# Calc Prep

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## Contents

# 1 Laws of algebra

**1.1. Terminology and Notation.** In this section we review the notations used in algebra. Some are peculiar to these notes. For example the notation $A := B$ indicates that the equality holds *by definition* of the notations involved. (See for example Paragraph 1.2 which follows.) Two other notations which will become important when we solve equations are $\implies$ and $\iff$. The notation $P \implies Q$ means that $P$ *implies* $Q$ i.e. *"If $P$, then $Q$"*. For example, $x = 2 \implies x^2 = 4$. (Note however that the converse statement $x^2 = 4 \implies x = 2$ is not always true since it might be that $x = -2$.) The notation $P \iff Q$ means $P \implies Q$ and $Q \implies P$, i.e. *"$P$ if and only if $Q$"*. For example $3x - 6 = 0 \iff x = 2$. The notations $\implies$ and $\iff$ are explained more carefully in Paragraphs 8.2 and 8.3 below.

**1.2. Implicit Multiplication.** In mathematics the absence of an operation symbol usually indicates multiplication: $ab$ mean $a \times b$. Sometimes a dot is used to indicate multiplication and in computer languages an asterisk is often used.

$$ab := a \cdot b := a * b := a \times b$$

**1.3. Order of operations.** Parentheses are used to indicate the order of doing the operations: in evaluating an expression with parentheses the innermost matching pairs are evaluated first as in

$$((1+2)^2+5)^2 = (3^2+5)^2 = (9+5)^2 = 14^2 = 196.$$

There are conventions which allow us not to write the parentheses. For example, multiplication is done before addition

$$ab + c \qquad \text{means } (ab) + c \text{ and not } a(b+c),$$

and powers are done before multiplication:

$$ab^2c \qquad \text{means } a(b^2)c \text{ and not } (ab)^2c.$$

In the absence of other rules and parentheses, the left most operations are done first.

$$a - b - c \qquad \text{means } (a-b) - c \text{ and not } a - (b - c).$$

The long fraction line indicates that the division is done last:

$$\frac{a+b}{c} \qquad \text{means } (a+b)/c \text{ and not } a + (b/c).$$

In writing fractions the length of the fraction line indicates which fraction is evaluated first:

$$\frac{a}{\frac{b}{c}} \qquad \text{means } a/(b/c) \text{ and not } (a/b)/c,$$

$$\frac{\frac{a}{b}}{c} \qquad \text{means } (a/b)/c \text{ and not } a/(b/c).$$

The length of the horizontal line in the radical sign indicates the order of evaluation:

$$\sqrt{a+b} \qquad \text{means } \sqrt{(a+b)} \text{ and not } (\sqrt{a}) + b.$$

$$\sqrt{a} + b \qquad \text{means } (\sqrt{a}) + b \text{ and not } \sqrt{(a+b)}.$$

**1.4. The Laws of Algebra.** There are four fundamental operations which can be performed on numbers.

1. Addition. The **sum** of $a$ and $b$ is denoted $a + b$.
2. Multiplication. The **product** of $a$ and $b$ is denoted $ab$.
3. Reversing the sign. The **negative** of $a$ is denoted $-a$.
4. Inverting. The **reciprocal** of $a$ (for $a \neq 0$) is denoted by $a^{-1}$ or by $\dfrac{1}{a}$.

These operations satisfy the following laws.

| | | |
|---|---|---|
| Associative | $a + (b + c) = (a + b) + c$ | $a(bc) = (ab)c$ |
| Commutative | $a + (b + c) = (a + b) + c$ | $a(bc) = (ab)c$ |
| Identity | $a + 0 = 0 + a = a$ | $a \cdot 1 = 1 \cdot a = a$ |
| Inverse | $a + (-a) = (-a) + a = 0$ | $a \cdot a^{-1} = a^{-1} \cdot a = 1$ |
| Distributive | $a(b + c) = ab + ac$ | $(a + b)c = ac + bc$ |

The operations of **subtraction** and **division** are then defined by

$$a - b := a + (-b) \qquad\qquad a \div b := \frac{a}{b} := a \cdot b^{-1} = a \cdot \frac{1}{b}.$$

All the rules of calculation that you learned in elementary school follow from the above fundamental laws. In particular, the Commutative and Associative Laws say that you can add a bunch of numbers in any order and similarly you can multiply a bunch of numbers in any order. For example,

$$(A+B)+(C+D) = (A+C)+(B+D), \qquad (A\cdot B)\cdot(C\cdot D) = (A\cdot C)\cdot(B\cdot D).$$

**1.5.** Because both addition and multiplication satisfy the commutative, associative, identity, and inverse laws, there are other analogies:

| | | |
|---|---|---|
| **(i)** | $-(-a) = a$ | $(a^{-1})^{-1} = a$ |
| **(ii)** | $-(a+b) = -a-b$ | $(ab)^{-1} = a^{-1}b^{-1}$ |
| **(iii)** | $-(a-b) = b-a$ | $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$ |
| **(iv)** | $(a-b)+(c-d) = (a+c)-(b+d)$ | $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ |
| **(v)** | $a-b = (a+c)-(b+c)$ | $\frac{a}{b} = \frac{ac}{bc}$ |
| **(vi)** | $(a-b)-(c-d) = (a-b)+(d-c)$ | $\frac{a/b}{c/d} = \frac{a}{b} \cdot \frac{d}{c}$ |

These identities are proved in the Guided Exercises. (An **identity** is an equation which is true for all values of the variables which appear in it.)

**1.6.** Here are some further identities which are proved using the distributive law.

| | |
|---|---|
| **(i)** $a \cdot 0 = 0$ | **(ii)** $-a = (-1)a$ |
| **(iii)** $a(-b) = -ab$ | **(iv)** $(-a)(-b) = ab$ |
| **(v)** $\frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd}$ | **(vi)** $(a+b)(c+d) = ab+ad+bc+bd$ |
| **(vii)** $(a+b)^2 = a^2+2ab+b^2$ | **(viii)** $(a+b)(a-b) = a^2-b^2$ |

These are also proved in the Guided Exercises.

**1.7.** The following **Zero-Product Property** will be used to solve equations.

$$pq = 0 \iff p = 0 \text{ or } q = 0 \text{ (or both).}$$

Proof: If $p = 0$ (or $q = 0$) then $pq = 0$ by (i) in Paragraph 1.6 . Conversely, if $p \neq 0$ then $q = p^{-1}pq = p^{-1}0 = 0$.

**Definition 1.8.** For a natural number $n$ and any number $a$ the $n$th **power** of $a$ is

$$a^n := \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}}$$

The zeroth power is

$$a^0 := 1$$

and negative powers are defined by

$$a^{-n} := \frac{1}{a^n}.$$

**1.9.** The following laws are easy to understand when $m$ and $n$ are integers. In Theorem 4.1 below we will learn that these laws also hold whenever $a$ and $b$ are positive real numbers and $m$ and $n$ are *any* real numbers, not just integers.

| | | | |
|---|---|---|---|
| **(i)** | $a^m a^n = a^{m+n}$ | e.g. | $a^2 a^3 = (aa)(aaa) = a^5$ |
| **(ii)** | $(a^m)^n = a^{mn}$ | e.g. | $(a^2)^3 = (aa)(aa)(aa) = a^6$ |
| **(iii)** | $\dfrac{a^m}{a^n} = a^{m-n}$ | e.g. | $\dfrac{a^2}{a^5} = a^{-3} = \dfrac{1}{a^3}$ |
| **(iv)** | $(ab)^m = a^m b^m$ | e.g. | $(ab)^2 = (ab)(ab) = (aa)(bb) = a^2 b^2$ |
| **(v)** | $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$ | e.g. | $\left(\dfrac{a}{b}\right)^2 = \dfrac{a}{b} \cdot \dfrac{a}{b} = \dfrac{aa}{bb} = \dfrac{a^2}{b^2}$ |

# 2 Kinds of Numbers

**2.1.** We distinguish the following different kinds of numbers.

- The **natural numbers** are $1, 2, 3 \ldots$.
- The **integers** are $\ldots - 3, -2, -1, 0, 1, 2, 3 \ldots$.
- The **rational numbers** are ratios of integers like $3/2$, $14/99$, $-1/2$.

- The **real numbers** are numbers which have an infinite decimal expansion like

$$\frac{3}{2} = 1.5000\ldots, \quad \frac{14}{99} = 0.141414\ldots, \quad \sqrt{2} = 1.4142135623730951\ldots.$$

- The **complex numbers** are those numbers of form $z = x + iy$ where $x$ and $y$ are real numbers and $i$ is a special new number called the **imaginary unit** which has the property that

$$\boxed{i^2 = -1;}$$

Every integer is a rational number (because $n = n/1$), every rational number is a real number (see Remark 2.3 below), and every real number is a complex number (because $x = x + 0i$). A real number which is not rational is called **irrational**.

**2.2.** Each kind of number enables us to solve equations that the previous kind couldn't solve:

- The solution of the equation $x + 5 = 3$ is $x = -2$ which is an integer but not a natural number.
- The solution of the equation $5x = 3$ is $x = \frac{3}{5}$ which is a rational number but not an integer.
- The equation $x^2 = 2$ has two solutions $x = \sqrt{2}$. The number $\sqrt{2}$ is a real number but not a rational number. (See Remark 2.5 below.)
- The equation $x^2 = 4$ has two real solutions $x = \pm 2$ but the equation $z^2 = -4$ has no real solutions because the square of a nonzero real number is always positive. However it does have two complex solutions, namely $z = \pm 2i$.

We will not use complex numbers until Section 22 but may refer to them implicitly as in

*The equation $x^2 = -4$ has no (real) solution.*

**Remark 2.3.** It will be proved in Theorem 21.6 that a real number is rational if and only if its decimal expansion eventually repeats periodically forever as in the following examples:

$$\frac{1}{3} = 0.3333\ldots, \qquad \frac{17}{6} = 2.83333\ldots,$$
$$\frac{7}{4} = 1.250000\ldots, \qquad \frac{1}{7} = 0.142857\,142857\,142857\ldots.$$

**Remark 2.4.** Unless the decimal expansion of a real number is eventually zero, as in $\frac{1}{2} = 0.5000\ldots$, any finite part of the decimal expansion is close to, but not exactly equal to, the real number. For example 1.414 is close to the square root of two but not exactly equal:

$$(1.414)^2 = 1.999396 \neq 2, \qquad (\sqrt{2})^2 = 2.$$

If we compute the square root to more decimal places we get a better approximation, but it still isn't exactly correct:

$$(1.4142135623730951)^2 = 2.0000000000000001448106923536440 1.$$

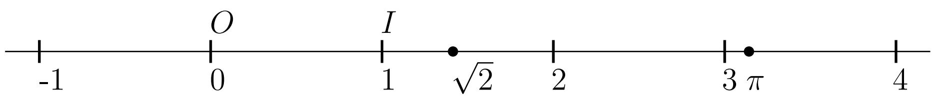
**Remark 2.5.** Here is a proof that the square root of 2 is irrational. If it were rational there would be integers $m$ and $n$ with

$$\left(\frac{m}{n}\right)^2 = 2.$$

By canceling common factors we may assume that $m$ and $n$ have no common factors and hence that they are not both even. Now $m^2 = 2n^2$ so $m^2$ is even so $m$ is even, say $m = 2p$. Then $4p^2 = (2p)^2 = m^2 = 2n^2$ so $2p^2 = n^2$ so $n^2$ is even so $n$ is even. This contradicts the fact $m$ and $n$ are not both even.

# 3 Coordinates on the Line and Order

**3.1.** The choice of two points $O$ and $I$ on a line $\ell$ determines a correspondence between the points of the line and the real numbers as indicated in the following picture.




The correspondence is called a **coordinate system** on the line $\ell$ and the line $\ell$ is called a **number line**. When the point $A$ corresponds to the number $a$ we say that the number $a$ is the **coordinate** of the point $A$. The **positive numbers** are the real numbers on the same side of 0 as 1 and the **negative numbers** are on the other side. We usually draw the number line as above so that it is horizontal and 1 is to the right of 0. We write say $a$ is **less than** $b$ and write $a < b$ $b$ is to the right of $a$, i.e. when $b - a$ is positive. it is equivalent to say that $b$ is **greater than** $a$ or $a$ to the left of $b$ and to write $b > a$. The notation $a \leq b$ means that $a$ is less than or equal to $b$ i.e. either $a < b$ or else $a = b$. Similarly, $b \geq a$ means that $b$ is greater than or equal to $a$ i.e. either $b > a$ or else $b = a$. Thus when $a < b$, a number [1]

$$c \text{ is between } a \text{ and } b \iff a < c < b.$$

Sometimes we insert the word **strictly** for emphasis: $a$ is strictly less than $b$ means that $a < b$ (not just $a \leq b$).

**3.2.** The **order relation** just described is characterized by the following.

**(Trichotomy)** Every real number is either positive, negative, or zero (and no number satisfies two of these conditions).

**(Sum)** The sum of two positive numbers is positive.

**(Product)** The product of two positive numbers is positive.

This characterization together with the notation explained in the previous paragraph implies the following:

[1] The notation $\iff$ is an abbreviation for "if and only if".

**(i)** Either $a < b$, $a = b$, or $a > b$.

**(ii)** If $a < b$ and $b < c$, then $a < c$.

**(iii)** If $a < b$, then $a + c < b + c$.

**(iv)** If $a < b$ and $c > 0$, then $ac < bc$.

**(v)** If $a < b$ and $c < 0$, then $ac > bc$.

**(vi)** If $0 < a < b$, then $0 < \frac{1}{b} < \frac{1}{a}$.

**3.3. Interval Notation**. The **open interval** $(a, b)$ is the set of all real numbers $x$ such that $a < x < b$, and the **closed interval** $[a, b]$ is the set of all real numbers $x$ such that $a \leq x \leq b$. Thus

$$x \text{ is in the set } (a, b) \iff a < x < b$$

and

$$x \text{ is in the set } [a, b] \iff a \leq x \leq b.$$

These notations are extended to include **half open intervals** and **unbounded intervals** as in

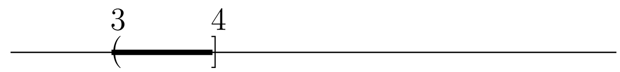
$$\begin{aligned} x \text{ is in the set } (a, b] &\iff a < x \leq b, \\ x \text{ is in the set } (a, \infty) &\iff a < x, \\ x \text{ is in the set } (-\infty, a] &\iff x \leq a, \qquad \text{etc.} \end{aligned}$$

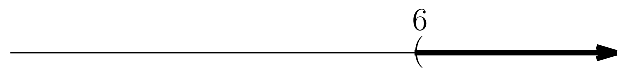
The **union** symbol $\cup$ is used to denote a set consisting of more than one interval as in

$$x \text{ is in the set } (a, b) \cup (c, \infty) \iff \text{either } a < x < b \text{ or else } c < x.$$

The symbol $\infty$ is pronounced **infinity** and is used to indicate that an interval is unbounded. It is not a number so we never write $(c, \infty]$.

**3.4.** Here are some pictures of sets. In these pictures the set is indicated by thickening the corresponding part(s) of the line, parentheses are used to indicated an endpoint which is not in the interval, brackets are used to indicated an endpoint which is in the interval, and the arrow indicates that the interval extends to infinity.

$(3, 4]$ 


$(6, \infty)$ 


$(3, 4] \cup (6, \infty)$ 


**3.5.** The **absolute value** of a real number $a$ is defined by $|a|$ and defined by

$$|a| := \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Since the $\sqrt{\ }$ denotes the nonnegative square root, an equivalent definition is

$$\boxed{|a| = \sqrt{a^2}}$$

The **distance** between two numbers $a$ and $b$ is the absolute value $|a - b|$ of their difference. Thus the absolute value of a real number is its distance from 0.

**3.6.** The average $c = \frac{1}{2}(a + b)$ of two real numbers $a$ and $b$ is the midpoint of the interval $[a, b]$. The following calculation shows that the distance from $a$ to $c$ is the same as the distance from $b$ to $c$:

$$|a - c| = \left| a - \frac{a + b}{2} \right| = \left| \frac{2a - (a + b)}{2} \right| = \left| \frac{a - b}{2} \right|$$

and

$$|b - c| = \left| b - \frac{a+b}{2} \right| = \left| \frac{2b - (a+b)}{2} \right| = \left| \frac{b-a}{2} \right|$$

so $|a - c| = \frac{1}{2}|a - b| = \frac{1}{2}|b - a| = |b - c|$.

**Remark 3.7.** The number line described in Paragraph 3.1 is completely characterized by the following properties: (i) The coordinate of the point $O$ is the number 0. (ii) The coordinate of the point $I$ is the number 1. (iii) If the coordinates of the points $A$ and $B$ are the numbers $a$ and $b$, then the coordinate of the **midpoint** $M$ of the segment $AB$ is the average $\frac{1}{2}(a + b)$. (iv) If three distinct points $A$, $B$, $C$ have coordinates $a$, $b$, $c$ respectively, and $C$ lies on the line segment $AB$, then the real number $c$ lies between the real numbers $a$ and $b$.

# 4 Exponents

The proof of the following theorem requires a more careful definition of the set of real numbers than we have given and is best left for more advanced courses.

**Theorem 4.1.** *Suppose that $a$ is a positive real number. Then there is one and only one way to define $a^x$ for all real numbers $x$ such that*

**(i)** $a^{x+y} = a^x \cdot a^y, \quad a^0 = 1, \quad a^1 = a, \quad 1^x = 1.$

**(ii)** *If $a > 1$ and $x < y$ then $a^x < a^y$.*

**(iii)** *If $a < 1$ and $x < y$ then $a^x > a^y$.*

*With this definition, the laws of exponents in Paragraph 1.9 continue to hold when $a$ and $b$ are positive real numbers and $m$ and $n$ are arbitrary real numbers. The number $a^x$ is positive (when $a$ is positive) regardless of the sign of $x$.*

**4.2.** In particular by property (v) in Paragraph 1.9 we have $(a^x)^y = a^{xy}$ so $(a^{m/n})^n = a^m$ and $(a^m)^{1/n} = a^{m/n}$. Hence for positive numbers $a$ and $b$ we have

$$b = a^{m/n} \iff b^n = a^m.$$

When $m = 1$ and $n$ is a natural number the number $a^{1/n}$ is called the $n$th **root** (**square root** if $n = 2$ and **cube root** if $n = 3$) and is sometimes denoted

$$\sqrt[n]{a} := a^{1/n}.$$

When $n$ is absent, $n = 2$ is understood:

$$\sqrt{a} := a^{1/2}.$$

**Remark 4.3.** A number $b$ is said to be an **$n$th root** of $a$ iff $b^n = a$. When $n$ is odd, every real number $a$ has exactly one (real) $n$th root and this is denoted by $\sqrt[n]{a}$. When $n$ is even, a positive real number $a$ has two (real) $n$th roots (and $\sqrt[n]{a}$ denotes the one which is positive) but a negative number has no real $n$th roots. (In trigonometry it is proved that every nonzero complex number has exactly $n$ distinct complex $n$th roots.)

The equation $b^2 = 9$ has two solutions, namely $b = 3$ and $b = -3$ and each is "a" square root of 9 but only $b = 3$ is "the" square root of 9. However $-2$ is the (only) real cube root of $-8$ because $(-2)^3 = -8$. The number $-9$ has no real square root (because $b^2 = (-b)^2 > 0$ if $b \neq 0$) but does have two complex square roots (because $(3i)^2 = (-3i)^2 = -9$). For most of this course we only use real numbers and we say that

$$\sqrt{a} \text{ is undefined when } a < 0$$

and that

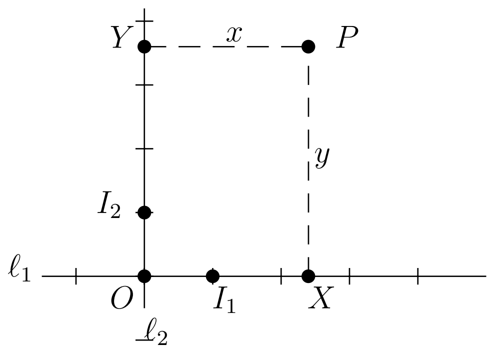
*you can't take the square root of a negative number.*

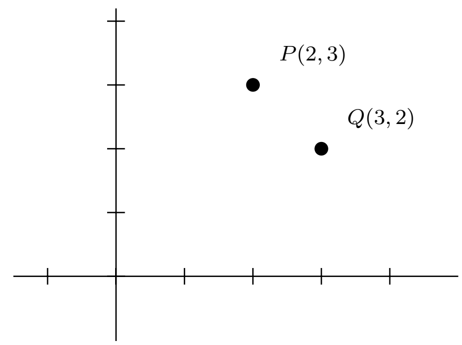
Also $\sqrt{a}$ always denotes the nonnegative square root: thus $(-3)^2 = 3^2 = 9$ but $\sqrt{9} = 3$ and $\sqrt{9} \neq -3$.

# 5 Coordinates in the Plane and Graphs

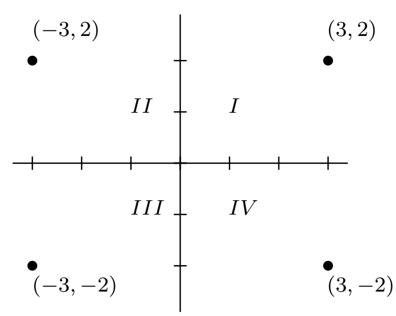
**5.1.** In Paragraph 3.1 we saw that a choice of two points $O$ and $I$ on a line $\ell$ determined a correspondence between the set of real numbers and the points of $\ell$. Similarly two perpendicular lines $\ell_1$ and $\ell_2$ intersecting in a point $O$ and points $I_1$ on $\ell_1$ and $I_2$ on $\ell_2$ determine a correspondence between pairs $(x, y)$ of real numbers and points $P$ on the plane of $\ell_1$ and $\ell_2$. Given a point $P$ in the plane we draw a line parallel to $\ell_1$ through $P$ intersecting $\ell_2$ in $Y$

and a line parallel to $\ell_2$ through $P$ intersecting $\ell_1$ in $X$. The point $X$ has coordinate $x$ in $\ell_1$ and the point $Y$ has coordinate $y$ in $\ell_2$. The point $P$ then corresponds to the pair $(x, y)$. We say that the **coordinates** of $P$ are $(x, y)$. The notation $P(x, y)$ is used as an abbreviation for the more cumbersome phrase "the point $P$ whose coordinates are $(x, y)$." The lines $\ell_1$ and $\ell_2$ are called the **coordinate axes** and the point $O$ is called the **origin**. The point $I_1$ has coordinates $(1, 0)$, the point $I_2$ has coordinates $(0, 1)$, and the origin $O$ has coordinates $(0, 0)$. This correspondence between points $P$ in the plane and pairs $(x, y)$ is called a **rectangular coordinate system** (or sometimes a **Cartesian coordinate system** in honor of it's inventor René Descartes).

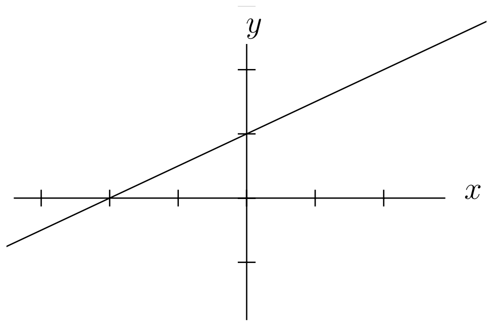


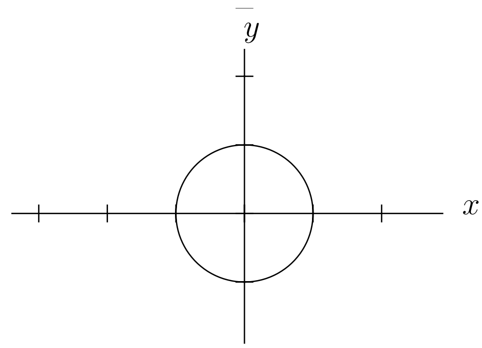

The coordinate axes divide the plane into four parts called **quadrants**. The first quadrant is the set of points $P(x, y)$ where both coordinates are positive and the remaining quadrants are numbered consecutively in the counter clockwise direction as in the diagram.




**Definition 5.2.** The **graph** of an equation in the variables $x$ and $y$ is the set of all points $P(x, y)$ such that $(x, y)$ satisfies the equation. Here are two examples.




$x - 2y + 2 = 0$




$x^2 + y^2 = 1$

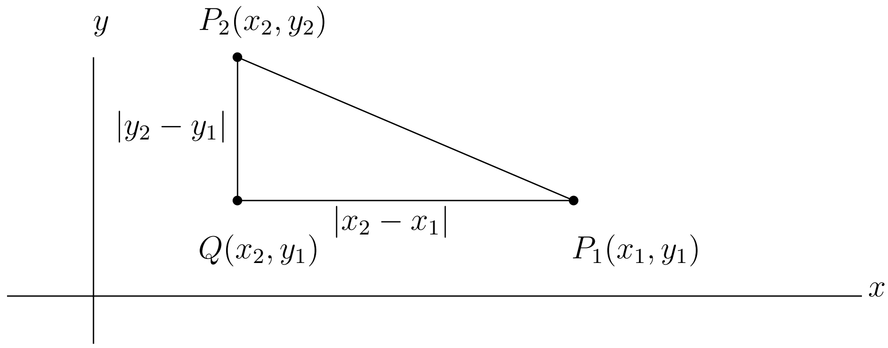
**Remark 5.3.** We can use any two variables, but usually we use $x$ and $y$. We always label the axes and write the equation to avoid confusion. When using the variables $x$ and $y$ we will usually call the first coordinate of a point the **$x$-coordinate** and the second coordinate the **$y$-coordinate**. For example, the $x$-coordinate of the point $P(2, 3)$ is 2 and the $y$-coordinate is 3. Also we call the horizontal axis (i.e. the line $\ell_1$ in Paragraph 5.1) the **$x$-axis** and the vertical axis (i.e. the line $\ell_2$) the $y$-axis.

When we are using coordinates to study geometry it is most natural to make the scale on the $x$-axis the same as the scale on the $y$-axis, i.e. to choose the points $O$, $I_1$, $I_2$ in 5.1 so that the distance from $O$ to $I_1$ is the same as the distance from $O$ to $I_2$ and that this distance is one unit. We will often assume this without comment. In other problems this is not so natural. (See for example Remark 12.11 below.)

**Theorem 5.4 (The Distance Formula).** *The distance $d(P_1, P_2)$ between the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by*

$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

*Proof.* The point $Q(x_2, y_1)$ has the same $y$-coordinate as $P_1$ and the same $y$-coordinate as $P_2$. The points $P_1$, $P_2$, $Q$ are the vertices of a right triangle with legs $QP_1$ and $QP_2$ of length $a = |x_1 - x_2|$ and $b = |y_1 - y_2|$. (See the picture below.) The length $c$ of the hypotenuse $P_1P_2$ is the distance $d(P_1, P_2)$ from $P_1$ to $P_2$. Hence the distance formula follows from the Pythogorean Theorem $a^2 + b^2 = c^2$. $\square$




**5.5.** The **circle** with center $C$ and radius $r$ is the set of all points $P$ such that the distance $d(C, P)$ from $C$ to $P$ is exactly $r$. If $C = C(h, k)$ and $P = P(x, y)$ the condition that $P$ lies on this circle may be written

$$\sqrt{(x-h)^2 + (y-k)^2} = d(C, P) = r.$$

Squaring both sides we see that the circle is the graph of the equation

$$\boxed{(x-h)^2 + (y-k)^2 = r^2.}$$

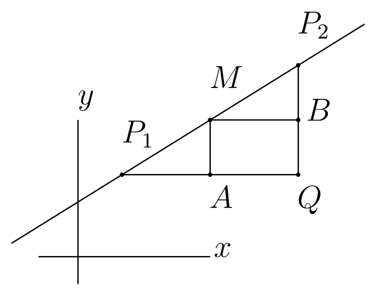
**Theorem 5.6** (**The Midpoint Formula**). *The midpoint $M$ of the the line segment joining the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is computed by averaging the coordinates:*

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

*Proof.* There are two ways to prove this. First we can compute the distance $d(P_1, M)$ from $P_1$ to $M$ and the distance $d(P_2, M)$ from $P_2$ to $M$ and check that

$$d(P_1, M) = d(P_2, M) = \tfrac{1}{2}\, d(P_1, P_2).$$

The second way is to use congruent triangles as in the diagram to the right. The point $Q(x_2, y_1)$ is the vertex of a right triangle $P_1QP_2$. The point $A(\frac{1}{2}(x_1 + x_2), y_1)$ is the midpoint of the line segment $P_1Q$ and the point $B(x_1, \frac{1}{2}(y_1+y_2))$ is the midpoint of the line segment $P_1Q$. (See Paragraphs 3.1 and 3.6.) The point $M$ is a vertex of the congruent right triangles $P_1AM$ and $MBP_2$ so $d(P_1M) = d(P_2M)$. □




# 6 Lines

**6.1.** The graph of an equation of form

$$Ax + By + C = 0$$

(where not both $A$ and $B$ are zero) is a **line**. If $A = 0$, the equation may be written in the form $y = b$ with $b = -C/B$ and the line is **horizontal**. If $B = 0$, the equation may be written in the form $x = a$ with $a = -C/A$ and the line is **vertical**.

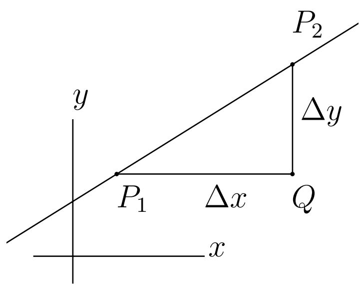
**Definition 6.2.** The **slope** of the line through the distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

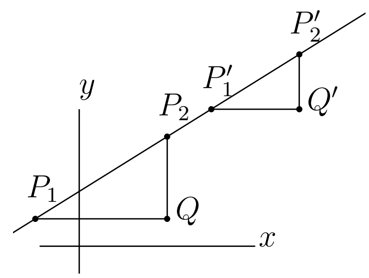
$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

In calculus this is sometimes written as

$$m = \frac{\Delta y}{\Delta x}, \qquad \Delta y = y_2 - y_1, \quad \Delta x = x_2 - x_1$$

and described as the "change in $y$ divided by the change in $x$".




As we move from $P_1$ to $P_2$ the $x$-coordinate "runs" form $x_1$ to $x_2$ and the $y$-coordinate "rises" from $y_1$ to $y_2$, so the slope is sometimes described as **rise over run**. If the $y$ coordinate decreases as the $x$ coordinate increases, the slope is negative.

The slope depends on the coordinate system but is independent of the choice of the pair of distinct points on the line. If $P_1'(x_1', y_1')$ and $P_2'(x_2', y_2')$ are two other points on the line and $Q = Q(x_1, y_2)$, $Q' = Q'(x_2', y_1')$ then

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2' - y_1'}{x_2' - x_1'}$$

as the triangles $P_1QP_2$ and $P_1'Q'P_2'$ are similar right triangles.

**6.3.** A **vertical line** is one which is parallel to the $y$-axis. The $x$-coordinate is constant along such a line so it has and equation of form $x = a$. For any two points on a vertical line we have $\Delta x = a - a = 0$. Hence the slope of a vertical line is not defined since we never divide by zero. A **horizontal line** is one which is parallel to the $y$-axis. It has an equation of form $y = b$. The slope of a horizontal line is zero (which is defined).

**6.4. The Point-Slope Equation of a Line**. Let $P_0(x_0, y_0)$ be a point on a line $\ell$ of slope $m$. Since any two points on the line can be used to define the slope we see that a point $P(x, y)$ which llies on the line $\ell$ satisfies

$$\frac{y - y_0}{x - x_0} = m.$$

This equation has one minor flaw; it doesn't work when $(x, y) = (x_0, y_0)$ (never divide by zero). To remedy this multiply by $(x - x_0)$ and add $y_0$ to both sides:

$$y = y_0 + m(x - x_0).$$

This is the **point-slope form** of the equation for *line through $P_0(x_0, y_0)$ with slope $m$*; this form makes it obvious that the point $P_0(x_0, y_0)$ lies on the line. For example, the equation for the line through $P_0(2, 3)$ and $P_1(4, 11)$ is

$$\frac{y-3}{x-2} = \frac{11-3}{4-2} = 4, \qquad \text{or} \qquad y = 3 + 4(x-2),$$

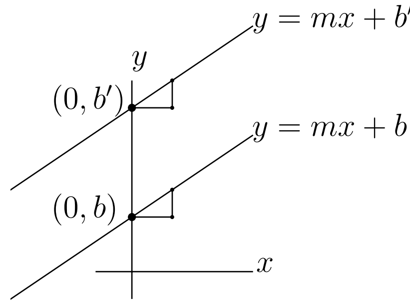
When $x_0 = 0$ and $y_0 = b$ the point-slope form becomes

$$y = mx + b.$$

This is called the **slope-intercept** form of the equation for the line because the point $(0, b)$ is the point where the line intercepts (intersects) the $y$-axis.

**Theorem 6.5** (**Parallel Lines**). *Two lines have the same slope if and only if they are parallel (or the same).*

*Proof.* Assume two lines have the same slope $m$. Then their slope intercept forms are $y = mx + b$ and $y = mx + b'$ If some point $(x, y)$ lies on both lines then $mx+b = y = mx+b'$ so $b = b'$ and the lines are the same, Hence if $b \neq b'$ the lines do not intersect, i.e. they are parallel. Conversely if the lines are parallel, then in the picture to the right the two triangles are similar so the hypotenuses are parallel.

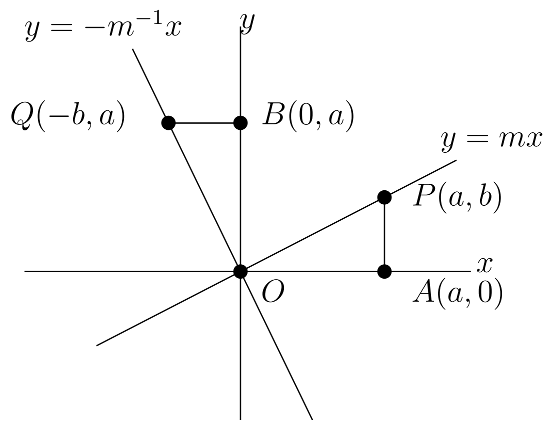



□

**Theorem 6.6** (**Perpendicular Lines**). *Two lines are perpendicular if and only if the slope of one is the negative reciprocal of the slope of the other.*

*Proof.* If we draw the parallels to the two lines through the origin we get two new lines with (by Theorem 6.5) the same slopes and which are perpendicular if and only if the original lines are. Hence we might as well assume the two lines pass through the origin. The diagram shows two lines with slopes $m$

and $-m^{-1}$ and passing through the origin. The point $P(a,b)$ lies on the line $y = mx$ so $b = ma$, and hence $-a = -m^{-1}b$ so the point $(-b,a)$ lies on the line $y = -m^{-1}x$. Hence the two right triangles $OAP$ and $OBQ$ are congruent. The two acute angles in a right triangle are complementary so the angle $AOP$ is the complement of the angle $BQO$. Since $QB$ is parallel to the $x$-axis the angle $QO$ makes with the $x$-axis is the complement to $POA$. Hence $POQ$ is a right angle.




□

# 7 Parabolas and Ellipses

**Definition 7.1.** Let $\ell$ be a line and $F$ be a point not on $\ell$. The **parabola** with **focus** $F$ and **directrix** $\ell$ is the set of points $P$ which are equidistant from $F$ and $\ell$. The line through $F$ and perpendicular to $\ell$ is called the **axis** of the parabola and the point where the axis intersects the parabola is called the **vertex** of the parabola.

**Theorem 7.2** (**Parabola Formula**). *If the directrix is parallel to the $x$-axis then the parabola is the graph of the equation of form*

$$\boxed{y = a(x-h)^2 + k.}$$

*The vertex is $V(h,k)$.*

*Proof.* Guided Exercise. □

**Example 7.3.** If the directrix is the horizontal line $y = -\frac{1}{4}$ and the focus is the point $F(\frac{1}{4}, 0)$, then the parabola is the graph of the equation

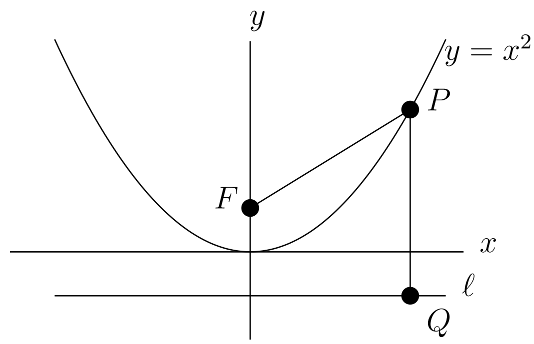
$$y = x^2.$$

To see this assume that the point $P(x,y)$ lies on the parabola. The distance from the point $P(x,y)$ to the point $F$ is

$$d(P,F) = \sqrt{(x^2 + (y - \tfrac{1}{4})^2}.$$

The distance from $P$ to $\ell$ is

$$d(P,\ell) = \left|y + \tfrac{1}{4}\right|.$$

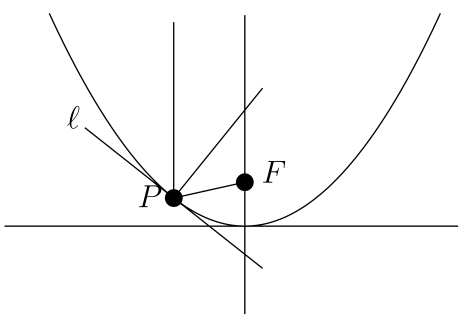


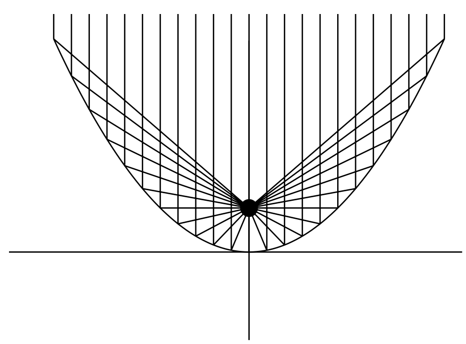

Squaring both sides of the equation $d(P,\ell) = d(P,F)$ gives

$$\left(y + \tfrac{1}{4}\right)^2 = x^2 + (y - \tfrac{1}{4})^2.$$

Expand and cancel to get $y = x^2$. In this example, the vertex is at the origin.

**Remark 7.4.** A **parabolic mirror** is a mirror in the shape of a parabola. If the sun's rays are directed perpendicular to the directrix, i.e. parallel to the axis, then they reflect so that the angle of incidence equals the angle of reflection. Using calculus it can be proved that all the rays pass through the focus. The diagram shows the tangent line $\ell$ to the parabola at the point $P$ and the focus $F$ of the parabola. The line perpendicular to $\ell$ through $P$ is called the *normal*. The ray from the sun is the vertical line through $P$. The angle between the vertical and the normal is called the *angle of incidence* and the angle between the normal and the reflected ray is called the *angle of reflection*. For the parabola, the reflected ray goes through the focus.



The same process works in reverse. Movie projectors used to have a carbon arc lamp located at the focus of a parabolic mirror so that the light rays would reflect from the mirror and pass through the lens in parallel supplying a uniformly illuminated image.

**Definition 7.5.** Let $F_1$ and $F_2$ be two points in the plane an $a$ be a number greater than the distance $d(F_1, F_2)$ from $F_1$ to $F_2$. The **ellipse** with **foci** $F_1$ and $F2$ and **diameter** $2a$ is the set of all points $P$ such that

$$d(F_1, P) + d(F_2, P) = 2a,$$

i.e. the sum of the distances from $P$ to the two given points $F_1$ and $F_2$ is the constant $2a$. A picture appears in Example 7.7 below. Note: The word *foci* is the plural of *focus*.

**Theorem 7.6** (**Standard Formula for Ellipse**)**.** *If the foci are $F_1(-c, 0)$ and $F_2(c, 0)$ where $c > 0$ then the ellipse with foci $F_1$ and $F_2$ and diameter $2a$ is the graph of the equation*

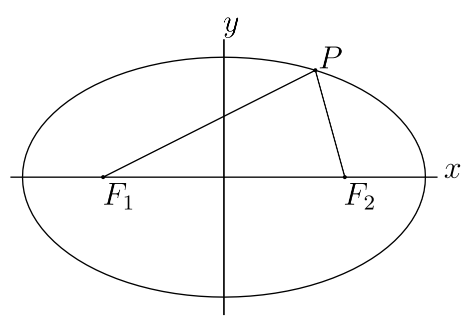
$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

*where $b^2 = a^2 - c^2$.*

*Proof.* Guided Exercise. □

**Example 7.7.** Assume that the foci are $F_1(-3, 0)$ and $F_2(3, 0)$ and that the diameter is 10. Then the equation $d(F_1, P) + d(F_2, P) = 2a$ is

$$\sqrt{(x+3)^2 + y^2} + \sqrt{(x-3)^2 + y^2} = 10.$$

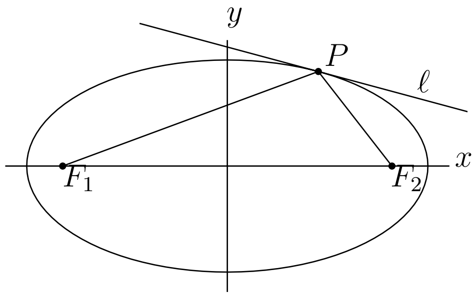



According to Theorem 7.6 this last equation can be simplified to

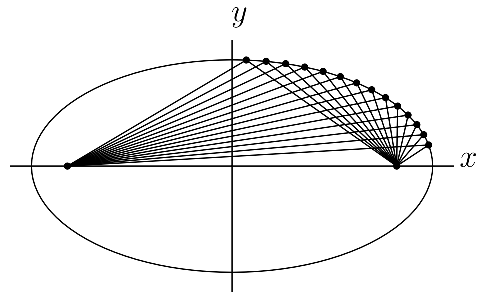
$$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$

You can derive the latter equation from the former by subtracting one of the square roots from both sides and squaring. This still leaves a square root on the right, but if you subtract all the other terms on the right from both sides and square again you get an equation with no square roots. After a bit more algebraic manipulation, the equation simplifies as claimed.

**Remark 7.8.** An ellipse has a reflective property much like a parabola does. When a light ray (or sound wave) strikes a surface, it bounces back so that the angle of incidence equals the angle of reflection. If the surface is elliptical, a ray emanating from one focus bounces off the ellipse and passes through the other focus. (This is often proved in second semester calculus.)




$\angle(\ell, F_1P) = \angle(\ell, F_2P)$




A Whispering Gallery

A **whispering gallery** works on this principle. There is one in the Museum of Science and Industry in Chicago. The entire room is in the shape of an ellipse. If you stand at one focus and your friend stands at the other, each of you can hear the other whisper, even though people only a few yards away cannot hear you. There are also whispering galleries in the United States Capitol, the rotundas of the Texas State Capitol and the Missouri State Capitol, St. Peter's Basilica in the Vatican City, St Paul's Cathedral in London, and many other places.

# 8 Solving Equations

**Definition 8.1.** A number $a$ is called a **solution** of an equation containing the variable $x$ if the equation becomes a true statement when $a$ is substituted for $x$. A solution of an equation is sometimes also called a **root** of the equation. Two equations are said to be **equivalent** iff they have exactly the same solutions. We will sometimes use the symbol $\iff$ to indicate that two equations are equivalent.

**8.2.** Usually two equations are equivalent because one can be obtained from the other by performing an operation to both sides of the equation which can be reversed by another operation of the same kind. For example, the equations $3x + 7 = 13$ and $x = 2$ are equivalent because

$$\begin{aligned} 3x + 7 = 13 &\iff \quad 3x = 6 \text{ (subtract 7 from both sides)}, \\ &\iff \quad x = 2 \text{ (divide both sides by 3)}. \end{aligned}$$

The reasoning is reversible: we can go from $x = 2$ to $3x = 6$ by multiplying both sides by by 3 and from $3x = 6$ to $3x+7 = 13$ by adding 7 to both sides.

**8.3.** We use the symbol $\implies$ when we want to assert that one equation **implies** another but do not want to assert the converse. The guiding principal here is

> *If an equation $E'$ results from an equation $E$ by performing the same operation to both sides, then $E \implies E'$, i.e. every solution of $E$ is a solution of $E'$.*

If the operation is not "reversible" as explained above, there is the possibility that the set of solutions gets bigger in which case the new solutions are called **extraneous solutions**. (They do not solve the original equation.) The simplest example of how an extraneous solution can arise is

$$x = 3 \implies x^2 = 9 \qquad \text{(square both sides)}$$

but the operation of squaring both sides is not reversible: it is incorrect to conclude that $x^2 = 9$ implies that $x = 3$. What *is* correct is that $x^2 = 9 \iff x = \pm 3$, i.e. *either* $x = 3$ *or else* $x = -3$. When solving an equation you may use operations which are not reversible provided that you

| |
|---|
| *Always check your answer!* |

(In addition to catching mistakes, this will show you which – if any – of the solutions you found are extraneous.)

**8.4.** Here are two ways in which extraneous solutions can arise:

**(i)** Squaring both sides of an equation.

**(ii)** Multiplying both sides of of an equation by a quantity not known to be nonzero.

As an example of (i) consider the equation

$$\sqrt{10-x} = -x-2.$$

Squaring both sides gives the quadratic equation $10-x = x^2+4x+4$ which has two solutions $x=-6$ and $x=1$. Now $\sqrt{10-(-6)} = -(-6)-2$ but $\sqrt{10-1} \neq -1-2$. ($\surd$ means the positive square root.) Thus $x=-6$ is the only solution of the original equation and $x=1$ is an extraneous solution.

As an example of (ii) consider

$$\frac{1}{x-1} = 2 + \frac{1}{x-1}.$$

This equation has no solution: if it did we would subtract $(x-1)^{-1}$ from both sides and deduce that $0=2$ which is false. But if we multiply both sides by $x-1$ we get $1 = 2(x-1)+1$ which has the (extraneous) solution $x=1$.

**Remark 8.5.** The symbols $\iff$ and $\implies$ relate equations (or more generally statements) not numbers. The notation $P \implies Q$ means that $Q$ is true if $P$ true. The notation $x \implies 5$ is nonsense. On exams always write complete sentences (e.g. $x=2$ or "the answer is 2") never just numbers (nouns are neither true nor false).

**8.6.** The definition of equivalent equations given in 8.1 applies to equations in two variables where the solution set is (usually) infinite: A pair of numbers $(a,b)$ is a **solution** of an equation containing the variables $x$ and $y$ if the equation becomes a true statement when $a$ is substituted for $x$ and $b$ is substituted for $y$. By Definition 5.2

*Two equations are equivalent if and only if they have the same graph.*

Again we usually show that two equations are equivalent by transforming one to the other by operations which can be reversed by another operation of the same kind. As a simple example

$$6x+3y=12 \iff 3y=-6x+12 \iff y=-2x+4$$

so the equations have the same graph, namely the line of slope $-2$ through the point $(0,4)$.

# 9 Systems of Equations

**9.1.** A **solution** to a **system** of two equations in two variables $x$ and $y$ is a pair $(a, b)$ of numbers which is a solution of each of the two equations. Each equation has a graph and the solutions to the system are the point where the two graphs intersect. Two systems are **equivalent** iff they have exactly the same solutions. As in Paragraphs 8.2 and 8.6 we usually show that two systems are equivalent by performing an operation on one system which can be reversed by another operation of the same kind.

**Example 9.2.** As an example we solve the system $3x+3y = 21$, $x+2y = 11$.

$$\begin{aligned} \begin{array}{r} 3x+3y=21 \\ x+2y=11 \end{array} \quad &\iff \quad \begin{array}{l} x+y=7 \\ x+2y=11 \end{array} \qquad (1) \\ &\iff \quad \begin{array}{l} x+y=7 \\ y=4 \end{array} \qquad (2) \\ &\iff \quad \begin{array}{l} x=3 \\ y=4 \end{array} \qquad (3) \end{aligned}$$

In step (1) we divided the first equation by 3; this can be undone by multiplying the first equation by 3. In step (2) we subtracted the first equation from the second; this can be undone by adding the first equation to the second. In step (3) we subtracted the second equation from the first; this can be undone by adding the second equation to the first. In each step we replace a system of two equations in $x$ and $y$ by an equivalent system, i.e. we replace a pair of lines through the point $(x, y) = (3, 4)$ by another pair of lines through that point.

**9.3.** The process of simplifying a system of equations by multiplying one of the equations by a nonzero number or replacing one of the equations by its difference with another is called **Gaussian Elimination**. The guiding principle is to “eliminate” one of the variables from the first equation and the other variable from the second equation leaving us with an equivalent system of form $x = a$, $y = b$.

Gaussian Elimination can be used for any number of equations in any number of variables. For problems as in the example (two equations in two unknowns) which are intended to be worked by hand, there is no reason to supply so much detail. For much larger problems (100 equations in 100 unknowns) which will be solved on a computer, Gaussian Elimination is used because we can prove it gets the correct answer.

**9.4.** Another nice feature of Gaussian Elimination is that it handles gracefully the case where the two lines determined by the two equations are parallel or the same. If the two equations correspond to lines with the same slope, Gaussian Elimination will lead to an equivalent system of form

$$\begin{aligned} ax + by &= c \\ 0 &= d. \end{aligned}$$

If $d \neq 0$ then the original lines are parallel but distinct and there is no solution. If $d = 0$, then the original lines are the same and there are infinitely many solutions (any point on the line solves both equations). A system with no solutions is said to be **inconsistent**.

# 10 Symmetry

**Definition 10.1.** The **reflection** of a point $P$ in a line $\ell$ is the point $Q$ on the other side of the line at the same distance from the line. The **reflection** of a point $P$ in a through the point $O$ is the point $Q$ on the the line $OP$ at the same distance from $O$ as is $P$. For example:

1. The reflection of $(x, y)$ in the $y$-axis is $(-x, y)$.

2. The reflection of $(x, y)$ in the $x$-axis is $(x, -y)$.

3. The reflection of $(x, y)$ in the line $y = x$ is $(y, x)$.

4. The reflection of $(x, y)$ about the origin is $(-x, -y)$.

A graph is **symmetric about the line** $\ell$ iff whenever a point $P$ lies on the graph so does its reflection in $\ell$. A graph is **symmetric about the point** $O$ iff whenever a point $P$ lies on the graph so does its reflection through $O$.

**10.2. Symmetry tests**. Combining these definitions we see that

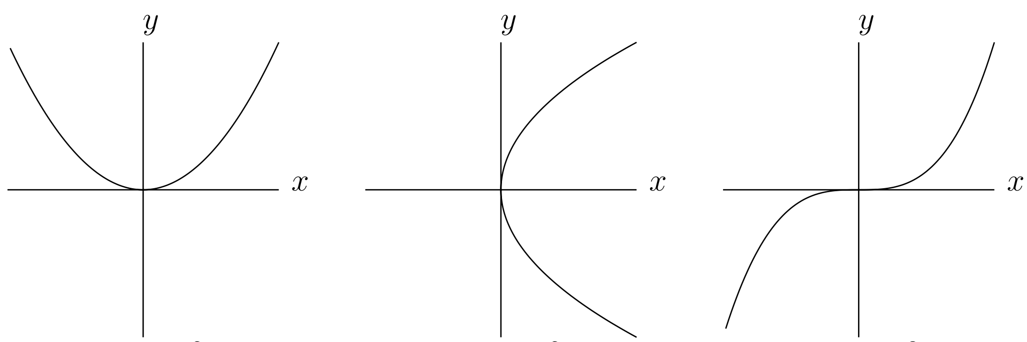
1. The graph of an equation is symmetric about the $y$-axis if and and only if the equation that results by replacing $x$ by $-x$ yields an equivalent equation.

2. The graph of an equation is symmetric about the $x$-axis if and and only if the equation that results by replacing $y$ by $-y$ yields an equivalent equation

3. The graph of an equation is symmetric about the $y$-axis if and only if the equation that results by replacing $x$ and $y$ by $-x$ and $-y$ yields an equivalent equation.

For example, the graph of $y = x^2$ is symmetric about the $y$-axis as $(-x)^2 = x^2$ and similarly the graph of $x = y^2$ is symmetric about the $x$-axis. The graph of $y = x^3$ is symmetric about the origin as $(-x)^3 = -x^3$ so

$$y = x^3 \iff (-y) = (-x)^3.$$




$y = x^2$ $\quad$ $x = y^2$ $\quad$ $y = x^3$

Note that it might be better to say"replacing $x$ by $(-x)$" rather than "replacing $x$ by $-x$" as $(-x)^2 = x^2 \neq -x^2$.

# 11 Completing the Square

**11.1.** By **completing the square** we mean the identity

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$$

which is easily proved by expanding the right hand side. In this section we use this identity for three purposes: to solve a quadratic equation (the Quadratic Formula), to find the center of a circle, and to find the vertex of a parabola.

**Theorem 11.2** (**Quadratic Formula**). *The solutions of the quadratic equation*

$$ax^2 + bx + c = 0$$

*are*

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.}$$

*Proof.* Guided Exercise. □

**Theorem 11.3** (**Center of a Circle**). *If $a \neq 0$, the equation*

$$ax^2 + ay^2 + bx + cy + d = 0$$

*is equivalent to the equation*

$$(x - h)^2 + (y - k)^2 = R$$

*where*

$$h = -\frac{b}{2a}, \qquad k = -\frac{d}{2a}, \qquad R = \frac{b^2 + c^2}{4a^2} - d.$$

*Hence the graph is a circle of radius $\sqrt{R}$ if $R > 0$, the single point $(h, k)$ if $R = 0$, and empty if $R < 0$.*

*Proof.* Guided Exercise. □

**Theorem 11.4** (**Vertex Formula**). *If $a \neq 0$, the equation*

$$y = ax^2 + bx + c$$

*is equivalent to the equation*

$$y = a(x - h)^2 + k$$

*where*

$$h = -\frac{b}{2a}, \qquad k = -\frac{b^2 - 4ac}{4a}.$$

*Hence the graph is a parabola with vertex $V(h, k)$.*

*Proof.* Guided Exercise. □

**11.5. Translating the Axes** The substitutions

$$u = x - h, \qquad v = y - k$$

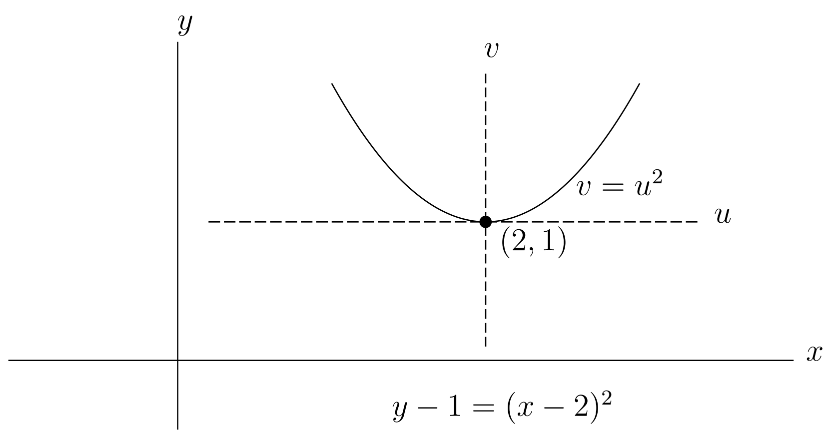
transform the equation

$$v = u^2$$

into the equation

$$y - h = (x - k)^2.$$

The point $(u, v) = (0, 0)$ on the former graph corresponds to the point $(x, y) = (h, k)$ on the latter. This is the vertex of the parabola. To draw the graph draw the graph of $v = u^2$ first and then draw in the $x$ and $y$ axes.




## 12 Functions

**Definition 12.1.** A **function** is a rule which produces an output $f(x)$ from an input $x$. The set of inputs $x$ for which the function is defined is called the **domain** and $f(x)$ (pronounced "$f$ of $x$") is the **value** of $f$ at $x$. The set of all possible outputs $f(x)$ as $x$ runs over the domain is called the **range** of the function.

**12.2. Functional Notation**. We usually define a function $f$ by writing an equation

$$f(x) = \text{ some expression in } x.$$

It is then understood that $f(a)$ denotes the result of substituting $a$ for $x$ in the expression. If $a$ is itself an expression, it should be surrounded by parentheses before doing the substitution. Thus if $f(x) = x^2$, then $f(p+q) = (p+q)^2$ not $p + q^2$.

**12.3.** If a function $f(x)$ is given by an expression in the variable $x$ and the domain is not explicitly specified, then the domain is understood to be the set of all $x$ for which the expression is meaningful. For example, for the function $f(x) = 1/x^2$ the domain is the set of all nonzero real numbers $x$ (the value $f(0)$ is not defined because we don't divide by zero) and the range is the set of all positive real numbers (the square of any nonzero number is positive). The domain and range of the square root function $\sqrt{x}$ is the set of all nonnegative numbers $x$. The domain of the function $y = \sqrt{1-x^2}$ is the interval $[-1, 1]$, i.e. $\sqrt{1-x^2}$ is meaningful only if $-1 \le x \le 1$ (otherwise the input to the square root function is negative). The range of the function is the interval $[0, 1]$ as $0 \le \sqrt{1-x^2} \le 1$.

**Definition 12.4.** The **graph** of a function $f$ is the graph of the equation

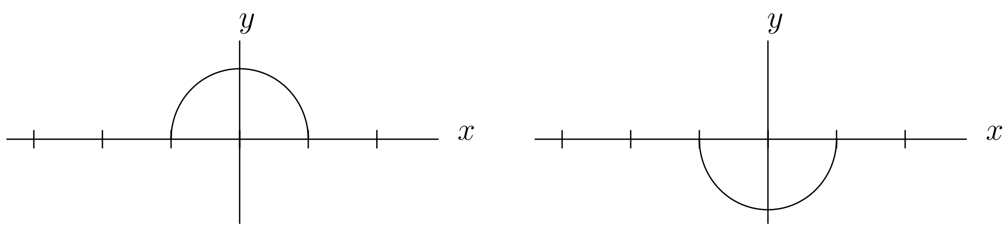
$$y = f(x),$$

i.e. the set of all points $P(x, y)$ whose coordinates $(x, y)$ satsfy the equation $y = f(x)$. According to Definition 5.2, the graph of an equation is the set of all points $P(x, y)$ in the $(x, y)$-plane whose coordinates $(x, y)$ satisfy the equation. The graph of a function $y = f(x)$ is a special case.To decide if a graph is the graph of a function we apply the

**Vertical Line Test**. *A graph is the graph of a function $f$ if and only if every vertical line $x = a$ intersects the graph in at most one point. Then, if the number $a$ is in the domain of $f$, the vertical line $x = a$ intersects the graph $y = f(x)$ in the point $P(a, f(a))$.*

**Example 12.5.** The graph of the equation $x^2 + y^2 = 1$ is a circle; it is not the graph of a function since the vertical line $x = 0$ (the $y$-axis) intersects the graph in two points $P_1(0, 1)$ and $P_2(0, -1)$. This graph is however the union of two different graphs each of which is the graph of a function:

$$x^2 + y^2 = 1 \iff \text{either} \quad y = \sqrt{1-x^2} \quad \text{or} \quad y = -\sqrt{1-x^2}.$$




$y = \sqrt{1 - x^2}$ $\qquad\qquad$ $y = -\sqrt{1 - x^2}$

**Remark 12.6.** To find the domain of a function we project its graph on the horizontal axis. To find the range of a function we project its graph on the vertical axis. More precisely,

> The domain of the function $f$ is the set of all real numbers $a$ such that the vertical line $x = a$ intersects the graph $y = f(x)$. The range of the function $f$ is the set of all real numbers $b$ such that the horizontal line $y = b$ intersects the graph $y = f(x)$.

**12.7.** You can add, subtract, multiply and divide two functions:

$$(f+g)(x) := f(x) + g(x), \qquad (f-g)(x) := f(x) - g(x),$$
$$(f \cdot g)(x) := f(x) \cdot g(x), \qquad (f/g)(x) := \frac{f(x)}{g(x)}.$$

In each case the domain is the intersection of the domains of $f$ and $g$; in the case where the functions are divided those $x$ where $g(x) = 0$ must also be removed from the domain.

**Definition 12.8.** The **composition** $g \circ f$ of two functions $g$ and $f$ is defined by

$$(g \circ f)(x) := g(f(x)).$$

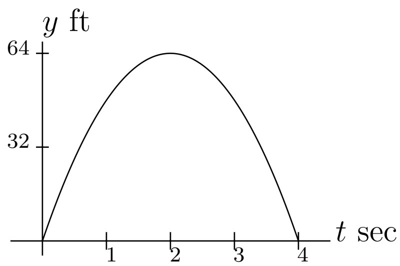
**Remark 12.9.** Composition of functions is not a commutative operation. For example, if $f(x) = x^2$ and $g(x) = x + 1$, then $(g \circ f)(x) = x^2 + 1$ and $(f \circ g)(x) = (x+1)^2$.

**12.10.** Two functions are said to be **equal** when they have the same domain and give the same output for every input. A consequence of this definition is that it doesn't matter what letters we use to define a function. For example, the functions

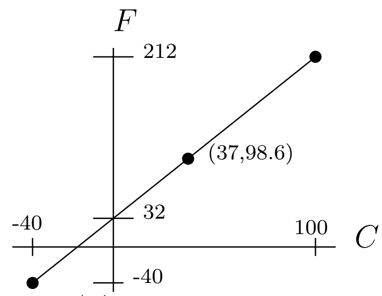
$$f(x) = x^2, \qquad g(u) = u^2$$

are equal since the domain of each is the set $(-\infty, \infty)$ of all real numbers and $f(t) = g(t)$ for all real numbers $t$. In story problems, we usually use letters which suggest the meaning like $t$ for time or $A$ for area.

**Remark 12.11.** When we are using coordinates to study geometry it is most natural to make the scale on the $x$-axis the same as the scale on the $y$-axis. In other problems this is not so natural. Here is a graph showing the height $y$ in feet of an object $t$ seconds after it is thrown into the air and another graph the showing relation between the temperature $F$ in Fahrenheit and the temperature $C$ in Celsius. Both graphs are graphs of functions and the units on the vertical axis are different from the units on the horizontal axis.




(i) $y = 16t^2 - 4t$




(ii) $F = 1.8C + 32$

# 13 Inverse Functions

**Definition 13.1.** Two functions $f$ and $g$ are said to be **inverse functions** iff the graphs the equations $y = f(x)$ and $x = g(y)$ are the same, i.e. iff

$$y = f(x) \iff x = g(y).$$

We also say that $g$ is the **inverse** of $f$ and write

$$g = f^{-1}.$$

**Example 13.2.** Since

$$y = 3x + 7 \iff x = \frac{y - 7}{3}$$

the functions $f(x) = 3x + 7$ and $g(y) = (y - 7)/3$ are inverse functions.

**Remark 13.3.** Be careful not to confuse $f(x)^{-1}$ and $f^{-1}(x)$. For example, if $f(x) = 3x + 7$, then

$$f(x)^{-1} = \frac{1}{3x+7}, \qquad \text{but} \qquad f^{-1}(x) = \frac{x-7}{3}.$$

**13.4.** If $f$ and $g$ are inverse functions, then the range of $f$ is the domain of $g$ and the domain of $f$ is the range of $g$. To decide if a function has an inverse we apply the

**Horizontal Line Test**. *A function $y = f(x)$ has an inverse if and only if every horizontal line $y = b$ intersects the graph in at most one point. Then the horizontal line $y = b$ intersects the graph $y = f(x)$ in the point $P(f^{-1}(b), b)$.*

**13.5.** A function $f$ is said to be **one-to-one** iff

$$f(x_1) = f(x_2) \implies x_1 = x_2.$$

Saying that a function is one-to-one is just another way of saying its graph satisfies the horizontal line test, i.e. that the function has an inverse. The function $f(x) = x^2$ is not one-to-one since $f(3) = f(-3)$ but $3 \neq -3$.

**Definition 13.6.** A function $f$ is said to be **increasing** iff

$$x_1 < x_2 \implies f(x_1) < f(x_2).$$

A function $f$ is said to be **decreasing** iff

$$x_1 < x_2 \implies f(x_1) > f(x_2).$$

**Theorem 13.7.** *If a function either increasing or decreasing it is one-to-one and therefore has an inverse.*

*Proof.* Assume $f$ is increasing and that $f(x_1) = f(x_2)$. Then it is not the case that $x_1 < x_2$ since that would imply $f(x_1) < f(x_2)$ and it is not the case that $x_1 > x_2$ since that would imply $f(x_1) > f(x_2)$. The only possibility us that $x_1 = x_2$. □

**Remark 13.8.** Just because a function has an inverse doesn't mean that we can find a formula for the inverse. For example, the function $f(x) = x^5 + x$ is increasing and therefore has an inverse $f^{-1}$, but in graduate courses in algebra it is proved that there is no elementary formula for $f^{-1}(y)$, i.e. there is no expression for $f^{-1}(y)$ involving only the operations we have have defined in these notes. Put another way, there is no nice formula for the solution of the equation $3 = x^5 + x$.

Even if we can't find a formula for the inverse of a function, we can still compute its value for any given input to any degree of accuracy with (say) a computer. So we can give the inverse a name and compute with it using the rules of algebra. This is exactly what we shall do in Section 17. with the exponential function

$$f(x) = 2^x.$$

This function is increasing so has an inverse function. and the inverse function is denoted by $\log_2(y)$:

$$y = 2^x \iff x = \log_2(y).$$

**13.9.** In Section 12.3 we said that, unless the contrary is stated, the domain of a function $f(x)$ which is defined by an expression in $x$ is the set of $x$ for which the expression is meaningful. However, we may decide to restrict the domain to a smaller set in order to define an inverse function. For example, in Section 13.5 we saw that the function $f(x) = x^2$ is not one-to-one since $f(3) = f(-3)$ and hence this function does not have an inverse function. But we can define

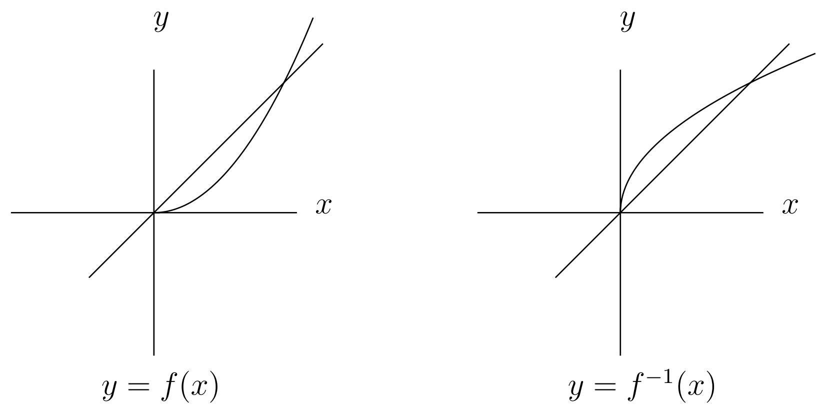
$$q(x) = x^2, \qquad x \geq 0$$

so the domain is $[0, \infty)$. Then

$$y = q(x) \iff x = \sqrt{y}$$

so $q(x)$ and $\sqrt{y}$ are inverse functions. (This device is used in trigonometry to define the inverse trigonometric functions.) If we were to program a computer to compute the function $q(x)$, asking the computer to compute $q(-3)$ would produce an error message like *"function undefined for this input"*.

**13.10.** If the function $f$ has an inverse, then the graph of the equation $y = f(x)$ is the same as the graph of the equation $x = f^{-1}(y)$. Of course the graph of the equation $y = f^{-1}(x)$ is (usually) different. The graphs of the equations $y = f(x)$ and $y = f^{-1}(x)$ are obtained from each other by interchanging the $x$-axis and the $y$-axis, i.e. by reflecting in the line $y = x$.




# 14 Average Rate of Change

**Definition 14.1.** The **average rate of change** of the function $f$ over the interval $[a, b]$ is the slope of the line joining the point $(a, f(a))$ to the point $(b, f(b))$. When $y = f(x)$ the average rate of change is often written as

$$\frac{\Delta y}{\Delta x} = \frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a},$$

i.e. the average rate of change is the change $\Delta y = f(b) - f(a)$ in $y$ divided by the change $\Delta x = b - a$ in $x$.

**Example 14.2.** The average rate of change of the function $f(x) = x^2$ on the interval $[a, b]$ is

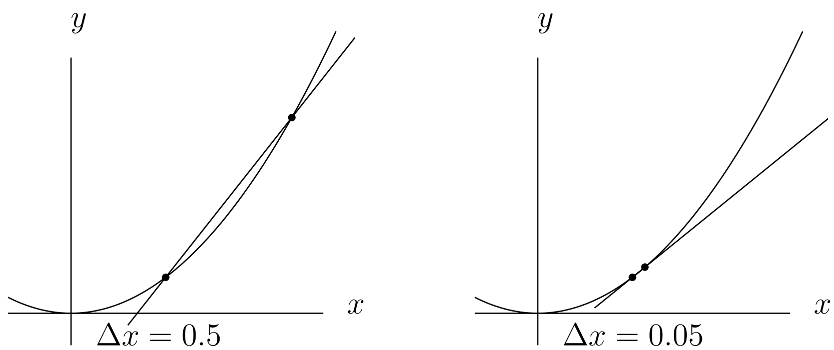
$$\frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a} = \frac{b^2 - a^2}{b - a} = \frac{(b + a)(b - a)}{b - a} = b + a.$$

Note that the $b - a$ in the denominator cancels out. To see way this happens we will repeat the calculation using the notation $h = b - a = \Delta x$ so that $b = a + h$. When we expand the numerator the terms which don't contain an $h$ cancel. Then the $h$ in the denominator cancels with (some of) the $h$'s in the numerator as follows:

$$\frac{\Delta f}{\Delta x} = \frac{(a + h)^2 - a^2}{h} = \frac{a^2 + 2ah + h^2 - a^2}{h} = \frac{2ah + h^2}{h} = 2a + h.$$

This cancellation will always happen when you are asked to simplify an average rate of change.

**14.3.** The average rate of change of a function $f$ over the interval $[a, b]$ is undefined when $a = b$ (zero divided by zero is nonsense) but for $b$ very close to $a$ the average rate of change will usually be close to a number called the **instantaneous rate of change**. This is the slope of the tangent line to the graph $y = f(x)$ at the point $(a, f(a))$.




In calculus the instantaneous rate of change is called the **derivative**. In the previous paragraph we saw that for the function $f(x) = x^2$ the average rate of change over the interval $[a, b]$ is $a+b$. This is $2a+h$ where $h = \Delta x = b-a$. The instantaneous rate of change at the point $(a, f(a))$ is $2a$ which is obtained from the average rate of change by taking $h = 0$.

**14.4.** Suppose you are traveling from Madison to Milwaukee by automobile via the interstate highway I-94. Your position is a function $s = f(t)$ of the time $t$. The value $s$ is the number on the mile marker at the side of the road. (Along much of the road there is a mile marker every tenth of a mile, but imagine there is one every few feet.) The change $\Delta s = f(t + \Delta t) - f(t)$ is also the change in the odometer reading in your car. If the time interval $\Delta t$ is so short that your speed doesn't change much in the time interval, the average rate of change $\Delta s/\Delta t$ is your speed as shown on the speedometer. The speed is the instantaneous rate of change of the position.

# 15 Polynomials

**15.1.** For any graph the points where it intersects the $x$-axis are called the **$x$-intercepts** and the points where it intersects the $y$-axis are called the **$y$-**

**intercepts**. The equation of the $x$-axis is $y = 0$ so we find the $x$-intercepts of the graph of an equation by plugging in $y = 0$ and solving for $x$. Similarly the equation of the $y$-axis is $x = 0$ so we find the $y$-intercepts by plugging in $x = 0$ and solving for $y$. If the graph is the graph of a function $y = f(x)$, then (assuming that 0 is in the domain) the $y$-intercept is the point $(0, f(0))$, and the $x$-intercepts are the points $(r, 0)$ such that $f(r) = 0$. The numbers $r$ such that $f(r) = 0$ are called the **zeros** (and sometimes also the **roots**) of the function.

**Definition 15.2.** A **polynomial** is a function of form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where the exponents on $x$ are nonnegative integers. A **polynomial equation** is an equation of form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0.$$

Its solutions are the zeros of the polynomial and are also called the **roots** of the polynomial. When $a_n \neq 0$ we say that the **degree** of the polynomial is $n$. The constants $a_0, a_1, \ldots, a_n$ are called the **coefficients**. For any polynomial $f(0) = a_0$ (the **constant term**) so the $y$-intercept of the graph is the point $(0, a_0)$.

**15.3.** A polynomial of degree one (or zero) is called **linear** (since its graph is a line), a polynomial of degree two is called **quadratic**, and a polynomial of degree three is called **cubic**. In Paragraph 6.4 we used the notation $f(x) = mx + b$ (slope-intercept form) rather than $f(x) = a_1 x + a_0$ for a linear function and in Theorem 11.2 we wrote the quadratic equation as $ax^2 + bx + c = 0$ rather than as $a_2 x^2 + a_1 x + a_0 = 0$.

**15.4.** It is easy to graph a linear function: just find two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on the graph and draw the line through them. It is also not hard to graph a quadratic function $f(x) = ax^2 + bx + c$. The graph is a parabola which opens up (like $y = x^2$) if $a > 0$ and down (like $y = -x^2$) if $a < 0$. By the Quadratic Formula in Theorem 11.2 the graph has two $x$-intercepts if the **discriminant** $b^2 - 4ac$ is positive, one $x$-intercept if the discriminant is zero, and no $x$-intercept if the discriminant is negative.

**Remark 15.5.** When $a = 1$ and $b$ and $c$ are integers we could try to solve the quadratic equation $x^2 + bx + c = 0$ by factoring. If

$$x^2 + bx + c = (x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1 r_2$$

(for all $x$) then

$$b = -(r_1 + r_2), \qquad c = r_1 r_2.$$

If we suspect that the roots are integers, we can try all possible ways of factoring $c$. For example, to solve $x^2 - 5x + 6 = 0$ we try $(r_1, r_2) = (1, 6)$, $(-1, -6)$, $(2, 3)$, $(-2, -3)$. Only $(r_1, r_2) = (2, 3)$ gives $-(r_1 + r_2) = -5$ so

$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

and the solutions of $x^2 - 5x + 6 = 0$ are $x = 2$ and $x = 3$. Of course, there is usually no reason to suspect that the roots are integers so it is best to use the Quadratic Formula (Theorem 11.2) to solve a quadratic equation.

**15.6.** A polynomial inequality like

$$x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 > 0$$

is easy to solve if we can factor the left hand side. Then we can write it in the form

$$(x - r_1)(x - r_2) \cdots (x - r_n) > 0.$$

On each of the intervals $(-\infty, r_1), (r_1, r_2), \ldots, (r_{n-1}, r_n), (r_n, \infty)$ the sign of the polynomial is constant so we can compute the sign by evaluating the polynomial at some point in the interval. For example,

$$(x - 1)(x - 4)(x - 9) > 0 \iff x \text{ in } (1, 4) \cup (9, \infty)$$

and

$$(x - 1)(x - 4)(x - 9) \geq 0 \iff x \text{ in } [1, 4] \cup [9, \infty)$$

As a check we evaluate
at $x = 0$ in $(-\infty, 1)$ and get $(0 - 1)(0 - 4)(0 - 9) = -36 < 0$,
at $x = 2$ in $(1, 4)$ and get $(2 - 1)(2 - 4)(2 - 9) = 14 > 0$,
at $x = 5$ in $(4, 9)$ and get $(5 - 1)(5 - 4)(5 - 9) = -16 < 0$, and
at $x = 10$ in $(9, \infty)$ and get $(10 - 1)(10 - 4)(10 - 9) = 54 > 0$.

**15.7.** To graph a polynomial $f(x)$ we use the following rules:

**(i)** The sign of the polynomial does not change between two adjacent roots. To determine this sign we can evaluate the polynomial at *any* number in between.

**(ii)** If the polynomial can be factored so that

$$f(x) = a_n(x - r_1)(x - r_2) \cdots (x - r_n)$$

where $r_1 < r_2 < \cdots < r_n$ are the roots, then between each pair of roots $r_k, r_{k+1}$ the graph reverses direction exactly once, i.e. either the function value $f(x)$ increases on some interval $[r_k, c_k]$ and then decreases on the interval $[c_k, r_{k+1}]$ or it decreases on some interval $[r_k, c_k]$ and then increases on the interval $[c_k, r_{k+1}]$

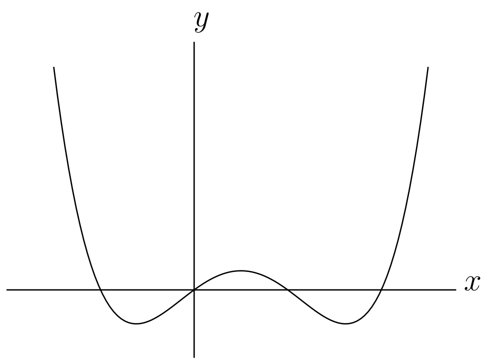
**(iii)** The absolute value of $f(x)$ is large when the absolute value of $x$ is large. Whether $f(x)$ is large positive or large negative depends only on the sign of the coefficient of $x^n$ (where $n$ is the degree) and on whether $n$ is odd or even.

In calculus you will learn how to determine on which intervals the function is increasing and you will learn enough to understand why (ii) is true.

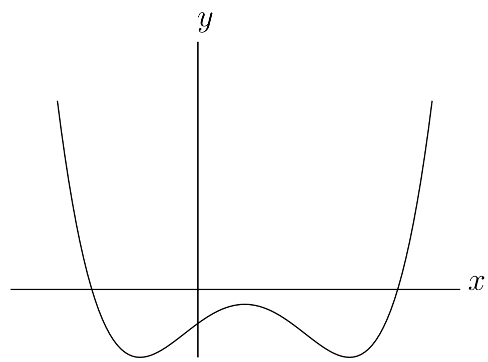
Below we have graphed the polynomial

$$f(x) = x^4 - 2x^3 - x^2 + 2x = (x + 1)(x - 0)(x - 1)(x - 2)$$

which factors completely as in item (ii). The degree is 4 and the roots are $-1, 0, 1, 2$. In each of the intervals $(-1, 0)$, $(0, 1)$, $(1, 2)$ between roots the function reverses direction exactly once. Next to the graph $y = f(x)$ is the graph $y = f(x) - 1$. The degree is still 4 but the graph $y = f(x) - 1$ has two $x$-intercepts (not 4 as does $y = f(x)$) and between them the function reverses direction three times. According to item (ii) this means that the polynomial $f(x) - 1$ does not factor completely (into real linear polynomials).




$y = f(x)$




$y = f(x) - 1$

# 16 Rational Functions

**Definition 16.1.** A **rational function** is a quotient

$$f(x) = \frac{g(x)}{h(x)}$$

of two polynomials. The domain is the set of all real numbers where the denominator $h(x)$ os not zero. Since the denominator $h(x)$ is not zero for $x$ in the domain, a zero of $f$ is the same as a zero of the numerator $g$. A point where the denominator is zero (and the numerator is not zero) is called a **pole** of $f$.

**16.2.** To graph a rational function

$$f(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$$

we use the following rules:

**(i)** On any interval which contains neither a zero nor a pole, the sign of $f(x)$ does not change. To determine this sign we can evaluate the rational function at *any* number in the interval.

**(ii)** If $n > m$ (and $a_n, b_m \neq 0$) then the absolute value of $f(x)$ is arbitrarily large when the absolute value of $x$ is sufficiently large.

**(iii)** If $n \leq m$ (and $a_n, b_m \neq 0$) then $f(x)$ is arbitrarily close to $c$ when the absolute value of $x$ is sufficiently large, where $c = a_n/b_m$ if $n = m$ and $c = 0$ is $n < m$.

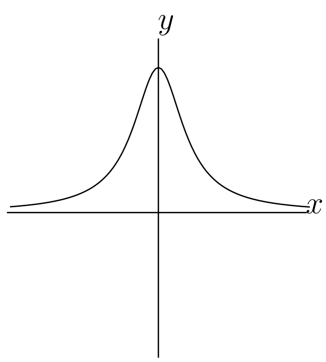
(iv) If $p$ is a pole, then the absolute value of $f(x)$ is arbitrarily large when $x$ is sufficiently close to $p$.

In case (iii) the horizontal line $y = c$ is called a **horizontal asymptote** of the graph of $y = f(x)$. The vertical line $x = p$ as in case (iv) is called a **vertical asymptote** of the graph.

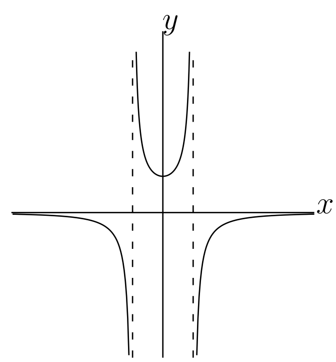
If a nonzero number is divided by a relatively small number, the result is large. This is why the absolute value of $f(x)$ is large when $x$ is near a pole. This is why (i) is true.

The reason why (ii)-(iv) are true is as follows. With $q(x), r(x), h(x)$ as in the Division Algorithm the term $r(x)/h(x)$ is very small when the absolute value of $x$ is large. This is because the degree of $r(x)$ is less than the degree of $h(x)$ so the absolute value of $r(x)$ is much smaller than the absolute value of $h(x)$ when the absolute value of $x$ is large. Hence the ratio $r(x)/h(x)$ is small and $f(x)$ is close to $q(x)$. When $m = n$, the quotient $q(x)$ is the constant $a_n/b_n$, when $n < m$ $q(x)$ is the zero polynomial, and when $n < m$, the quotient $q(x)$ has positive degree and hnce has large absolute value when the absolute value of $x$ is large.

**Example 16.3.** The $x$-axis is a horizontal asymptote of the graph $y = \dfrac{4}{1+x^2}$ and also of the graph $y = \dfrac{1}{1-x^2}$. The lines $x = \pm 1$ are vertical asymptotes of the graph $y = \dfrac{1}{1-x^2}$.




$y = 4/(1 + x^2)$




$y = 1/(1 - x^2)$

# 17 Exponentials and Logarithms

**17.1.** Each positive number $a$ distinct from one determines a function

$$f(x) = a^x$$

called the **exponential function base** $a$. It is characterized by the following properties:

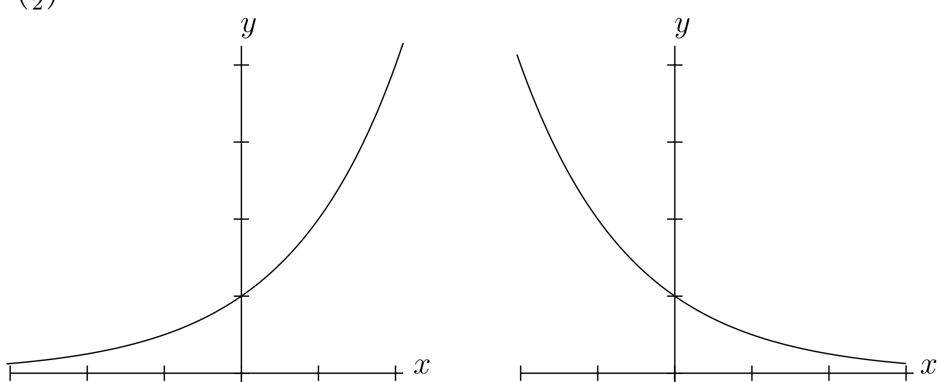
**(i)** The domain is the set of all real numbers and the range is the set of all positive real numbers.

**(ii)** $a^{u+v} = a^u \cdot a^v, \quad a^0 = 1.$

**(iii)** If $a > 1$, the exponential function base $a$ is increasing.

**(iv)** If $0 < a < 1$, the exponential function base $a$ is decreasing.

This was stated as Theorem 4.1. A careful definition of the exponential function requires concepts which are normally taught in calculus. All the exponent laws in Paragraph 1.9 which hold hold when the exponents are integers continue to hold for real exponents. Here are the graphs $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$.




$y = 2^x$ $\qquad$ $y = (1/2)^x$

If $a > 1$ then $0 < 1/a = a^{-1} < 1$ and $(1/a)^x \left(a^{-1}\right)^x = a^{-x}$. Hence the graph of $y = (1/a)^x$ is obtained from the graph of $y = a^x$ by reflection in the $y$-axis.

**17.2.** For $a > 1$ as $x$ increases through positive values the function value $y = a^x$ increases very rapidly; one calls this **exponential growth**. For example, $2^{10} = 1024$ while $2^{20} = 1,048,576$, i.e. doubling the input from ten to twenty changes the output from about one thousand to about one million. On the other hand as $x$ decreases through negative values the value $y = a^x$ approaches the $x$-axis very rapidly, but (since $a^x > 0$) never intersects it. For $0 < a < 1$, the situation is reversed: as $x$ increases, the function value $y = a^x$ approaches the $x$-axis very rapidly – this is called **exponential decay** – while as $x$ decreases through negative values the function value becomes large very rapidly. Whether or not $a > 1$ or $a < 1$ the function value is always positive.

**Remark 17.3.** We say that the $x$-axis is a **horizontal asymptote** for the graph $y = a^x$ but the situation is not exactly the same as it was for the graph of a rational function. When a rational function has a horizontal asymptote, the graph is close to the asymptote whenever the absolute value of $x$ is large whether $x$ is positive or $x$ is negative. For the exponential function, the graph approaches the $x$-axis when $x$ is large positive or large negative, but not both.

**Definition 17.4.** The **logarithm base** $a$ is the inverse function to the exponential function base $a$. It is denoted by $\log_a$ so

$$\boxed{y = \log_a(x) \iff x = a^y.}$$

i.e.

$$\boxed{y = \log_a(a^y) \qquad x = a^{\log_a(x)}.}$$

**17.5.** The exponential function satisfies the identity

$$a^u a^v = a^{u+v}$$

which says that "exponentiation changes addition into multiplication". Analogously, "the logarithm changes multiplication into addition". If $U = a^u$ and $V = a^v$ then $UV = a^{u+v}$. But these equations also say that $u = \log_a(U)$, $v = \log_a(V)$, and $u + v = \log_a(UV)$. Hence

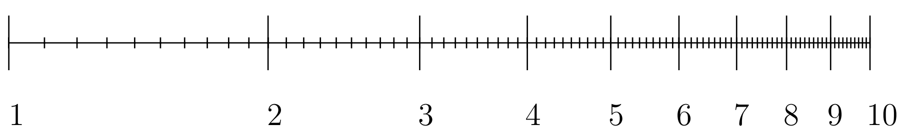
$$\log_a(UV) = \log_a(U) + \log_a(V).$$

The other laws of exponentiation imply in the same way corresponding laws for logarithms as follows:

| | | |
|---|---|---|
| **(i)** | $a^u a^v = a^{u+v}$ | $\log_a(UV) = \log_a(U) + \log_a(V)$ |
| **(ii)** | $a^0 = 1$ | $0 = \log_a(1)$ |
| **(iii)** | $(a^u)^p = a^{pu}$ | $\log_a(U^p) = p\log_a(U)$ |
| **(iv)** | $\dfrac{a^u}{a^v} = a^{u-v}$ | $\log_a\left(\dfrac{U}{V}\right) = \log_a(U) - \log_a(V)$ |

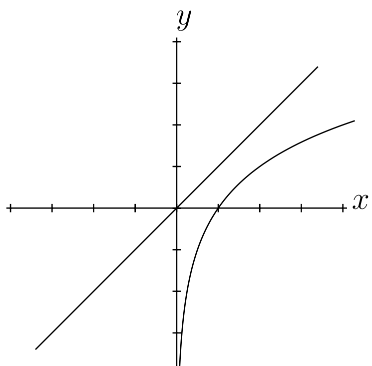
**Remark 17.6.** Before the advent of electronic computers, logarithms and exponentials were used to do arithmetic. To multiply two numbers $U$ and $V$ one would look up their logarithms in a table, add the logarithms, and then find the number in the table with (almost) the same logarithm as the sum. That number was the product $UV$. (The point is that addition of long numbers is much easier than multiplication.)

Scientists and engineers used a device called a **slide rule** to perform these operations quickly. Essentially it consisted of two rulers marked with numbers spaced according to their logarithms as in this diagram:

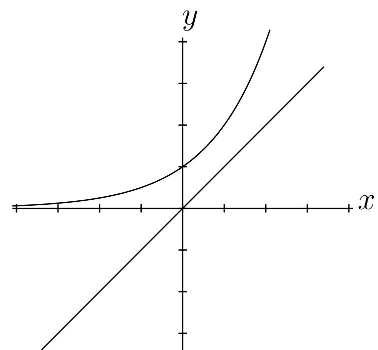



To compute the product of (say) $2\times3$ you place the 1 on one of the logarithmic rulers above the 2 on the other. The number on the bottom ruler below the 3 on the top ruler is then at a distance of $\log(2)+\log(3) = \log(2\times3)$ from the 1 on the bottom ruler and using the markings on the ruler you can read off the product. The above diagram was created with a computer and is quite accurate. You can see how it works by marking off the distance between the 1 and the 2 on a piece of paper. You will see that this distance is exactly the same as the distance between the 3 and the 6.

**17.7.** The equations $y = \log_a(x)$ and $x = a^y$ have the same graph. (This is always true for inverse functions.) The graph of $y = \log_a(x)$ is obtained from the graph of $y = a^x$ by reflecting in the line $y = x$, i.e. interchanging the $x$-axis and the $y$-axis.




$y = \log_2(x)$




$y = 2^x$

# 18 Exponential Growth and Decay

**Definition 18.1.** A quantity $N$ is said to obey an **exponential law** iff its value at time $t$ is given by the formula

$$N = N_0 a^t$$

where $a$ is a positive real number distinct from 1. Since $a^0 = 1$ it follows that $N_0$ is the value of $N$ at time $t = 0$ and since $a^{t+1} = a^t \cdot a$ it follows that $N$ changes by a factor of $a$ in each time interval of length 1. When $a > 1$ we say that $N$ **grows exponentialy** while if $a < 1$ we say that $N$ **decays exponentially**.

**18.2.** Many phenomena are governed by exponential growth laws. To name a few:

1. Money invested in a bank account grows exponentially at 6% per year (or whatever the interest rate is). If you invest $B_0$ dollars in a bank account that pays 6% per year in $t$ years your balance will be

$$B = B_0(1.06)^t \text{ dollars.}$$

2. The population of the world grows exponentially at 1.5% per year. (This will not continue forever.) As long as this growth rate continues the population $P$ of the world in $t$ years will be

$$P = P_0(1.015)^t$$

where $P_0$ is the population today. Other populations (e.g. bacteria) obey the same law (albeit with a different growth rate).

3. The amount of radioactivity in a radioactive material decays exponentially (at a rate that depends on the material). For example carbon-14 decays at 0.012% per year so that amount $N$ of carbon-14 in a sample of organic material is given by

$$N = N_0(0.988)^t$$

where $N_0$ is the amount of carbon-14 that was in the sample $t$ years ago. (Since the percentage of carbon-14 in living material is known, this formula can be used to estimate how long it has been since the organic material was alive.)

**18.3.** It is easy to understand why a quantity might grow or decay exponentially. Consider for example a population of bacteria. At any moment a certain percentage of the population will be at the right stage of development to subdivide. Each bacterium which subdivides adds a new bacterium to the population. If (say) 2% of the bacteria subdivide every hour, then in the first hour the population grows from $N_0$ to $N_1 = N_0(1.02)$. In the second hour the population grows from $N_1$ to $N_2 = N_1(1.02)$ so $N_2 = N_0(1.02)^2$. In general, after $t$ hours the population will be $N_0(1.02)^t$.

# 19 The Natural Logarithm

**19.1.** The irrational number

$$e = 2.7182818284590451\ldots$$

is most often used as the base for the exponential function. The exponential function base $e$ is called the **natural exponential** and the inverse function

$$\ln(x) := \log_e(x)$$

is called the **natural logarithm**. We shall explain what is natural about the natural logarithm in two ways.

**19.2.** First we note that the average rate of change (see Section 14) for the natural exponential on the interval $[x, x+h]$ is given by

$$\frac{e^{x+h} - e^x}{h} = e^x \left( \frac{e^h - 1}{h} \right)$$

It is proved in calculus that the number $e$ is characterized by the fact that the ratio $\left( \frac{e^h - 1}{h} \right)$ is very close to one when $h$ is small. This means that the average rate of change over a short interval $[x, x+h]$ (the *instantaneous rate of change*) is the value $e^x$ of the natural exponential.

Now consider a quantity $N$ governed by an exponential law

$$N = N_0 a^t$$

and let $r = \ln(a)$ be the natural logarithm of the base $a$. Then $a = e^r$ so $a^t = (e^r)^t = e^{rt}$ and the exponential law takes the form

$$N = N_0 e^{rt}.$$

Now consider the average rate of change of $N$ over a short time interval $[t, t+\Delta t]$ of duration $\Delta t$. Using the above formula with $x = rt$ and $h = r\Delta t$ we get

$$\frac{\Delta N}{\Delta t} = \frac{N_0 e^{rt + r\Delta t} - e^{rt}}{\Delta t} = N_0 e^{rt} \left( \frac{e^{r\Delta t} - 1}{\Delta t} \right) = Nr \left( \frac{e^h - 1}{h} \right)$$

which says that the instantaneous rate of change is $Nr$. For example, if $r = 0.06$ this says that the instantaneous rate of change is 6% of $N$. (Remember: "per" means divide, "of" means times, and "cent" means 100 so 6 % means 6/100 and 6% of $N$ is $0.06N$.) In summary

> The instantaneous rate of change of a quantity governed by the exponential law $N = N_0 e^{rt}$ is $100r\%$ (of $N$).

**19.3.** The second way of understanding what is natural about the natural exponential function is to consider a bank account which earns a nominal interest rate of (say) 6% per year. The interest rate is called nominal because the actual amount of interest paid depends on how often the bank pays the interest. An interest rate of 6% per year is the same as an interest rate of $6\%/12 = 0.5\%$ per month but if the bank pays the interest every month, the account will grow by slightly more than 6% in a year. Here is how a few different banks might pay interest.

(i) The Alaska Bank pays interest compounded annually. This means that every year it looks at the balance in an account and adds 6%. Thus an initial deposit of $B_0$ dollars grows to

$$B = B_0(1.06)^t$$

dollars after $t$ years (assuming no other deposits are made during this period).

(ii) The Minnesota Bank pays interest compounded monthly. This means that every month it looks at the balance in an account and adds $6/12\% = 0.5\%$. Thus an initial deposit of $B_0$ dollars grows to

$$B = B_0(1.005)^{12t}$$

dollars after $12t$ months $= t$ years (assuming no other deposits are made during this period.)

(iii) The Wisconsin Bank pays interest compounded weekly. This means that every week it looks at the balance in an account and adds $6/52\% = 0.001154\%$. Thus an initial deposit of $B_0$ dollars grows to

$$B = B_0\left(1 + \frac{0.06}{52}\right)^{52t}$$

dollars after $52t$ weeks $= t$ years.

(iv) The Delaware Bank pays interest compounded daily. This means that every day it looks at the balance in an account and adds $6/365\%$. Thus an initial deposit of $B_0$ dollars grows to

$$B = B_0\left(1 + \frac{0.06}{365}\right)^{365t}$$

dollars after $365t$ days $= t$ years.

The general formula for the balance $B$ in a bank account after $t$ years if the balance is initially $B_0$, the interest rate is $r$ per year, and the interest is **compounded** (i.e. paid) $m$ times per year is

$$B = B_0 \left(1 + \frac{r}{m}\right)^{mt}.$$

In all cases the formula is of the form

$$B = B_0 a^t$$

where $a = (1 + r/m)^m$ an the compounding period is $1/m$ years i.e. the interest is compounded $m$ times per year; When $m$ becomes infinite we say that the interest is **compounded continuously** and in calculus it is proved that this formula works with $a = e^r$ so the balance after $t$ years is given by

$$B = B_0 e^{rt}.$$

The following table shows the values of $a^t = \left(1 + \frac{r}{m}\right)^{mt}$ for $t = 2$, $r = 0.05$, $n = mt$, and various values of the number $m$ of compounding periods per year.

| $m$ | $\left(1 + \frac{r}{m}\right)^{mt} = \left(1 + \frac{0.05}{m}\right)^{2m}$ |
|---|---|
| 1 | 1.102500000000000 |
| 12 | 1.104941335558328 |
| 52 | 1.105117820169223 |
| 365 | 1.105163349128883 |
| $\infty$ | 1.105170918075648 |

Thus an account with a starting value of \$1000 at 5% per year will, in two years, earn \$102.50 in interest if the interest is compounded annually and \$105.17 in interest if the interest is compounded continuously.

**Remark 19.4.** Presumably in the old days the interest was credited to the account at the the end of each compounding period and if the account was closed in the middle of a compounding period the amount of interest earned since the beginning of the compounding period was forfeited. Today most banks would use the formula $B = B_0\left(1+\frac{r}{m}\right)^{mt}$ for all values of $t$ not just when $mt$ is an integer. Of course, the contract signed when the account was opened will make this precise.

**19.5.** The **doubling time** of a quantity $N = N_0e^{rt}$ which is increasing exponentially is the time $t$ such that $N = 2N_0$. Since

$$2N_0 = N_0e^{rt} \implies 2 = e^{rt} \implies \ln 2 = rt$$

the doubling time is $t = (\ln 2)/r$. Similarly, the **half life** of a quantity $N = N_0e^{rt}$ which is decreasing exponentially is the time $t$ such that $N = N_0/2$, i.e. $t = -(\ln 2)/r$.

# 20 Sequences and Series

**Definition 20.1.** A **sequence** is a function whose domain is a set of integers usually either the natural numbers $1, 2, 3, \ldots$ or the nonnegative integers $0, 1, 2, 3, \ldots$. The output of the sequence when the input is $n$ is called the $n$th **term** of the sequence.

**20.2.** Usually the input is denoted by a letter like $n$ near the middle of the alphabet and the output for input $n$ is denoted by making $n$ a subscript rather than by surrounding $n$ by parentheses, i.e. we write $a_n$ rather than $a(n)$. Here are some examples.

$$a_n = 5 + 3n, \qquad b_n = 2^n, \qquad c_n = n^2.$$

**Definition 20.3.** An **arithmetic sequence** is one of form

$$a_n = a + nd$$

where $a$ and $d$ are constants. An arithmetic sequence has the property that the difference of successive terms is a constant:

$$a_{n+1} - a_n = (a + (n+1)d) - (a + nd) = d.$$

**Definition 20.4.** A **geometric sequence** is one of form

$$g_n = ar^n$$

where $a$ and $r$ are constants. A geometric sequence has the property that the ratio of successive terms is a constant:

$$\frac{g_{n+1}}{g_n} = \frac{ar^{n+1}}{ar^n} = r.$$

**Example 20.5.** The sequence $c_n = n^2$ is neither arithmetic nor geometric. The sequence is not arithmetic because the difference

$$c_{n+1} - c_n = (n+1)^2 - n^2 = 2n + 1$$

is not constant (i.e. it depends on $n$) and the sequence is not geometric because the ratio

$$\frac{c_{n+1}}{c_n} = \frac{(n+1)^2}{n^2} = 1 + \frac{2}{n} = \frac{1}{n^2}$$

is not constant (i.e. it also depends on $n$).

**Definition 20.6.** Every sequence determines another sequence called the corresponding **series**. The $n$th term of the series is the sum of the first $n$ terms of the sequence. Thus if the $n$th term of the sequence is $a_n$, then the $n$th term of the series is

$$S_n = a_1 + a_2 + \cdots + a_n.$$

**Definition 20.7.** A series is often written using **Sigma notation** as in

$$S_n = \sum_{k=1}^{n} a_k.$$

More generally

$$\sum_{k=m}^{n} a_k := a_m + a_{m+1} + \cdots + a_n.$$

**Theorem 20.8.** *The sum of the first $n$ terms of a geometric sequence is given by*

$$\sum_{k=1}^{n} ar^k = a\,\frac{r^{n+1} - r}{r - 1}.$$

*Proof.* Suppose for example that $n = 3$ (the general case is similar). Then

$$\sum_{k=1}^{3} r^k = r + r^2 + r^3$$

and multiplying by $r$ gives

$$r\sum_{k=1}^{3} r^k = r(r + r^2 + r^3) = r^2 + r^3 + r^4$$

so

$$(1-r)\sum_{k=1}^{3} r^k = \sum_{k=1}^{3} r^k - r\sum_{k=1}^{3} r^k = (r + r^2 + r^3) - (r^2 + r^3 + r^4) = r - r^4.$$

Now divide by $(1-r)$ and multiply by $a$ to get the formula. □

**20.9. Mortgages**. A family takes out a 30 year $100,000 mortgage to buy a house. The interest rate is 6% per year and they will repay the loan in 360 equal monthly payments. To compute the monthly payment imagine that the family has taken 360 loans

$$100000 = L_1 + L_2 + \cdots + L_{360}$$

and they will repay the $k$th loan at the end of the $k$th month with the monthly payment of $a$ dollars. The monthly interest rate is $6\%/12 = 0.005$ so the amount owed on the $k$th loan after $k$ months is $(1.005)^k L_k$. This is the amount of the $k$th payment so $a = (1.005)^k L_k$ so $L_k = a(1.005)^{-k}$ so

$$100000 = L_1 + L_2 + \cdots + L_{360} = a(1.005)^{-1} + a(1.005)^{-2} + \cdots + a(1.005)^{-360}.$$

Using Sigma notation and Theorem 20.8 with $r = (1.005)^{-1}$ this may be written

$$100000 = \sum_{k=1}^{360} a(1.005)^{-k} = a\frac{(1.005)^{-361} - (1.005)^{-1}}{(1.005)^{-1} - 1}$$

To evaluate the expression on the right multiply top and bottom by 1.005 to get

$$\frac{(1.005)^{-361} - (1.005)^{-1}}{(1.005)^{-1} - 1} = \frac{(1.005)^{-360} - 1}{1 - 1.005} = \frac{1 - (1.005)^{-360}}{0.005} = 166.79$$

so $100000 = 166.79a$ so the monthly payment is

$$a = 100000/166.79 = 599.56.$$

## 21 Infinite Series

**Definition 21.1.** For a sequence $a_n$ the notation

$$\lim_{n\to\infty} a_n = b$$

means that that the numbers $a_n$ are arbitrarily close to the number $b$ when $n$ is sufficiently large. The expression on the left is called the the **limit** of $a_n$ as $n$ becomes infinite. For a series we also use the notation

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k.$$

**Theorem 21.2.** *If the ratio $r$ of successive terms in a geometric series is less than one in absolute value, then the sum of the infinite geometric series is*

$$\sum_{k=1}^{\infty} ar^k = \frac{ar}{1-r}.$$

*Proof.* By Theorem 20.8

$$\sum_{k=1}^{n} ar^k = \frac{ar - ar^{n+1}}{1-r} = \frac{ar}{1-r} + cr^n$$

where $c = ar/(1-r)$. In Section 17 we saw that for $|r| < 1$ the graph of the exponential function $y = r^x$ decays exponentially as $x$ becomes large positive. This implies that

$$\lim_{n\to\infty} r^n = 0$$

so

$$\sum_{k=1}^{\infty} ar^k = \lim_{n\to\infty} \sum_{k=1}^{n} ar^k = \frac{ar}{1-r} + c \lim_{n\to\infty} r^n = \frac{ar}{1-r}.$$

□

**Example 21.3.** (Zeno's Paradox) To travel one mile I must first travel the first half mile, then half of the remaining distance, then half of the remaining distance, and so on. How can I ever go whole distance? The answer is that the sum of all the distances is one:

$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}, \qquad \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{3}{4}, \qquad \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}, \qquad \ldots.$$

The numerator is only one less than the denominator:

$$\sum_{k=1}^{n}\left(\frac{1}{2}\right)^k = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n} = \frac{2^n-1}{2^n} = 1-2^{-n}.$$

The finite sums are getting closer to one and the infinite sum is

$$\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1.$$

**Example 21.4.** We use Theorem 21.2 to prove that the infinite repeating decimal $0.1\,36\,36\,36\,36\,36\,36\ldots.$ is equal to $3/22$. The first step is to make the decimal look like a geometric series.

$$\begin{aligned}
0.1\,36\,36\,36\,36\,36\,36\ldots &= \frac{1}{10} + 36\big(0.0\,01\,01\,01\,01\,01\ldots\big)\\
&= \frac{1}{10} + \frac{36}{10}\big(0.01\,01\,01\,01\,01\ldots\big)\\
&= \frac{1}{10} + \frac{36}{10}\big(10^{-2}+10^{-4}+10^{-6}+10^{-8}+\cdots\big)\\
&= \frac{1}{10} + \frac{36}{10}\sum_{k=1}^{\infty}\left(\frac{1}{100}\right)^k
\end{aligned}$$

Now by Theorem 20.8

$$\sum_{k=1}^{\infty}\left(\frac{1}{100}\right)^k = \frac{0.01}{1-0.01} = \frac{0.01}{0.99} = \frac{1}{99}$$

so

$$0.1\,36\,36\,36\,36\ldots = \frac{1}{10}+\frac{36}{10}\cdot\frac{1}{99}.$$

Now we do the arithmetic:

$$\frac{1}{10}+\frac{36}{10}\cdot\frac{1}{99} = \frac{99+36}{990} = \frac{135}{990} = \frac{15}{110} = \frac{3}{22}.$$

**21.5.** The concept of an infinite sum makes the definition of decimal expansion more precise. If $x$ is a real number between 0 and 1 it has a decimal expansion

$$x = \sum_{k=1}^{\infty} d_k\,10^{-k}$$

where each $d_k$ is an integer between 0 and 9. For most real numbers the digits $d_k$ won't follow any pattern, but

**Theorem 21.6.** *A real number $x$ is rational if and only if has a repeating decimal expansion like the one in Example 21.4.*

*Proof.* A rational number is a ratio $p/q$ of two integers $p$ and $q$. To see why the decimal expansion eventually repeats periodically imagine computing the decimal expansion by long division. At each step in the long division algorithm we compute the next digit in the quotient, multiply that digit by $q$, subtract the product to get the next remainder, and bring down the next digit from the dividend. The remainder is smaller than $q$, otherwise we would have used a larger digit in the quotient we are computing. Once we are computing digits of the right of the decimal point the digit we bring down from the dividend is always zero and since the remainder is always less than $q$ we will eventually find ourselves redoing what we have already done.

The proof that a real number with a repeating decimal expansion is rational is just like the computation in Example 21.4. We first write the number as the sum of a finite decimal and a negative power of ten times a geometric series. We then use Theorem 21.2 and do the arithmetic. □

# 22 Complex Numbers

**Definition 22.1.** The **complex numbers** are those numbers of form

$$z = x + iy$$

where $x$ and $y$ are real numbers and $i$ is a special new number called the **imaginary unit** which has the property that $i^2 = -1$.The real number $x$ is called the **real part** of $z$ and the real number $y$ is called the **imaginary part** of $z$. Two complex numbers are equal iff their real parts are equal and their imaginary parts are equal. The complex number

$$\bar{z} := x - iy$$

is called the **conjugate** of $z$.

**22.2.** The arithmetic operations are performed by treating $i$ as a variable and then replacing $i^2$ by $-1$ if necessary. This makes clear that the commutative, associative, and distributive laws hold, that the additive inverse is$-z = -(x_i y) = (-x) + (-y)i$ so that the law $z + (-z) = 0$, and that the rule for multiplying complex numbers is

$$\begin{aligned} zw &= (x+iy)(u+iv) \\ &= xu + (xv+yu)i + yvi^2 \\ &= xu + (xv+yu)i - yv \\ &= (xu - yv) + i(xv + yu) \end{aligned}$$

where $z = x + iy$ and $w = u + iv$ and $x$ $y$, $u$, $v$ are real. Using the algebraic law $(a+b)(a-b) = a^2 - b^2$ we see that the product of a complex number and its conjugate is

$$z\bar{z} = (x+iy)(x-iy) = x^2 - i^2y^2 = x^2 + y^2$$

which is a positive real number if $z \neq 0$. Hence the multiplication inverse is defined by

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}.$$

**22.3.** Conjugation satisfies the following laws.

1. A complex number is real if and only if it is equal to its conjugate:

$$x + iy = x - iy \iff y = 0 \iff z = x.$$

2. The conjugate of the conjugate is the original number.

$$\bar{\bar{z}} = \overline{(x - iy)} = x + iy = z.$$

3. Doing an arithmetic operation and then taking the conjugate gives the same result as taking the conjugates first and then doing the operation:

$$\overline{z \pm w} = \bar{z} \pm \bar{w}, \qquad \overline{zw} = \bar{z}\bar{w}, \qquad \overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}.$$

These are all easy to prove. For example if $z = x+iy$, $\bar{z} = x-iy$, $w = u+iv$, $\bar{w} = u - iv$ then

$$\begin{aligned} \bar{z}\bar{w} &= (x-iy)(u-iv) \\ &= (xu - yv) - (xv + yu)i \\ &= \overline{(xu - yv) + (xv + yu)i} \\ &= \overline{zw} \end{aligned}$$

## 23 Division of Polynomials

**Theorem 23.1** (**Division Algorithm**). *Let $p(x)$ and $d(x)$ be polynomials and assume that $d(x)$ is not the zero polynomial. Then the are unique polynomialsq$(x)$ and $r(x)$ such that*

$$p(x) = d(x) \cdot q(x) + r(x)$$

*and either $r(x)$ is the zero polynomial or the degree of the remainder $r(x)$ is less than the degree of the divisor $d(x)$.*

*Proof.* Guided Exercise. □

**Remark 23.2.** An analogous statement holds for integers. Let $p$ and $d$ be be integers with $d > 0$. Then there are unique integers $q$ and $r$ with

$$p = d \cdot q + r, \qquad 0 \leq r < d.$$

For example,

$$23 = 7 \cdot 3 + 2, \qquad 0 \leq 2 < 7.$$

In the case of rational numbers (as opposed to rational functions) the condition that the degree of $r(x)$ is less than the degree of $d(x)$ is replaced by the condition that the remainder is less than the divisor. In fact there is a very strong analogy between integers and rational numbers on the one hand and polynomials and rational functions on the other.

## 24 The Fundamental Theorem of Algebra

**Theorem 24.1** (**Fundamental Theorem of Algebra**). *A polynomial equation (of positive degree) has a complex root.*

**Remark 24.2.** The proof of the Fundamental Theorem is rather difficult and is not ordinarily taught in undergraduate courses. The theorem is true even if the coefficients in the polynomial are complex. In this section we shall see what the theorem means for **real polynomials**, i.e. polynomials with real coefficients.

**Theorem 24.3** (**Remainder Theorem**). *If a polynomial $f(x)$ is divided by $(x - r)$, the remainder is $f(r)$*

*Proof.* By the Division Algorithm (Theorem 23.1) we have

$$f(x) = (x - r)q(x) + c \qquad (*)$$

where $c$ is a polynomial of degree less than the degree of the divisor $(x - r)$. But the degree of the divisor $(x - r)$ is one so the degree of $c$ is zero, i.e. $c$ is a constant. Now plug in $x = r$ to get $f(r) = (r - r)q(r) + c = 0 + c = c$ so $(*)$ becomes $f(x) = (x - r)q(x) + f(r)$. □

**Corollary 24.4** (**Factor Theorem**). *If $r$ is a root of the polynomial $f(x)$, then $(x - r)$ evenly divides $f(x)$, i.e. $f(x) = (x - r)q(x)$.*

*Proof.* By the Remainder Theorem just proved we have

$$f(x) = (x - r)q(x) + f(r).$$

Hence if $f(r) = 0$, then $f(x) = (x - r)q(x)$. □

**Theorem 24.5** (**Complete Factorization**). *Let $f(x)$ be a polynomial of degree $n$. Then*

$$f(x) = c(x - r_1)(x - r_2)\cdots(x - r_n)$$

*where $c$ is a constant and $r_1, r_2, \ldots, r_n$ are the roots of $f(x)$, i.e. the solutions of $f(x) = 0$.*

*Proof.* By the Fundamental Theorem of Algebra, there is a complex number $r_1$ such that $f(r_1) = 0$ so by the Factor Theorem just proved there is a polynomial $f_1(x)$ such that

$$f(x) = (x - r_1)f_1(x).$$

Now $f_1(x)$ is a polynomial of degree $n - 1$ so by the same argument there is a complex number $r_1$ and a polynomial of $f_2(x)$ with $f_1(x) = (x - r_2)f_2(x)$ and hence

$$f(x) = (x - r_1)f_1(x) = (x - r_1)(x - r_2)f_2(x).$$

Repeating $n$ times gives

$$f(x) = (x - r_1)(x - r_2)\cdots(x - r_n)c.$$

where $c = f_n(x)$ has degree 0, i.e. $c$ is constant. □

**Corollary 24.6.** *Every real polynomial may be written as a product of real linear polynomials and real quadratic polynomials with negative discriminant.*

*Proof.* If the coefficients of $f(x)$ are real and $r$ is a complex but non real root of $f(x)$ then $0 = \overline{f(r)} = f(\bar{r})$ so the conjugate $\bar{r}$ is also a root. Let $p$ and $q$ bet the real and imaginary parts of $r$ so $r = p + qi$ and $\bar{r} = p - qi$. The $r + \bar{r} = 2p$ and $r\bar{r} = p^2 + q^2$. The quadratic polynomial

$$(x - r)(x - \bar{r}) = x^2 - (r + \bar{r})x + r\bar{r} = x^2 + 2px + (p^2 + q^2)$$

has negative discriminant $b^2 - 4ac = 4p^2 - 4(p^2 + q^2) = -q^2$. Thus the non real complex roots occur in pairs in the complete factorization and may be combined to give the desired factorization into real (quadratic and linear) polynomials. □

# A Where to Look in the Textbook

This appendix tells you where to look in the course textbook

> David Cohen: *College Algebra* Fifth Edition, Thomson Brooks/Cole 2003.

for additional reading.

**1** (Laws of algebra) The material in this section is mostly review but some of it is discussed in Appendices A.2 and B.1 of the textbook. The material in Paragraph 1.1 Paragraph 1.2 Paragraph 1.3 Paragraph 1.4 Page A-5 in Appendix A.2 Paragraph 1.5 Paragraph 1.6 Page A-5 in Appendix A.2 Paragraph 1.7 Pages 17 and 77, Definition 1.8 Appendix B.2 Page A-8, Paragraph 1.9

**2** (Kinds of Numbers) Most of the material in this section is covered in Section 1.1 of the textbook. Page 2. Paragraph 2.2 Remark 2.3 Page 729. Remark 2.5 See Page A-7 Appendix A.3 of the textbook for the proof that $\sqrt{2}$ is irrational.

**3** (Coordinates on the Line and Order) Interval notation is explained in Section 1.1 (Page 3). Paragraph 3.1 Page 1. Paragraph 3.2 Paragraph 3.3 Paragraph 3.4 Page 4 Paragraph 3.5 Section 1.2 Page 6. Paragraph 3.6 Section 1.2 Page 9. The properties of inequalities are reviewed at the beginning of Section 2.5 of the textbook.

**4** (Exponents) See Appendices B.2 and B.3 in the textbook. Theorem 4.1 Paragraph 4.2

**5** (Coordinates in the Plane and Graphs) Coordinates in the plane are introduced in Sections 1.2 of the textbook Paragraph 5.1 Definition 5.2 Page 36. The Distance Formula 5.4 first appears on Page 23-24. Paragraph 5.5 Pages 68-70. The Midpoint Formula 5.6 is on Page 27. Definition 7.1 Page 66, Page 636. Theorem 7.2 Page 638. Example 7.3 Page 639. Remark 7.4 Page 644, Page 652 (Exercise 19). Definition 7.5 Page 653. Theorem 7.6 Page 655. Example 7.7 Page 656.

**6** (Lines) This material is covered in section 1.6 of the textbook. Paragraph 6.1 Page 55. Definition 6.2 Page 48. Paragraph 6.3 Page 52. Paragraph 6.4 Page 55. Theorem 6.5 Page 56, Theorem 6.6 Page 56.

**8** (Solving Equations) Definition 8.1 appears on Page 13 of the textbook. Extraneous solutions are explained on pages 15-16. Paragraph 8.2 Paragraph 8.3 Paragraph 8.4 Page 104 Remark 8.5 Paragraph 8.6 This material is reviewed in Section 1.3 of the textbook.

**9** (Systems of Equations) Systems of two linear equations in two unknowns are reviewed in Section 6.1 of the textbook. Paragraph 9.1 Example 9.2 Paragraph 9.3 Paragraph 9.4

**10** (Symmetry) Definition 10.1 Section 1.7 Page 63. Paragraph 10.2 Page 64.

**11** (Completing the Square) Paragraph 11.1 Theorem 11.2 Pages 18 and 87. Theorem 11.3 Page 70. Theorem 11.4 Page 273. Paragraph 11.5 Pages 198-209.

**12** (Functions) Definition 12.1 Section 3.1 Page 160. Paragraph 12.2 Section 3.1 Page 167. Paragraph 12.3 12.3 Section 3.1 Pages 162-163. Definition 12.4 Section 3.2 Page 173. The Vertical Line Test 12.4 Section 3.2 Page 174. Example 12.5 Example 2 Page 174. Remark 12.6 Section 3.2 Pages 174-175. Paragraph 12.7 Section 3.5 Page 209. Definition 12.8 Section 3.5 Page 211. Remark 12.9 Section 3.5 Example 2 Page 211. Paragraph 12.10

**13** (Inverse Functions) Definition 13.1 Section 3.6 Page 222. Example 13.2 Section 3.6 Example 2 Page 223. Remark 13.3 Paragraph 13.4 Section 3.6 Example 2 Page 229. The Horizontal Line Test 13.4 Section 3.6 Page 229. Paragraph 13.5 Section 3.6 Page 229. Definition 13.6 186, 244 Theorem 13.7 Remark 13.8 Paragraph 13.9 Paragraph 13.10 Section 3.6 Page 227.

**14** (Average Rate of Change) Definition 14.1 Section 3.3 Page 189. Example 14.2 Paragraph 14.3 Paragraph 14.4

**15** (Polynomials) Section 4.6 Page 326. Paragraph 15.1 Definition 15.2 Paragraph 15.3 Paragraph 15.4 15.4 Remark 15.5 Paragraph 15.6 Paragraph 15.7

**16** (Rational Functions) Definition 16.1 Section 4.7 Theorem 23.1 345, 569-571 Paragraph 16.2 Section 4.7 Example 8 Page 350. Example 16.3

**17** (Exponentials and Logarithms) Paragraph 17.1 Section 5.1 Paragraph 17.2 Remark 17.3 Definition 17.4 Section 5.3 Paragraph 17.5 Box 17.5 Remark 17.6 Paragraph 17.7

**18** (Exponential Growth and Decay) Section 5.7. Definition 18.1 Paragraph 18.2 Paragraph 18.3

**19** (The Natural Logarithm) Paragraph 19.1 Section 5.2 Paragraph 19.2 Paragraph 19.3 Remark 19.4 Paragraph 19.5

**22** (Complex Numbers) Definition 22.1 Section 7.1 Paragraph 22.2 Paragraph 22.3

**23** (Division of Polynomials) Theorem Division Algorithm 23.1 Remark 23.2 An analogous statement holds for integers.

**24** (The Fundamental Theorem of Algebra) Theorem Fundamental Theorem of Algebra] 24.1 Theorem Remainder Theorem] 24.3 Corollary Factor Theorem] 24.4 Theorem Complete Factorization] 24.5 Corollary 24.6

**20** (Sequences and Series) Section 9.3. Definition 20.1 Paragraph 20.2 Definition 20.3 Section 9.4. Definition 20.4 Section 9.5. Example 20.5 Definition 20.6 Definition 20.7 Section 9.3 Theorem 20.8 Section 9.5. Paragraph 20.9 (Extra)

**21** (Infinite Series) Definition 21.1 Theorem 21.2 Example 21.3 Example 21.4 Paragraph 21.5 Theorem 21.6 Page 729.