The Differentiation Laws

(Proofs)

September 21, 2000

§1 The notation

$$\lim_{x\to a} F(x) = L$$

is read "the limit of $F(x)$ as x approaches a is L." It means $F(x)$ gets closer and closer to L as x gets closer and closer to a. Sometimes the textbook writes this as

$$F(x) \to L \qquad \text{as} \qquad x \to a;$$

this is read as "$F(x)$ approaches L as x approaches a" or "$F(x)$ goes to L as x goes to a". I like to explain limits by writing

$$F(x) \approx L \quad \textit{when} \quad x \approx a.$$

The notation $A \approx B$ means A is approximately equal to B.

§2 Here are two ways of writing the definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}, \tag{1}$$

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}. \tag{2}$$

To use equation (1) to estimate $f'(4)$ we might take $x = 4$ and $h = 0.001 \approx 0$, so $x + h = 4.001$ and

$$f'(4) = f'(x) \approx \frac{f(x+h) - f(x)}{h} = \frac{f(4.001) - f(4)}{0.001}.$$

To use equation (2) to estimate $f'(4)$ we might take $a = 4$ and $x = 4.001 \approx 4$, so $x - a = 0.001$ and

$$f'(4) = f'(a) \approx \frac{f(x) - f(a)}{x - a} = \frac{f(4.001) - f(4)}{0.001}.$$

This illustrates that the two formulas are different ways of expressing the same thing. In equation (1) the variable h is a **dummy variable** whereas the variable x is a **free variable**. If all occurrences of dummy variable in an expression are changed to another letter, the meaning of the expression is unchanged:

$$\lim_{k\to 0} \frac{f(x+k) - f(x)}{k} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

A free variable is used to assert that an equation is valid for a range of values; if the free variable is changed on one side of an equation it must be changed on the other side as well. Thus (1) could be written

$$f'(y) = \lim_{h\to 0} \frac{f(y+h) - f(y)}{h}.$$

One can substitute a number for a free variable as in

$$f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h},$$

but substituting a number for a dummy variable yields nonsense. In equation (2), x is the dummy variable and a is the free variable.

§3 Definition. The **derivative** $f'(a)$ is defined by

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}. \tag{D}$$

A function is **differentiable** at a iff this limit exists and is finite. A function is differentiable iff it is differentiable at every a in its domain. A function f is **continuous** at a iff

$$\lim_{x\to a} f(x) = f(a). \tag{C}$$

A function is continuous iff it is continuous at every a in its domain. A function is continuous on a set (e.g. interval) I iff it is defined[1] and continuous at every point a of I.

[1] i.e. I is a subset of the domain of f

§4 The Limit Laws. These are

$$\lim_{x\to a} c = c \qquad \text{constant law}$$

$$\lim_{x\to a} x = a \qquad \text{identity law}$$

$$\lim_{x\to a}(F(x)+G(x)) = \lim_{x\to a} F(x) + \lim_{x\to a} G(x) \qquad \text{sum law}$$

$$\lim_{x\to a}(F(x)-G(x)) = \lim_{x\to a} F(x) - \lim_{x\to a} G(x) \qquad \text{difference law}$$

$$\lim_{x\to a}(F(x)\cdot G(x)) = \lim_{x\to a} F(x) \cdot \lim_{x\to a} G(x) \qquad \text{product law}$$

$$\lim_{x\to a} \frac{F(x)}{G(x)} = \frac{\lim_{x\to a} F(x)}{\lim_{x\to a} G(x)} \qquad \text{quotient law}$$

$$\lim_{x\to a} g(F(x)) = g\left(\lim_{x\to a} F(x)\right) \qquad \text{continuity law}$$

assuming that the limits on the right exists. (In the constant law c denotes a constant function, i.e. the function whose value is always the number c. In the quotient law we must also assume that the limit in the denominator is nonzero. In the continuity law the function g is continuous.)

§5 Theorem. *A differentiable function is continuous.*

Proof: Assume that the limit (D) exists. Then

$$
\begin{aligned}
\left(\lim_{x\to a} f(x)\right) - f(a) &= \left(\lim_{x\to a} f(x)\right) - \left(\lim_{x\to a} f(a)\right) && \text{(const)} \\
&= \lim_{x\to a}(f(x)-f(a)) && \text{(diff)} \\
&= \lim_{x\to a} \frac{f(x)-f(a)}{x-a}\cdot(x-a) && \text{(hsa)} \\
&= \lim_{x\to a} \frac{f(x)-f(a)}{x-a}\cdot \lim_{x\to a}(x-a) && \text{(prod)} \\
&= \left(\lim_{x\to a} \frac{f(x)-f(a)}{x-a}\right)\cdot 0 && \text{(const, diff, ident)} \\
&= f'(a)\cdot 0 && \text{(definition)} \\
&= 0 && \text{(hsa)}
\end{aligned}
$$

so Equation (C) holds. (In this proof *const* means *constant law*, *diff* means *difference law*, *ident* means *identity law*, *prod* means *product law*, and *hsa* means *high school algebra.*)

§6 The differentiation formulas are used to differentiate cu, $u+v$, $u-v$, $u \cdot v$, and u/v when you know how to differentiate u and v. Where c is a constant, u and v are functions, and $'$ denotes differentiation the differentiation formulas are

$$c' = 0 \qquad \text{(constant formula)}$$

$$(u \pm v)' = u' \pm v' \qquad \text{(sum formula)}$$

$$(uv)' = u'v + uv' \qquad \text{(product formula)}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} \qquad \text{(quotient formula)}$$

The constant formula and the product formula imply

$$(cu)' = cu'$$

Using the product formula repeatedly gives

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

Using the quotient formula (with numerator 1) proves that the formula works for negative integers. The quotient formula can be derived[2] from the product formula as follows: if $w = u/v$ then $wv = u$ so $w'v + wv' = u'$ so

$$w' = \frac{u' - wv'}{v} = \frac{u' - (u/v)v'}{v} = \frac{u'v - uv'}{v^2}.$$

Using these formulas you can easily differentiate any rational function (i.e. ratio of polynomials).

[2]Unlike the argument below, this argument does not prove that w is differentiable if u and v are.

§7 Proof of the constant formula. Suppose that $f(x) = c$ for all x where c is a constant. Then

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} && \text{(definition)} \\ &= \lim_{x \to a} \frac{c - c}{x - a} && \text{(hypothesis)} \\ &= \lim_{x \to a} 0 && \text{(hsa)} \\ &= 0 && \text{(const)} \end{aligned}$$

§8 Derivative of identity function. Suppose that $f(x) = x$ for all x. Then

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} && \text{(definition)} \\ &= \lim_{x \to a} \frac{x - a}{x - a} && \text{(hypothesis)} \\ &= \lim_{x \to a} 1 && \text{(hsa)} \\ &= 1 && \text{(const)} \end{aligned}$$

§9 Proof of the sum formula. Suppose that $f(x) = u(x) + v(x)$ for all x where u and v are differentiable. Then

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} && \text{(definition)} \\
&= \lim_{x\to a} \frac{(u(x)+v(x))-(u(a)+v(a))}{x-a} && \text{(hypothesis)} \\
&= \lim_{x\to a} \left(\frac{u(x)-u(a)}{x-a} + \frac{v(x)-v(a)}{x-a} \right) && \text{(hsa)} \\
&= \lim_{x\to a} \frac{u(x)-u(a)}{x-a} + \lim_{x\to a} \frac{v(x)-v(a)}{x-a} && \text{(limit law)} \\
&= u'(a) + v'(a) && \text{(definition)}
\end{aligned}$$

§10 Proof of the product formula. Suppose that $f(x) = u(x)v(x)$ for all x where u and v are differentiable. Then

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} && \text{(definition)} \\
&= \lim_{x\to a} \frac{u(x)\cdot v(x)-u(a)\cdot v(a)}{x-a} && \text{(hypothesis)} \\
&= \lim_{x\to a} \left(\left(\frac{u(x)-u(a)}{x-a} \right) \cdot v(a) + u(x) \cdot \left(\frac{v(x)-v(a)}{x-a} \right) \right) && \text{(hsa)} \\
&= \left(\lim_{x\to a} \frac{u(x)-u(a)}{x-a} \right) \cdot v(a) + u(a) \cdot \left(\lim_{x\to a} \frac{v(x)-v(a)}{x-a} \right) && \text{(limit laws)} \\
&= u'(a)v(a) + v'(a)u(a) && \text{(definition)}
\end{aligned}$$

(In the fourth step the theorem that a differentiable function is continuous is also used.)

§11 Proof of the quotient formula. Suppose that $f(x) = u(x)/v(x)$ for all x where u and v are differentiable and $v(a) \neq 0$. Then

$$
\begin{aligned}
f'(a) = & \\
&= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} && \text{(definition)} \\
&= \lim_{x\to a} \frac{(u(x)/v(x)) - (u(a)/v(a))}{x - a} && \text{(hypothesis)} \\
&= \lim_{x\to a} \frac{u(x)v(a) - u(a)v(x)}{v(x)v(a)(x - a)} && \text{(hsa)} \\
&= \lim_{x\to a} \left(\frac{u(x) - u(a)}{x - a} \cdot \frac{v(a)}{v(x)v(a)} - \frac{u(a)}{v(x)v(a)} \cdot \frac{v(x) - v(a)}{x - a} \right) && \text{(hsa)} \\
&= \left(\lim_{x\to a} \frac{u(x) - u(a)}{x - a} \right) \cdot \frac{v(a)}{v(a)^2} - \frac{u(a)}{v(a)^2} \cdot \left(\lim_{x\to a} \frac{v(x) - v(a)}{x - a} \right) && \text{(lim law)} \\
&= u'(a) \cdot \frac{v(a)}{v(a)^2} - \frac{u(a)}{v(a)^2} \cdot v'(a) && \text{(definition)} \\
&= \frac{u'(a)v(a) - u(a)v'(a)}{v(a)^2} && \text{(hsa)}
\end{aligned}
$$

(In the fifth step the theorem that a differentiable function is continuous is also used.)

One of the following questions will appear on the first exam.

§12 Question. Define *differentiable function*, *continuous function*, and prove that a differentiable function is continuous. In your proof you may use (without proof) the limit laws and high school algebra. **Answer**: Write Equation (D), Equation (C), and the proof in §5.

§13 Question. State and prove the formula for the derivative of a constant function. In your proof you may use (without proof) the limit laws and high school algebra. **Answer**: Write what is in §7.

§14 Question. State and prove the formula for the derivative of the function $f(x) = x$. In your proof you may use (without proof) the limit laws and high school algebra. **Answer**: Write what is in §8.

§15 Question. State and prove the formula for the derivative of the sum of two functions. In your proof you may use (without proof) the limit laws and high school algebra. **Answer**: Write what is in §9.

§16 Question. State and prove the formula for the derivative of the product of two functions. In your proof you may use (without proof) the limit laws, the theorem that a differentiable function is continuous, and high school algebra. **Answer**: Write what is in §10.

§17 Question. State and prove the formula for the derivative of the quotient of two functions. In your proof you may use (without proof) the limit laws, the theorem that a differentiable function is continuous, and high school algebra. **Answer**: Write what is in §11.