Math 221 – Final Exam (Two Hours) – Wednesday May 15, 2002
Answers

I. (60 points.) **(a)** Find the second derivative d^2y/dx^2 of $y = x^7$ with respect to x.

Answer: $\dfrac{dy}{dx} = 7x^6$, $\dfrac{d^2y}{dx^2} = 42x^5$.

(b) Evaluate $\displaystyle\int_{\pi/4}^{\pi/3} \sin t\, dt$

Answer: $\displaystyle\int_{\pi/4}^{\pi/3} \sin t\, dt = -\cos t\Big|_{t=\pi/4}^{t=\pi/3} = \cos(\pi/4) - \cos(\pi/3) = \frac{\sqrt{2}}{2} - \frac{1}{2}.$

(c) Evaluate $\displaystyle\int_{\ln 3}^{\ln 6} 8e^x\, dx.$

Answer: $\displaystyle\int_{\ln 3}^{\ln 6} 8e^x\, dx = 8e^x\Big|_{x=\ln 3}^{x=\ln 6} = 48 - 24.$

(d) Evaluate $\displaystyle\int_8^9 2^t\, dt$

Answer: $\displaystyle\int_8^9 2^t\, dt = \int_8^9 e^{(\ln 2)t}\, dt = \frac{e^{(\ln 2)t}}{\ln 2}\Big|_{t=8}^{t=9} = \frac{2^9 - 2^8}{\ln 2}.$

II. (40 points.) **(a)** Find the equation of the tangent line to the curve $3xy^2 - 2x^2y = 1$ at the point $(x, y) = (1, 1)$.

Answer: By implicit differentiation we have

$$3y^2 + 6xy\frac{dy}{dx} - 4xy - 2x^2\frac{dy}{dx} = 0 \qquad (*)$$

so

$$\frac{dy}{dx} = \frac{4xy - 3y^2}{6xy - 2x^2}$$

and

$$\frac{dy}{dx}\Big|_{(x,y)=(1,1)} = \frac{4-3}{6-2} = \frac{1}{4}$$

and the equation for the tangent line is

$$y - 1 = \frac{x-1}{4}.$$

(b) Find d^2y/dx^2 at this point.

Answer: Differentiating (*) gives

$$6y\frac{dy}{dx} + 6y\frac{dy}{dx} + 6\left(\frac{dy}{dx}\right)^2 + 6xy\frac{d^2y}{dx^2} - 4y - 4x\frac{dy}{dx} - 4x\frac{dy}{dx} - 2x^2\frac{d^2y}{dx^2} = 0.$$

Evaluating at $(x, y) = (1, 1)$ gives

$$6 \cdot \frac{1}{4} + 6 \cdot \frac{1}{4} + 6\left(\frac{1}{4}\right)^2 + 6\,\frac{d^2y}{dx^2}\bigg|_{(x,y)=(1,1)} - 4 - 4 \cdot \frac{1}{4} - 4 \cdot \frac{1}{4} - 2\,\frac{d^2y}{dx^2}\bigg|_{(x,y)=(1,1)} = 0$$

so

$$(6-2)\,\frac{d^2y}{dx^2}\bigg|_{(x,y)=(1,1)} = -\frac{6}{4} - \frac{6}{4} - 6 \cdot \left(\frac{1}{4}\right)^2 + 4 + 4 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} = 2.625$$

and the value of d^2y/dx^2 at $(x, y) = (1, 1)$ is $2.625/4 = 0.65625$.

III. (60 points.) **(a)** Explain what a Riemann sum is and write the definition of $\int_a^b f(x)\,dx$ as a limit of Riemann sums.

Answer: A Riemann sum is a finite sum

$$S = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$$

where $a = x_0 < x_1 < \cdots < x_n = b$ and $x_{k-1} \leq c_k \leq x_k$. The Riemann integral is the limit

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} S$$

where $\|P\| = \max\{x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}\}$.

(b) State the Fundamental Theorem of Calculus.

Answer: If $I(x) = \int_a^x f(t)\,dt$ then $I'(x) = f(x)$. Hence, if $F'(x) = f(x)$, then $\int_a^b f(t)\,dt = F(b) - F(a)$.

(c) Illustrate the Fundamental Theorem of Calculus by computing the definite integral $\int_0^2 x\,dx$ in two ways, first by computing the area under the graph by taking the limit of the sum of the areas of narrow rectangles and then by applying the Fundamental Theorem of Calculus to get same answer. Hint: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Answer: Let $c_k = x_k = 2k/n$. Then $x_k - x_{k-1} = 2/n$, $\|P\| = 2/n$, and

$$S = \sum_{k=1}^{n} \frac{2k}{n} \cdot \frac{2}{n} = \frac{4}{n^2} \cdot \frac{n(n+1)}{2} = 2\left(1 + \frac{1}{n}\right).$$

Hence

$$\int_0^2 x\,dx = \lim_{n\to\infty} 2\left(1 + \frac{1}{n}\right) = 2.$$

The function $F(x) = x^2/2$ has derivative $F'(x) = x$. Hence

$$\int_0^2 x\,dx = F(2) - F(0) = \frac{2^2}{2} - 0 = 2.$$

IV. (30 points.) **(a)** What does it mean for a function $f(x)$ to be continuous at $x = a$?

Answer: The function $f(x)$ is continuous at $x = a$ iff $\lim_{x\to a} f(x) = f(a)$.

(b) What does it mean for a function $f(x)$ to be differentiable at $x = a$?

Answer: The function $f(x)$ is differentiable at $x = a$ iff the limit

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

exists.

(c) Find the values of a and b so that the function

$$f(x) = \begin{cases} x^2 + 3x + a, & \text{if } x \le 1, \\ bx + 2, & \text{if } x > 1, \end{cases}$$

is differentiable for all values of x.

Answer: For $f(x)$ to be continuous we must have equality of the one sided limits

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1+} f(x).$$

But

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} x^2 + 3x + a = 4 + a, \qquad \lim_{x\to 1+} f(x) = \lim_{x\to 1+} bx + 2 = b + 2,$$

so $f(1) = 4 + a = b + 2$. For $f(x)$ to be differentiable we must have equality of the one sided limits

$$\lim_{x\to 1-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1+} \frac{f(x) - f(1)}{x - 1}.$$

The left limit is

$$\lim_{x\to 1-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1-} \frac{(x^2+3x+a)-(4+a)}{x-1} = 2+3=5$$

and the right limit is

$$\lim_{x\to 1+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1+} \frac{(bx+2)-(b+2)}{x-1} = b$$

so $f'(1) = b = 5$. Hence (as $4+a = b+2$) we have $a = -1$.

V. (50 points.) Graph $f(x) = xe^x$. Your graph should indicate the local maxima and minima (if any), the inflection points (if any), and the limits $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$.

Answer: For the limits at $\pm\infty$ we have

$$\lim_{x\to\infty} xe^x = \infty\cdot\infty = \infty$$

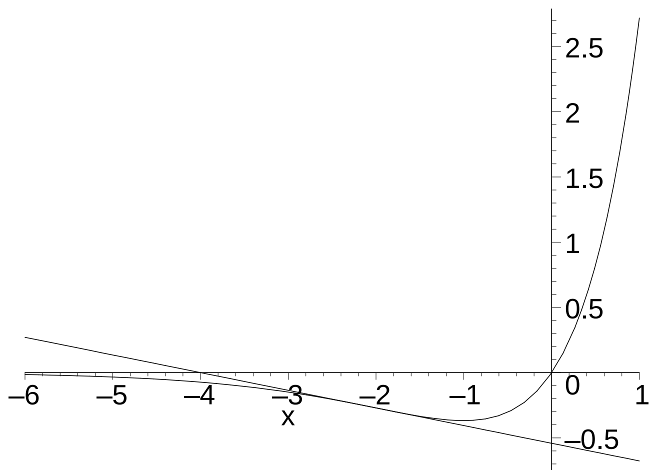
and, by l'Hôpital's rule,

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} = \frac{1}{\infty} = 0.$$

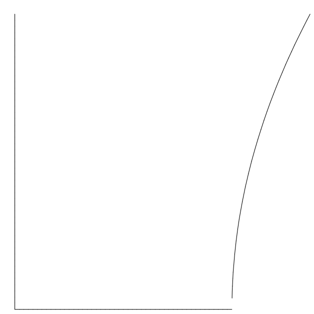
By the product rule we have $f'(x) = (1+x)e^x$ and $f''(x) = (2+x)e^x$. As $e^x > 0$ we have the following table:

x	$f(x)$	$f'(x)$	$f''(x)$	
$-\infty$	0			
$x < -2$		< 0	< 0	$\cap \searrow$
$x = -2$		< 0	$= 0$	inflection point
$-2 < x < -1$		< 0	> 0	$\cup \searrow$
$x = -1$		$= 0$	> 0	minimum
$-1 < x$		> 0	> 0	$\cup \nearrow$
∞	∞			

Here is the graph of $y = xe^x$ together with the tangent line at the point of inflection.



VI. (30 points.) A vase is constructed by rotating the curve $y = 10\sqrt{x-1}$ for $0 \le y \le 6$ around the y axis. It is filled with water to a height $y = h$ where $h < 6$.



(a) Find the volume of the water in terms of h.

Answer: Writing x as a function of y gives $x = 1 + \dfrac{y^2}{100}$. Using disks we have $dV = \pi x^2\,dy = \pi\left(1 + \dfrac{y^2}{100}\right)^2 dy$ so

$$V(h) = \int_0^h \pi\left(1 + \frac{y^2}{100}\right)^2 dy.$$

(b) If the vase is filling with water at the rate of 2 cubic units per second, how fast is the height of the water increasing when this height is 5 units?

Answer: We are given that $\frac{dV}{dt} = 2$. By the Fundamental Theorem

$$\frac{dV}{dh} = \pi \left(1 + \frac{h^2}{100}\right)^2$$

so by the Chain Rule

$$\frac{dh}{dt} = \frac{dV/dt}{dV/dh} = \frac{2}{\pi}\left(1 + \frac{h^2}{100}\right)^{-2}.$$

Hence when $h = 5$ we have

$$\left.\frac{dh}{dt}\right|_{h=5} = \frac{2}{\pi}\left(1 + \frac{25}{100}\right)^{-2} = \frac{32}{25\pi}.$$

VII. (30 points.) The population of California grows exponentially at a rate of 2% per year. The population of California on January 1, 1990 was 20,000,000.

(a) Write a formula for the population $N(t)$ of California t years after January 1, 1990.

Answer: $N = 20,000,000e^{0.02t}$.

(b) Each Californian consumes pizzas at the rate of 70 pizzas per year. At what rate is California consuming pizzas t years after 1990?

Answer: Let $P(t)$ denote the total number of pizzas consumed by Californians since some fixed date (say January 1, 1990) till t years after January 1, 1990. The rate of pizza consumption by all Californians is the rate per Californian times the number of Californians. Thus

$$\frac{dP}{dt} = 70N = 1,400,000,000e^{0.02t}$$

is the rate of pizza consumption by Californians at time t.

(c) How many pizzas were consumed in California from January 1, 1995 to January 1, 1999?

Answer: This is

$$P(9) - P(5) = \int_5^9 \frac{dP}{dt}\,dt = \left.\frac{1,400,000,000e^{0.02t}}{0.02}\right|_5^9$$

The answer is $70,000,000,000(e^{0.18} - e^{0.10})$ pizzas.

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There are  204 scores
grade      range       count      percent
   A     230...300     22         10.8%
  AB     210...229     20          9.8%
   B     170...209     44         21.6%
  BC     160...169     14          6.9%
   C     140...159     31         15.2%
   D     100...139     52         25.5%
   F       0... 99     21         10.3%
Mean score = 162.8. Mean grade = 2.15.
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