Math 221 – Exam II Tuesday Mar 23 – 5:30-7:00 PM

Answers

I. (25 points.) Find $\frac{dy}{dx}$. Note: The book sometimes writes $D_x y$ for $\frac{dy}{dx}$.

(a) $y = (x^2 - x + 1)^{-7}$

Answer: *Let* $u = x^2 - x + 1$. *Then* $y = (x^2 - x + 1)^{-7} = u^{-7}$ *so*

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{du}(u^{-7}) \cdot \frac{du}{dx} \\ &= (-7u^{-8}) \cdot (2x - 1) \\ &= \boxed{-7(x^2 - x + 1)^{-8} \cdot (2x - 1)} \end{aligned}$$

(b) $y = \sin^4(x^2 + 3x)$

Answer: *Let* $u = x^2 + 3x$, *then* $\sin^4(x^2 + 3x) = \sin^4(u)$. *Let* $v = \sin(u)$, *then* $\sin^4(u) = v^4$. *It follows that*

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(v^4)}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} \\ &= (4v^3) \cdot (\cos(u)) \cdot (2x + 3) \\ &= (4\sin^3(u)) \cdot (\cos(u)) \cdot (2x + 3) \\ &= \boxed{4\sin^3(x^2 + 3x) \cdot \cos(x^2 + 3x) \cdot (2x + 3)} \end{aligned}$$

(c) $y = \left(\frac{\sin(x)}{\cos(2x)}\right)^3$

Answer: *Let* $u = \frac{\sin(x)}{\cos(2x)}$, *then* $\left(\frac{\sin(x)}{\cos(2x)}\right)^3 = u^3$. *It follows that*

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(u^3)}{du} \cdot \frac{du}{dx} \\ &= (3u^2) \cdot \left(\frac{\cos(x)\cos(2x) - \sin(x)(-2\sin(2x))}{\cos^2(2x)}\right) \\ &= \boxed{3\left(\frac{\sin(x)}{\cos(2x)}\right)^2 \cdot \left(\frac{\cos(x)\cos(2x) + 2\sin(x)\sin(2x)}{\cos^2(2x)}\right)} \end{aligned}$$

II. (20 points.) Write $\frac{dy}{dx}$ in terms of x and y.

(a) $4x^3 + 7xy^2 = 2y^3$

Answer:

$$\begin{aligned}
\frac{d}{dx}[4x^3 + 7xy^2] &= \frac{d}{dx}[2y^3] \\
12x^2 + (7y^2 + 7x(2y\frac{dy}{dx})) &= 6y^2\frac{dy}{dx} \\
12x^2 + 7y^2 + 14xy\frac{dy}{dx} &= 6y^2\frac{dy}{dx} \\
12x^2 + 7y^2 &= (6y^2 - 14xy)\frac{dy}{dx} \\
\boxed{\frac{12x^2 + 7y^2}{6y^2 - 14xy}} &= \frac{dy}{dx}
\end{aligned}$$

(b) $xy + \sin(xy) = 1$

Answer:

$$\begin{aligned}
\frac{d}{dx}[xy + \sin(xy)] &= \frac{d}{dx}[1] \\
(y + x\frac{dy}{dx}) + (\cos(xy)(y + x\frac{dy}{dx})) &= 0 \\
(x + x\cos(xy))\frac{dy}{dx} &= -(y + y\cos(xy)) \\
\frac{dy}{dx} &= -\frac{y(1 + \cos(xy))}{x(1 + \cos(xy))} \\
\frac{dy}{dx} &= \boxed{-\frac{y}{x}}
\end{aligned}$$

III. (25 points.) Two particles move along a coordinate line. At the end of t seconds their directed distances from the origin, in feet, are given by $s_1 = 4t - 3t^2$ and $s_2 = t^2 - 2t$, respectively.

(a) When do they have the same velocity?

Answer: *The velocities are given respectively by* $v_1 = 4 - 6t$ *and* $v_2 = 2t - 2$. *Then* $v_1 = v_2$ *when* $4 - 6t = 2t - 2$, *hence* $8t = 6$, *so* $\boxed{t=3/4}$.

(b) When do they have the same speed?

Answer: *Based on the previous problem, we know that the particles have the same velocity at* $t = 3/4$, *so this is one answer. The other case where the speeds* *can be equal is if the velocities have the opposite sign but the same magnitude (i.e.* $v_1 = -v_2$*). Then* $4 - 6t = 2 - 2t$*, so* $2 = 4t$ *and* $t = 1/2$*. The answers are thus* $\boxed{t = 1/2 \text{ and } t = 3/4}$.

(c) When do they have the same position?

Answer: *The positions are the same when* $s_1 = s_2$*. Then* $4t - 3t^2 = t^2 - 2t$*, so* $4t^2 - 6t = 0$*. Factoring, we have* $2t(2t - 3) = 0$*, so* $\boxed{t = 0 \text{ and } t = 3/2}$ *are the solutions.*

IV. (25 points.) **(a)** Find the equation of the tangent line to the graph of the curve

$$x^2y^2 + 4xy = 12y$$

at the point $(2, 1)$.

Answer:

$$\begin{aligned}
\frac{d}{dx}[x^2y^2 + 4xy] &= \frac{d}{dx}[12y] \\
(2xy^2 + x^2(2y\frac{dy}{dx}) + (4y + 4x\frac{dy}{dx})) &= 12\frac{dy}{dx} \\
2xy^2 + 4y &= 12\frac{dy}{dx} - x^2(2y\frac{dy}{dx}) - 4x\frac{dy}{dx} \\
2xy^2 + 4y &= (12 - 4x - 2x^2y)\frac{dy}{dx} \\
\frac{2xy^2 + 4y}{12 - 4x - 2x^2y} &= \frac{dy}{dx}
\end{aligned}$$

At the point $(2, 1)$*, we have*

$$\frac{dy}{dx} = \frac{2(2)(1)^2 + 4(1)}{12 - 4(2) - 2(2)^2(1)} = \frac{4 + 4}{12 - 8 - 8} = \frac{8}{-4} = -2.$$

Now we know the slope, so the equation of the tangent line is $y - 1 = -2(x - 2)$*, or* $\boxed{y = -2x + 5}$.

(b) The equation in part (a) of this problem defines a differentiable function $f(x)$ implicitly for x near 2, i.e.

$$x^2f(x)^2 + 4xf(x) = 12f(x).$$

This function satisfies $f(2) = 1$. Find the linear function $L(x)$ which best approximates $f(x)$ for x near 2.

Answer: *This is the linear approximation* $L(x) = f(2) + f'(2)(x - 2)$*. Since the graph of the linear approximation is the tangent line, the answer is the same as the answer to part (a), namely* $\boxed{L(x) = -2x + 5}$

V. (20 points.) Each edge of a variable cube is increasing at a rate of 1 inch per second. How fast is the volume of the cube increasing when an edge is 3 inches long?

Answer: *The formula for the volume of a cube is* $V = l^3$*, where* l *is the edge length. We want to calculate* $\frac{dV}{dt}$ *when* $l = 3$*. Using the Chain Rule, we have*

$$\frac{dV}{dt} = \frac{dV}{dl} \cdot \frac{dl}{dt} = 3l^2 \cdot \frac{dl}{dt}.$$

We are given that $\frac{dl}{dt} = 1$ *inch/second, and are interested in the calculation for* $l = 3$*. Substituting values, we have*

$$\frac{dV}{dt} = 3(3\boldsymbol{in})^2 \cdot 1\boldsymbol{in/s} = \boxed{27\ \mathbf{in}^3/\mathbf{s}}$$

VI. (25 points.) Determine where the graph of the function

$$g(x) = 4x^3 - 3x^2 - 6x + 12$$

is increasing, decreasing, concave up and concave down. Then sketch the graph.

Answer: *To determine intervals of monotonicity we use the first derivative*

$$g'(x) = 12x^2 - 6x - 6 = 6(2x^2 - x - 1) = 6(2x + 1)(x - 1).$$

Then $g'(x) = 0$ *when* $x = -1/2, 1$*. The intervals having these endpoints are*

$$(-\infty, -1/2), \qquad (-1/2, 1), \qquad (1, \infty).$$

One each of these intervals the derivative $g'(x)$ *cannot change sign: else there would be more zeros. To find the sign of* $g'(x)$ *on such an interval we evaluate at one point; it does not matter which.*

- $g'(-1) > 0$ *so* $g'(x) > 0$ *on* $(-\infty, -1/2)$ *so* $\boxed{g \text{ is increasing on } (-\infty, -1/2);}$
- $g'(0) < 0$ *so* $g'(x) > 0$ *on* $(-1/2, 1)$ *so* $\boxed{g \text{ is decreasing on } (-1/2, 1);}$
- $g'(2) > 0$ *so* $g'(x) > 0$ *on* $(1, \infty)$ *so* $\boxed{g \text{ is increasing on } (1, \infty).}$

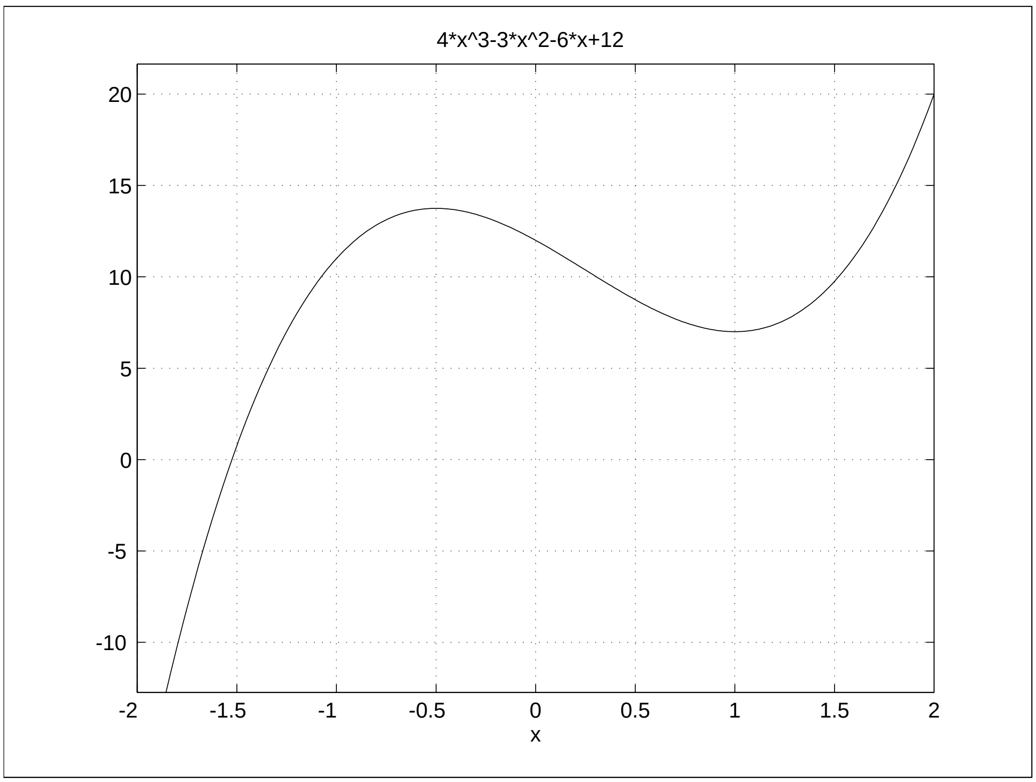
To determine intervals of concavity, we use the second derivative

$$g''(x) = 24x - 6 = 6(4x - 1).$$

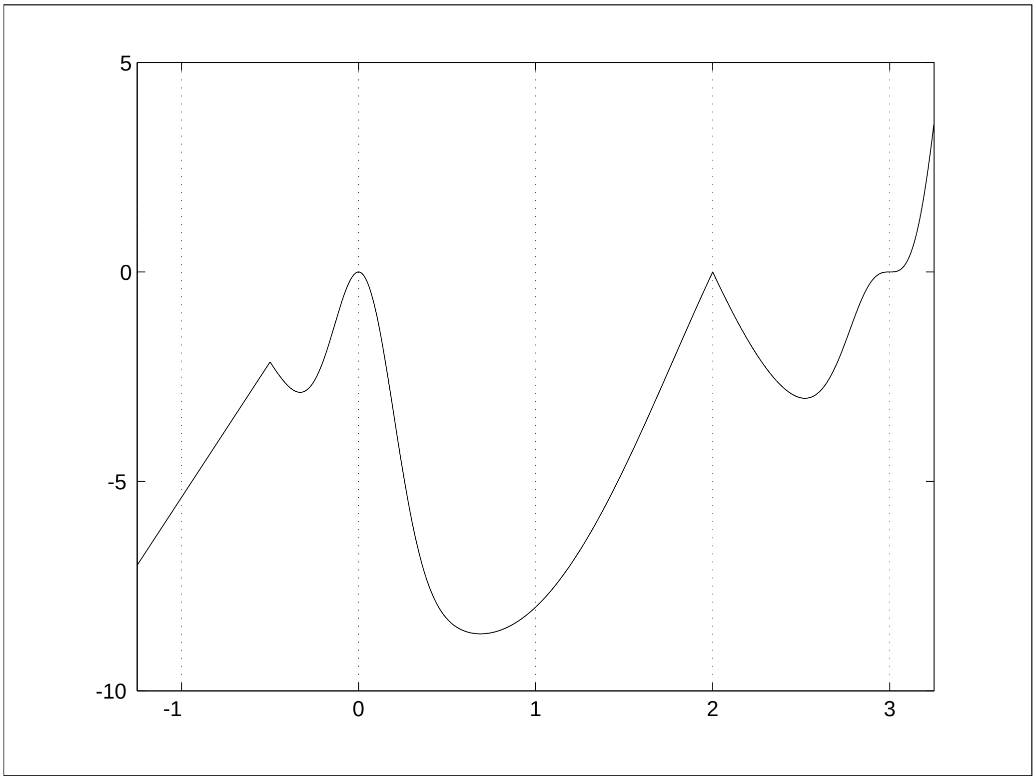
Then $g''(x) = 0$ *when* $x = 1/4$. *Thus*

- $g''(x) > 0$ *for* $x > 1/4$, *so* $\boxed{g \text{ is concave up on } (1/4, \infty)}$, *and*
- $g''(x) < 0$ *for* $x < 1/4$, *so* $\boxed{g \text{ is concave down on } (-\infty, 1/4)}$

The limits at $\pm\infty$ *are* $\lim_{x\to-\infty} g(x) = -\infty$ *and* $\lim_{x\to\infty} g(x) = \infty$. *Here is the graph.*



VII. (20 points.) Use the graph of $f(x)$ to determine the properties of f at the points given. For each point, write "Yes" or "No" in the box to indicate whether the point $x = c$ has the given property. (Write "Yes" in the column headed $f'(c)$ DNE if the derivative of $f(x)$ does not exist at $x = c$, i.e. if c is a singular point of $f(x)$.)



Point	$f'(c) > 0$	$f''(c) > 0$	$f'(c)$ DNE	$f'(c) = 0$	Inflection Pt.	Local Max
$c = -1$	Yes	No	No	No	No	No
$c = \ 0$	No	No	No	Yes	No	Yes
$c = \ 1$	Yes	Yes	No	No	No	No
$c = \ 2$	No	No	Yes	No	No	Yes
$c = \ 3$	No	No	No	Yes	Yes	No

VIII. (20 points.) A function $f(x)$ is concave up on an interval $a \le x \le b$ and c is a point between a and b.

(a) Write the equation for the secant line through the points $(a, f(a))$ and $(b, f(b))$ and then write an inequality which expresses the fact that the graph of $y = f(x)$ lies below this line for $a \le x \le b$,

Answer: *The point (x, y) lies on this line if and only if the slope from $(a, f(a))$ to (x, y) is the same as the slope from $(a, f(a))$ to $(b, f(b))$, i.e.*

$$\frac{y - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a}$$

so the secant line is $\boxed{y = f(a) + \left(\frac{f(b) - f(a)}{b - a}\right)(x - a).}$ *Since the graph lies below the secant line, the inequality*

$$\boxed{f(x) \le f(a) + \left(\frac{f(b) - f(a)}{b - a}\right)(x - a)}$$

holds for $a \le x \le b$.

(b) Write the equation for the tangent line at the point $(c, f(c))$ and then write an inequality which expresses the fact that the graph of $y = f(x)$ lies above this line for $a \le x \le b$,

Answer: *The point (x, y) lies on this line if and only if the slope from $(c, f(c))$ to (x, y) is the same as $f'(c)$, i.e.*

$$\frac{y - f(c)}{x - c} = f'(c)$$

so the tangent line is $\boxed{y = f(c) + f'(c)(x - c).}$ *Since the graph lies below the secant line, the inequality*

$$\boxed{f(c) + f'(c)(x - c) \le f(x)}$$

holds for $a \le x \le b$.

IX. (20 points.) A riverboat company offers a fraternal organization a Fourth of July excursion with the understanding that there will be at least 400 passengers. The price of each ticket will be \$12.00, and the company agrees to discount the price by \$0.02 for each passenger in excess of 400. Write an expression for the price function $p(x)$ of each ticket as a function of the number of passengers x, and find the number of passengers that makes the total revenue a maximum.

Answer: *Assuming $x \geq 400$, the price of a ticket will be*

$$p(x) = 12 - 0.02(x - 400) = \boxed{20 - \frac{x}{50}}.$$

The total revenue is

$$r(x) = x \cdot p(x) = 20x - \frac{x^2}{50}.$$

We are considering x in the interval $[400, \infty)$, so the maximum will occur either for $x = 400$ or at a point where $r'(x) = 0$. We calculate $r'(x) = 20 - \frac{2x}{50} = 20 - \frac{x}{25}$. This equals zero when $x = 20 \cdot 25 = 500$. We are too lazy to check values, so we use the second derivative test. Since $r''(x) = -\frac{1}{25} < 0$, we know that the function is concave down, hence $x = 500$ is a local maximum. Therefore, revenue is maximized when $\boxed{x = 500}$. We do not have to check the value of the endpoint because there is no local minimum in the interval $[400, 500]$.

Grader's Comments: I did not cover section 4.5 in lecture, but did say that the students were responsible for it. One of the TA's reported to me that some of his students believed they were not responsible for section 4.5 so I reemphasized in lecture that they were. As a result most students predicted that there would be a problem from section 4.5, had seen this problem solved, and remembered the answer. Many students did this problem well, but others wrote the correct answer with very little correct reasoning to support it (and a lot of incorrect reasoning). Life would have been a lot simpler had I changed the numbers in the problem.

Many students followed Example 2 page 189 of the text. In that example it is appropriate to express the x as a function of the price p rather than the reverse.

The wording of the problem above is not exactly the same as what appeared on the test; the wording that appeared on the test was the wording that appeared in the text (problem 12 on page 192). This wording differs from the above in that the phrase "the company agrees to discount the price by \$0.02 for each passenger in excess of 400" is replaced by "the company agrees to discount the price by \$0.20 for each 10 passengers in excess of 400". One can read this as implying that the price for 409 passengers is \$12 per passenger while the price for 410 passengers is \$11.80 per passenger. With this interpretation, the problem becomes much harder. The phrase in the box on page 189 of the text indicates that this interpretation is not intended.

Another thing I don't like about the problem is that there should be an upper bound for x: the boat won't carry a million passengers! (The lower bound of 400 is reasonable since the company wouldn't want to operate the boat at all if there are too few passengers.)

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__________ Thu Mar 25 16:04:50 2004
There are  202 scores
grade      range        count      percent
   A     180...200     17          8.4%
  AB     165...179     31         15.3%
   B     150...164     47         23.3%
  BC     120...149     62         30.7%
   C     100...119     26         12.9%
   D      80... 99     12          5.9%
   F       0... 79      7          3.5%
Mean score = 142.7. Mean grade = 2.66.
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