

Answer:

# Calculus 221 - Final Exam

Tuesday, December 22, 1998

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**I** Evaluate the following:

(a)  $\frac{dy}{dx}$  where  $y = \sqrt{1 - x^2}$ .

Answer:

(b)  $\frac{dy}{dx}$  where  $3x^2 + xy + y^2 = 9$ .

Answer:

(c)  $\lim_{n \rightarrow \infty} \frac{2^n}{n^2}$ .

Answer:

---

**II** Evaluate the following:

(a)  $\int_3^5 x^2 dx$ .

Answer:

(b)  $\int_3^5 \frac{dx}{1 + x^2}$ .

Answer:

(c)  $\int_3^5 \frac{x dx}{1 + x^2}$ .

Answer:

---

**III** Set up, BUT DO NOT EVALUATE, an integral for the volume of the solid obtained by rotating the region bounded by the curves

$$y = x - 1, \quad y = (x - 4)^2 + 1,$$

about the line  $y = 7$ . BE SURE TO SPECIFY THE LIMITS OF INTEGRATION IN YOUR INTEGRAL.

---

**IV** (a) Evaluate  $\lim_{h \rightarrow 0} \frac{1}{h} \int_2^{2+h} \frac{dx}{1 + x^3}$ .

Answer:

(b) Evaluate the derivative  $F'(x)$  of the function

$$F(x) = \int_{x^2}^{\sin x} \frac{dt}{1 + t^3}.$$

Answer:

---

**V** The velocity  $v$  of a particle at time  $t$  is

$$v = e^{-t^2}t.$$

How far does it go from time  $t = 3$  to time  $t = 5$ ?

---

**VI** Evaluate  $\sum_{k=32}^{487} (2 + 7k)^2$ . Your answer should not contain the summation symbol  $\sum$ , but you may leave any other arithmetic undone. You may use the formulas

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$$

---

**VII** Write a Riemann sum with 100 terms which approximates  $\int_3^5 \sin(x) dx$ . You need not simplify your answer. It should begin with  $\sum_{k=1}^{100}$ .

---

**VIII** (a) A hemispherical bowl of radius 10 inches is filled with water to a height of  $h$  inches. How many cubic inches of water are in the bowl?

Answer:

(b) If water is being added at a rate of 3 cubic inches per minute, how fast is the water level  $h$  rising when the water level is 7 inches?

Answer:

---

**IX** Graph  $y = x^4 - 2x^2$  after answering the following questions:

(a) On what intervals is the function increasing?

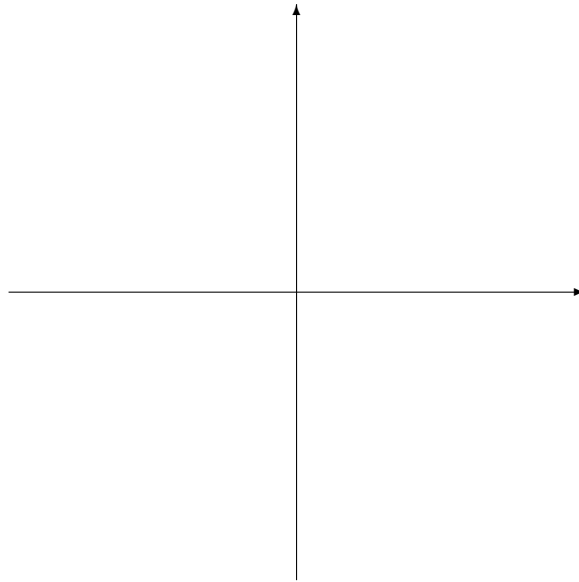
Answer:

(b) On what intervals is the function concave up?

Answer:

(c) What are the vertical and horizontal asymptotes?

Answer:



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**X** Prove that if

$$w(x) = u(x)v(x)$$

and  $u(x)$  and  $v(x)$  are differentiable, then

$$w'(x) = u'(x)v(x) + u(x)v'(x).$$

In your proof you may use (without proof) the limit laws and the fact that a differentiable function is continuous; however, you should indicate where these facts are used in your proof.

---

**XI** The population of California grows exponentially at a rate of 2% per year. The population of California on January 1, 1990 was 20,000,000.

(a) Write a formula for the population  $N(t)$  of California  $t$  years after January 1, 1990.

Answer:

(b) Each Californian consumes pizzas at the rate of 70 pizzas per year. At what rate is California consuming pizzas  $t$  years after 1990?

Answer:

(c) How many pizzas are consumed in California from January 1, 1995 to January 1, 1999?

Answer: