

Math 221 – Exam I – Friday February 20 – (50 Minutes)

Answers

I. (25 points.) Define *differentiable function*, *continuous function*, and prove that a differentiable function is continuous. In your proof you may use (without proof) the limit laws and high school algebra.

Grader's Comments: I assigned 5 points each for the definitions of "continuous function" and "differentiable function", and 15 points for the proof that a differentiable function is continuous.

Most students knew the definition of continuity at a point, namely $f(a) = \lim_{x \rightarrow a} f(x)$. Many did not remark that this must be true for all values of a in the domain of f , but I did not take off any points for that. Some instead characterized continuity as being able to graph the function without lifting one's pencil from the paper. This did not earn any points, as they were told specifically that this is NOT the definition.

For the definition of a differentiable function, many simply wrote down the definition of $f'(a)$. As long as they did this correctly, I did not take off any points, but wrote a remark that the limit must exist and be finite for all values of a in the domain of f .

For the proof, most students either managed to write down more or less correctly what was in the handout, or wrote nothing close to a proof, so the grading was close to binary. In some cases, the style was poor, or the logic was a bit mangled, e.g., no "=" signs where needed, missing limits, incorrect first line ($f'(a) = 0$ was common, followed by a more or less recitation of the rest of the proof). I took off a few points in these cases.

Some students, in lieu of a general proof, showed that some specific differentiable function is continuous. I did not give any points for this; after all, the handout told them exactly what to write.

II. (25 points.) Sketch the graph of

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } 0 \leq x < 1 \\ 1 + x & \text{if } x \geq 1 \end{cases}$$

Then find each of the following or state that it does not exist.

(a) $\lim_{x \rightarrow 0} f(x)$

(b) $\lim_{x \rightarrow 1} f(x)$

(c) $\lim_{x \rightarrow 2} f(x)$

(d) $\lim_{x \rightarrow 1^-} f(x)$

(e) $\lim_{x \rightarrow 1^+} f(x)$

(f) $f(1)$

Answer:

III. (25 points.) Find each limit or state that it doesn't exist. (Distinguish between an infinite limit and one which doesn't exist.)

(a) $\lim_{x \rightarrow \infty} \frac{x - 3}{\sqrt{x^2 - 9}}$.

Answer:

(b) $\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - 1}$

Answer:

IV. (25 points.) **(A)** If $f(3) = 7$, $f'(3) = 2$, $g(3) = 6$, $g'(3) = -10$, find

(a) $(f \cdot g)'(3)$

(b) $(f + g)'(3)$

(c) $(f/g)'(3)$

(B) If $f(5) = 9$, $g(9) = 4$, $g'(9) = -5$, $f'(4) = 7$, $f'(5) = 11$, and $h(x) = g(f(x))$, find $h'(5)$.

Answer:

V. (25 points.) **(A)** If $y = \frac{\sin x + \cos x}{\sin x}$ find $\frac{dy}{dx}$. (Remember $\frac{dy}{dx}$ and $D_x y$ are two notations for the same thing.)

Answer: By the quotient rule

$$\frac{dy}{dx} = \frac{(\cos x - \sin x) \sin x - (\sin x + \cos x) \cos x}{\sin^2 x} = \frac{\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}$$

Grader's Comments: Many students simplified first as in

$$y = 1 + \frac{\cos x}{\sin x}$$

and then incorrectly asserted that $\sin x / \cos x$ is the $\tan x$ rather than $\cot x$. Even if they did get $y = 1 + \cot x$ some forgot the negative sign in

$$\frac{dy}{dx} = -\csc^2 x.$$

(B) If $f(x) = \cos\left(\frac{3x^2}{x+2}\right)$, find $f'(x)$.

Answer: By the chain rule and the quotient rule

$$f'(x) = -\sin\left(\frac{3x^2}{x+2}\right) \frac{d}{dx}\left(\frac{3x^2}{x+2}\right) = -\sin\left(\frac{3x^2}{x+2}\right) \frac{6x(x+2) - 3x^2}{(x+2)^2}.$$

Grader's Comments: *The most common mistake was to use the (incorrect) formula $(g \circ h)'(x) = g'(h'(x))$ instead of the correct formula $(g \circ h)'(x) = g'(h(x))h'(x)$. These students wrote that the derivative is*

$$-\sin\left(\frac{6x(x+2) - 3x^2}{(x+2)^2}\right).$$

VI. (25 points.) Find the equation of the tangent line to the curve

$$y = (x^2 + 1)^3(x^4 + 1)^2$$

at the point $(x, y) = (1, 32)$.

Answer:

There are 208 scores

grade	range	count	percent
A	135...150	29	13.9%
AB	125...134	31	14.9%
B	115...124	43	20.7%
BC	100...114	52	25.0%
C	85... 99	27	13.0%
D	60... 84	18	8.7%
F	0... 59	8	3.8%

Mean score = 110.0. Mean grade = 2.67.
